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# Degree distance and edge-connectivity

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**Abstract**

Let $G$ be a finite connected graph. The degree distance $D'(G)$ of $G$ is defined as $\sum_{\{u,v\}\subseteq V(G)}(\deg u + \deg v)\, d_G(u,v)$, where $\deg w$ is the degree of vertex $w$ and $d_G(u,v)$ denotes the distance between $u$ and $v$ in $G$. In this paper, we give asymptotically sharp upper bounds on the degree distance in terms of order and edge-connectivity.

## 1 Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The *distance*, $d_G(u,v)$, between $u$ and $v$, in $G$, is the length of a shortest $u$-$v$ path in $G$. The *degree*, $\deg v$, of a vertex $v$ of $G$, is the number of edges incident with it. The *edge-connectivity*, $\lambda = \lambda(G)$, of $G$ is the minimum number of edges whose removal results in a disconnected or trivial graph. Vertex-connectivity is defined analogously.

Topological indices and graph invariants based on the distances between the vertices of a graph are widely used in theoretical chemistry for establishing relations between the structure and the properties of molecules. They give correlations with physical, chemical and thermodynamic parameters of chemical compounds [4, 8]. One such topological index is the degree distance. Formally, the *degree distance*, $D'(G)$, is defined as

$$D'(G) = \sum_{\{u,v\}\subseteq V(G)} (\deg u + \deg v) d_G(u,v).$$

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The degree distance seems to have been considered for the first time by Dobrynin and Kochetova [5] in 1994 and at the same time by Gutman [6]. After 1994 many authors reported on the degree distance; for example, Bucicovschi and Cioabă [2], Dankelmann, Gutman, Mukwembi and Swart [3], Tomescu [11, 12] and Hou and Chang [7]. Tomescu [11], in 1999, proposed the following attractive conjecture on the upper bound on degree distance in terms of order.

**Conjecture 1** *[11] Let $G$ be a connected graph of order $n$. Then*

$$D'(G) \leq \frac{n^4}{27} + O(n^3).$$

Whilst this 1999 conjecture of Tomescu was completely resolved in [9] by refining the standard method of dealing with degree distance developed in [3], not much work has been done on the upper bounds on degree distance in terms of other parameters. Two of the present authors [10] showed that

$$D'(G) \leq \frac{n^4}{9(\delta+1)} + O(n^3), \tag{1}$$

where $\delta$ is the minimum degree of $G$. Moreover, for a fixed $\delta$, the inequality is asymptotically sharp. The present authors [1] continued this study and improved the upper bound (1) for graphs with fixed vertex-connectivity. Precisely, they proved the asymptotically tight upper bound:

$$D'(G) \leq \frac{n^4}{27\kappa} + O(n^3), \tag{2}$$

for a $\kappa$-connected graph $G$ of order $n$. The two bounds, (1) and (2), solve completely the problem of bounding degree distance in terms of order and two classical connectivity measures, namely, minimum degree, and vertex-connectivity. In this paper, we are concerned with finding upper bounds on degree distance in terms order and the third connectivity measure, edge-connectivity.

For $\lambda \geq 8$, the bound is a direct consequence of (1) while the cases $\lambda \leq 7$ are more complicated. Thus for $\lambda \geq 8$, an application of the inequality, $\delta \geq \lambda$, to (1) yields the following proposition.

**Proposition 1** *Let $G$ be a $\lambda$-edge-connected graph, $\lambda \geq 8$, of order $n$. Then*

$$D'(G) \leq \frac{n^4}{9(\lambda+1)} + O(n^3).$$

*Moreover, for a fixed $\lambda$, this inequality is asymptotically sharp.*

The problem of getting better upper bounds of the degree distance in terms of order and edge-connectivity $\lambda$, where $2 \leq \lambda \leq 7$, turns out to be harder and requires some additional ideas apart from the standard method of treating degree distance that was introduced in [3]. We will therefore consider this problem separately as the subject of this article. Thus here we completely solve the problem of relating degree distance to order and each of the three classical connectivity measures, namely, minimum degree, vertex-connectivity and edge-connectivity.

The notation and terminology we use is as follows. The *diameter*, $\mathrm{diam}(G) = d$, of $G$ is the largest of the distances between two vertices in $G$. The *eccentricity*, $ec_G(v)$, of a vertex $v \in V(G)$ is the maximum distance between $v$ and any other vertex in $G$. For a vertex $v$ of $G$, we denote by $D(v)$ the total distance or the status of $v$, i.e, $D(v) = \sum_{x \in V(G)} d_G(v,x)$. The quantity $\deg vD(v)$ is denoted by $D'(v)$. We will often make use of Tomescu's observation [11] that the degree distance can equivalently be expressed as

$$D'(G) = \sum_{v \in V(G)} D'(v).$$

We denote the open neighbourhood of $v$ by $N(v)$, that is, $N(v) = \{x \in V(G) \mid d_G(x,v) = 1\}$. The closed neighbourhood of $v$ in $G$, i.e., $N(v) \cup \{v\}$, is denoted by $N[v]$. Let $G_1$ and $G_2$ be two vertex disjoint graphs. The *union*, $G_1 \cup G_2$, of $G_1$ and $G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join*, $G_1 + G_2$, of $G_1$ and $G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. For $k \geq 3$ vertex disjoint graphs $G_1, G_2, \ldots, G_k$, the *sequential join*, $G_1 + G_2 + \cdots + G_k$, is the graph

$$(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{k-1} + G_k).$$

For nonempty subsets $V_1, V_2 \subset V(G)$, we denote by $E(V_1, V_2)$ the set $\{e = xy \in E(G) \mid x \in V_1, y \in V_2\}$ of edges with one end in $V_1$ and the other end in $V_2$. For any $v \in V(G)$ with eccentricity $e$, let

$$N_i(v) := \{x \in V(G) | d_G(x,v) = i\}$$

for all $i = 0, 1, 2, \ldots, e$, and $k_i(v) = |N_i(v)|$. Where vertex $v$ is understood, we write $N_i$ and $k_i$ instead of $N_i(v)$ and $k_i(v)$, respectively. Where there is no danger of confusion, we simply write $d(u,v)$ instead of $d_G(u,v)$.

## 2 Results

We first illustrate that the bound presented in Proposition 1 is, for a fixed $\lambda$, asymptotically sharp. For positive integers $n$, $\lambda$ and $k$ with $k \equiv 1 \pmod 3$, consider the graph $G_{n,k,\lambda} = G_1 + G_2 + \cdots + G_k$, where $G_1 = K_{\lceil \frac{1}{2}(n - \frac{(k-2)(\lambda+1)}{3}) \rceil}$, $G_k = K_{\lfloor \frac{1}{2}(n - \frac{(k-2)(\lambda+1)}{3}) \rfloor}$, $G_2 = K_\lambda = G_{k-1}$ and for $3 \leq i \leq k-2$,

$$G_i = \begin{cases} K_{\frac{\lambda+1}{3}} & \text{if } \lambda \equiv 2 \bmod 3, \\ K_{\frac{\lambda}{3}} \text{ for } i = 0, 2 \bmod 3 \quad \text{and} \quad K_{\frac{\lambda}{3}+1} \text{ for } i = 1 \bmod 3 & \text{if } \lambda \equiv 0 \bmod 3, \\ K_{\frac{\lambda+2}{3}} \text{ for } i = 0, 2 \bmod 3 \quad \text{and} \quad K_{\frac{\lambda-1}{3}} \text{ for } i = 1 \bmod 3 & \text{if } \lambda \equiv 1 \bmod 3. \end{cases}$$

Then $D'(G_{n,k,\lambda}) = \dfrac{n^4}{9(\lambda+1)} + O(n^3)$, when $k = \frac{n}{\lambda+1} + O(1)$, confirming that the bound presented in Proposition 1 is, for a fixed $\lambda$, asymptotically sharp.

The following discussion is useful in this paper:

**Discussion 1** Let $G$ be a graph, $V_1, V_2 \subset V(G)$ with $V_1 \cap V_2 = \emptyset$. Clearly, $|E(V_1, V_2)| \leq |V_1||V_2|$. If $E(V_1, V_2)$ is a disconnecting set of $G$, then $|E(V_1, V_2)| \geq \lambda(G)$ so that $|V_1||V_2| \geq \lambda(G)$. Let $v \in V(G)$. Then $k_i k_{i+1} \geq \lambda$ for all $i = 1, 2, \ldots, ec_G(v) - 1$.

The following lemma follows from $ab \leq (\frac{a+b}{2})^2$. In other words, the geometric mean of two (positive) real numbers never exceeds their arithmetic mean.

**Lemma 1** *For positive integers $a$ and $b$,*

*(a)* $ab \geq 2$ *implies that* $a + b \geq 3$.

*(b)* $ab \geq 3$ *implies that* $a + b \geq 4$.

*(c)* $ab \geq 4$ *implies that* $a + b \geq 4$.

*(d)* $ab \geq 5$ *implies that* $a + b \geq 5$.

*(e)* $ab \geq 6$ *implies that* $a + b \geq 5$.

*(f)* $ab \geq 7$ *implies that* $a + b \geq 6$.

We now present a very simple, but important observation.

**Fact 1** *Let $G$ be a 2-edge-connected graph of order $n$ and diameter $d$. If $v \in V(G)$, then*

$$d \leq \frac{2}{3}(n - \deg\ v) + \frac{4}{3}.$$

*Proof of Fact 1:* Let $v_0$ be a vertex of $G$ of eccentricity $d$ and let $N_i = N_i(v_0)$. Let $v \in V(G)$. Then $v \in N_i$ for some $i \in \{0, 1, 2, \ldots, d\}$. Thus, $N(v) \subset N_{i-1} \cup N_i \cup N_{i+1}$, and recall by Lemma 1 $(a)$ that $|N_j \cup N_{j+1}| \geq 3$ for all $j = 1, 2, \ldots, d-1$. Hence,

$$\begin{aligned} n &\geq \left|\bigcup_{j=0}^{i-2} N_j\right| + \deg v + 1 + \left|\bigcup_{j=i+2}^{d} N_j\right| \\ &\geq \deg v + 1 + 3\left(\frac{d-2}{2}\right) \\ &\geq \deg v + \frac{3}{2}d - 2. \end{aligned}$$

Hence $d \leq \frac{2}{3}(n - \deg v) + \frac{4}{3}$, as required. □

We will need the following useful result.

**Proposition 2** *Let $G$ be a 2-edge-connected graph of order $n$ and diameter $d$. If $v \in V(G)$, then*

$$D(v) \leq d(n - \frac{3}{4}d - \deg\, v) + O(n).$$

**Proof:** Let $v \in V(G)$, denote the eccentricity of $v$ by $e$. For all $i = 1, 2, , \ldots, e$, let $N_i = N_i(v)$ and $|N_i| = k_i$. Clearly, $k_1 = \deg\, v$. Since $G$ is 2-edge-connected, then for all $i = 1, 2, \ldots, e-1$, $k_i k_{i+1} \geq 2$ and thus by Lemma 1 $(a)$, $k_i + k_{i+1} \geq 3$. Hence,

$$\begin{aligned} D(v) &= 1k_1 + 2k_2 + \cdots + ek_e \\ &\leq \begin{cases} \deg\, v + 2\cdot 1 + 3 \cdot 2 + \cdots + (e-2)\cdot 1 + (e-1)\cdot 2 \\ +e(n - \frac{3}{2}e - \deg\, v + 2) + O(n) \qquad \text{if } \ e \text{ is even,} \\ \\ \deg\, v + 2\cdot 1 + 3 \cdot 2 + \cdots + (e-2)\cdot 2 + (e-1)\cdot 1 \\ +e(n - \frac{3}{2}e - \deg\, v + \frac{5}{2}) + O(n) \qquad \text{if } \ e \text{ is odd,} \end{cases} \\ &\leq e\left(n - \frac{3}{4}e - \deg\, v\right) + O(n). \end{aligned}$$

Now consider $f(x) := x(n - \frac{3}{4}x - \deg\, v)$, where $x = e$. The function $f$ is increasing on $\left[1, \frac{2}{3}(n - \deg\, v)\right]$. Using Fact 1 and $1 \leq e \leq d$, we consider two cases. First if $d \leq \frac{2}{3}(n - \deg\, v)$, then $D(v) \leq f(d) + O(n) = d\left(n - \frac{3}{4}d - \deg\, v\right) + O(n)$. Secondly, if by Fact 1, $d = \frac{2}{3}(n - \deg\, v) + c$, where $0 \leq c \leq \frac{4}{3}$, then $f \leq f\left(\frac{2}{3}(n - \deg\, v)\right) = f(d-c)$. But

$$\begin{aligned} f(d-c) &= (d-c)\left(n - \frac{3}{4}(d-c) - \deg\, v\right) \\ &= d\left(n - \frac{3}{4}d - \deg\, v\right) + O(n). \end{aligned}$$

Hence, in both cases $D(v) \leq d\left(n - \frac{3}{4}d - \deg\, v\right) + O(n)$, as required. □

The standard technique of dealing with bounding degree distance presented in [3] does not account for the relationship between degree distance and edge-connectivity. In the next theorem, we will refine the vertex partitions used in [3] to adequately account for edge-connectivity. The diameter plays a crucial role and provides us with the following intermediate result.

**Theorem 1** *Let $G$ be a 2-edge-connected graph of order $n$ and diameter $d$. Then*

$$D'(G) \leq \begin{cases} \frac{1}{4}dn(n - \frac{3}{2}d)^2 + O(n^3) & \textit{if } d < \frac{n}{3}, \\ \\ \frac{3}{4}d^2(n - \frac{3}{2}d)^2 + O(n^3) & \textit{if } d \geq \frac{n}{3}. \end{cases}$$

*Moreover, this inequality is asymptotically sharp.*

**Proof:** Let $v_0$ be a vertex of $G$ of eccentricity $d$ and let $N_j = N_j(v_0)$. Recall that $|N_j \cup N_{j+1}| \geq 3$ for all $j = 0, 1, 2, \ldots, d-1$. For each set $B_i \in \{N_0 \cup N_1, N_2 \cup N_3, N_4 \cup N_5, \ldots\}$ choose any three elements $u_{i1}, u_{i2}, u_{i3} \in B_i$ and denote the set $\{u_{i1}, u_{i2}, u_{i3}\}$ by $A_i$, $i = 1, 2, \ldots, \lceil \frac{d+1}{2} \rceil$. Let $N := \cup_{i=1}^{\lceil \frac{d+1}{2} \rceil} A_i$.

**Claim 1** *Let $N$ be as above. Then*

$$\sum_{u \in N} D'(u) \leq O(n^3).$$

*Proof of Claim 1:* Partition $N$ as $N = U_1 \cup U_2 \cup \cdots \cup U_9$, where

$$\begin{aligned}
U_1 &= \{u_{11}, u_{41}, u_{71}, \ldots\},\\
U_2 &= \{u_{12}, u_{42}, u_{72}, \ldots\},\\
U_3 &= \{u_{13}, u_{43}, u_{73}, \ldots\},\\
U_4 &= \{u_{21}, u_{51}, u_{81}, \ldots\},\\
U_5 &= \{u_{22}, u_{52}, u_{82}, \ldots\},\\
U_6 &= \{u_{23}, u_{53}, u_{83}, \ldots\},\\
U_7 &= \{u_{31}, u_{61}, u_{91}, \ldots\},\\
U_8 &= \{u_{32}, u_{62}, u_{92}, \ldots\},\\
U_9 &= \{u_{33}, u_{63}, u_{93}, \ldots\}.
\end{aligned}$$

Then,

$$\sum_{u \in N} D'(u) = \sum_{u \in U_1} D'(u) + \sum_{u \in U_2} D'(u) + \ldots + \sum_{u \in N_9} D'(u).$$

For each $x, y \in U_i$, $i = 1, 2, \ldots, 9$, since $d(x, y) \geq 5$ we have $N(x) \cap N(y) = \emptyset$. It follows that $\sum_{x \in U_i} \deg x \leq n$ for $i = 1, 2, \ldots, 9$. From Proposition 2,

$$\begin{aligned}
D(x) &\leq d\left(n - \frac{3}{4}d - \deg x\right) + O(n)\\
&= O(n^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{u \in N} D'(u) &= \sum_{u \in N} D(u) \deg u\\
&\leq O(n^2) \left( \sum_{u \in U_1} \deg u + \sum_{u \in U_2} \deg u + \ldots + \sum_{u \in U_9} \deg u \right)\\
&\leq O(n^2)(9n)\\
&= O(n^3),
\end{aligned}$$

and Claim 1 is proven.

From here on-wards we partition the remaining vertices of $G$ analogously to the standard partitioning developed in [3]. Let $\mathcal{C}$ be a maximum set of disjoint pairs of vertices from $V - N$ which lie at a distance at least 3, i.e., if $\{a, b\} \in \mathcal{C}$, then $d(a, b) \geq 3$. If $\{a, b\} \in \mathcal{C}$ we say $a$ and $b$ are partners. Finally, let $K$ be the remaining vertices of $G$, i.e., $K = V - N - \{x : x \in \{a, b\} \in \mathcal{C}\}$. Let $|K| = k$, and $|\mathcal{C}| = c$. Then

$$n = 3\left\lceil \frac{d+1}{2} \right\rceil + 2c + k. \tag{3}$$

**Fact 2** *Let* $\{a, b\} \in \mathcal{C}$. *Then* $\deg a + \deg b \leq n - \frac{3}{2}d + O(1)$.

*Proof of Fact 2:* Note that, since $d(a, b) \geq 3$, $N[a] \cap N[b] = \emptyset$. Also, each of the two vertices, $a$ and $b$, can be adjacent to at most 9 vertices in $N$. Thus,

$$\begin{aligned} n &\geq \deg a + 1 + \deg b + 1 + |N| - 18 \\ &\geq \deg a + \deg b + \frac{3}{2}d + \frac{3}{2} - 16 \\ &= \deg a + \deg b + \frac{3}{2}d + O(1), \end{aligned}$$

and rearranging the terms completes the proof of Fact 2.

Now consider two cases.

Case 1: $k \leq 1$. For $x \in K$, $D(x) \leq (n-1)^2$, so $D'(x) \leq (n-1)^3$. Thus $\sum_{x \in K} D'(x) = O(n^3)$.

**Claim 2** *If* $\{a, b\} \in \mathcal{C}$, *then* $D'(a) + D'(b) \leq \frac{1}{2}dn(n - \frac{3}{2}d) + O(n^2)$.

*Proof of Claim 2:* By Proposition 2, $D(a) \leq d\left(n - \frac{3}{4}d - \deg a\right) + O(n)$. Hence,

$$D'(a) \leq \deg a \left(d\left(n - \frac{3}{4}d - \deg a\right)\right) + O(n^2).$$

Similarly, $D'(b) \leq \deg b \left(d\left(n - \frac{3}{4}d - \deg b\right)\right) + O(n^2)$. Thus,

$$\begin{aligned} D'(a) + D'(b) &\leq \deg a\left(d\left(n - \frac{3}{4}d - \deg a\right)\right) + \deg b\left(d\left(n - \frac{3}{4}d - \deg b\right)\right) + O(n^2) \\ &= d\left((\deg a + \deg b)\left(n - \frac{3}{4}d\right) - \left((\deg a)^2 + (\deg b)^2\right)\right) + O(n^2) \\ &\leq d\left((\deg a + \deg b)\left(n - \frac{3}{4}d\right) - \frac{1}{2}(\deg a + \deg b)^2\right) + O(n^2). \end{aligned}$$

Let $x = \deg a + \deg b$ and let $f(x) := d\left(x\left(n - \frac{3}{4}d\right) - \frac{1}{2}x^2\right)$. Then by Fact 2, $x \leq n - \dfrac{3}{2}d + O(1)$. A simple differentiation shows that $f$ is increasing for all $x \leq n - \frac{3}{4}d$.

Hence, $f$ attains its maximum for $x = n - \frac{3}{2}d + O(1)$. Thus,

$$\begin{aligned} D'(a) + D'(b) &\leq f\left(n - \frac{3}{2}d + O(1)\right) \\ &= \frac{1}{2}dn\left(n - \frac{3}{2}d\right) + O(n^2), \end{aligned}$$

and Claim 2 is proven.

From (3), we have $c = \frac{1}{2}\left(n - 3\lceil\frac{d+1}{2}\rceil - k\right)$. Hence since $k \leq 1$, we have $c = \frac{1}{2}\left(n - \frac{3}{2}d\right) + O(1)$. This, in conjunction with Claim 2, yields

$$\begin{aligned} \sum_{\{a,b\}\in\mathcal{C}} (D'(a) + D'(b)) &\leq c\left(\frac{1}{2}dn\left(n - \frac{3}{2}d\right) + O(n^2)\right) \\ &= \left(\frac{1}{2}\left(n - \frac{3}{2}d\right) + O(1)\right)\left(\frac{1}{2}dn\left(n - \frac{3}{2}d\right) + O(n^2)\right) \\ &= \frac{1}{4}dn\left(n - \frac{3}{2}d\right)^2 + O(n^3). \end{aligned}$$

Hence,

$$\begin{aligned} D'(G) &= \sum_{\{a,b\}\in\mathcal{C}} (D'(a) + D'(b)) + \sum_{x\in K} D'(x) + \sum_{u\in N} D'(u) \\ &\leq \frac{1}{4}dn\left(n - \frac{3}{2}d\right)^2 + O(n^3) + O(n^3) + O(n^3) \\ &= \frac{1}{4}dn\left(n - \frac{3}{2}d\right)^2 + O(n^3), \end{aligned}$$

which establishes the bound in the theorem for Case 1 and for $d < \frac{n}{3}$. For $d \geq \frac{n}{3}$,

$$\frac{1}{4}nd(n - \frac{3}{2}d)^2 \leq \frac{3}{4}d^2(n - \frac{3}{2}d)^2 + O(n^3),$$

and so the theorem is proved for Case 1.

Case 2: $k \geq 2$. Now the pairs of vertices in $\mathcal{C}$ will be partitioned further. Fix a vertex $z \in K$. For each pair $\{a, b\} \in \mathcal{C}$, choose a vertex closer to $z$, if $d(a, z) = d(b, z)$ arbitrarily choose one of the vertices. Let $A$ be the set of all these vertices closer to $z$, and $B$ be the set of partners of these vertices in $A$, so $|A| = |B| = c$. Furthermore, let $A_1(B_1)$ be the set of vertices $w \in A(B)$ whose partner is at a distance at most 9 from $w$. Let $c_1 = |A_1| = |B_1|$.

**Claim 3** *For all $u, v \in A \cup K$, $d(u, v) \leq 8$.*

*Proof of Claim 3:* Since $\mathcal{C}$ is a maximum set of pairs of vertices of distance at least 3, any two vertices of $K$ must be at a distance of at most 2. We show that $d(a, z) \leq 4$ for all $a \in A$. Suppose, to the contrary, that there exists a vertex $a \in A$ for which $d(a, z) \geq 5$. Let $b$ be the partner of $a$. By definition of $A$, $d(z, b) \geq 5$. Now consider another vertex $z' \in K$, $\quad z \neq z'$. Since $d(z,z') \leq 2$ we have $5 \leq d(b,z) \leq d(b,z') + d(z,z') \leq d(b,z') + 2$ which implies $d(b,z') \geq 3$. This contradicts the maximality of $\mathcal{C}$ since $\{a,b\}$ will be replaced by $\{a,z\}$ and $\{b,z'\}$. Hence $d(a,z) \leq 4$, for each $a \in A$. Thus for $u,v \in A$, $\quad d(u,v) \leq d(u,z) + d(z,v) \leq 8$.

**Claim 4** *For all $x \in K$,*

$$D'(x) \leq d\left(n - \frac{3}{2}d - c\right)\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2).$$

*Proof of Claim 4:* By Claim 3, all $c+k$ vertices in $A \cup K$ lie within a distance of 8 from each vertex $x \in K$. This implies that all the $c_1$ vertices in $B_1$ lie within a distance of $9+8$ from $x$. Thus, as in Proposition 2,

$$\begin{aligned} D(x) \quad &\leq \quad \begin{cases} 8(c+k) + 17c_1 + 18 + 2\cdot 19 + 20 + \cdots + d - 1 \\ \quad + \, d\left(n - c - c_1 - k - \frac{3}{2}d\right) \qquad \text{if } d \text{ is odd,} \\[2ex] 8(c+k) + 17c_1 + 18 + 2\cdot 19 + 20 + \cdots + 2(d-1) \\ \quad + \, d\left(n - c - c_1 - k - \frac{3}{2}d\right) \qquad \text{if } d \text{ is even,} \end{cases} \\ &= \quad d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2). \end{aligned}$$

In order to find a bound on the degree of $x$ we use a counting argument. Note that $x$ can have at most 9 neighbours in $N$. By definition of $A$ and $B$, $x$ cannot be adjacent to two vertices, $w$ and $z$, where $w \in A$ is a partner of $z \in B$ since $d(w,z) \geq 3$. Thus, $x$ is adjacent to at most $c$ vertices in $A \cup B$. It follows that

$$\begin{aligned} n \quad &\geq \quad \deg x + |N| - 9 + |A \cup B| - c \\ &= \quad \deg x + \frac{3}{2}d + \frac{3}{2} - 9 + c. \end{aligned}$$

Hence $\deg x \leq n - \frac{3}{2}d - c + \frac{15}{2}$. Therefore,

$$\begin{aligned} D'(x) \quad &= \quad \deg x D(x) \\ &\leq \quad d\left(n - \frac{3}{2}d - c\right)\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2), \end{aligned}$$

and this proves Claim 4.

We now turn to finding an upper bound on the contribution of the pairs in $\mathcal{C}$ to the degree distance. We abuse notation and write $\{a,b\} \in A_1 \cup B_1$ if $a$ and $b$ are partners, i.e., $\{a,b\} \in \mathcal{C}$, with $a \in A_1$ and $b \in B_1$. Note that

$$\sum_{\{a,b\}\in\mathcal{C}} (D'(a)+D'(b)) = \sum_{\{a,b\}\in A_1\cup B_1} (D'(a)+D'(b)) + \sum_{\{a,b\}\in(A-A_1)\cup(B-B_1)} (D'(a)+D'(b)).$$

We first consider the set $A_1 \cup B_1$.

**Claim 5** *Let $\{a, b\} \in \mathcal{C}$. If $d(a, b) \leq 9$, i.e., if $\{a, b\} \in A_1 \cup B_1$, then*

$$D'(a) + D'(b) \leq d\left(n - \frac{3}{2}d\right)\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2).$$

*Proof of Claim 5:* We first show that any two vertices in $A \cup K \cup B_1$ lie within a distance of 26 from each other. By Claim 3, any two vertices in $A \cup K$ lie within a distance of 8 from each other. Now assume that $b, v \in B_1$, and let $a$ and $u$ be the partners of $b$ and $v$ in $A_1$, respectively. Then $d(b, v) \leq d(b, a) + d(a, u) + d(u, v) \leq 9 + 8 + 9 = 26$. Thus any two vertices in $B_1$ are within a distance of 26 from each other. Now let $a \in A \cup K$ and $b \in B_1$, and let $u$ be the partner of $b$ in $A_1 \subseteq A$. Then $d(a, b) \leq d(a, u) + d(u, b) \leq 8 + 9 < 26$. Hence any two vertices in $A \cup K \cup B_1$ lie within a distance of 26 from each other.

Now let $w \in A_1 \cup B_1$. Since $w$ is in $A \cup Y \cup B_1$, all the $c + k + c_1 - 1$ vertices in $A \cup K \cup B_1$ lie within a distance of 26 from $w$. It follows, as in Proposition 2, that

$$\begin{aligned} D(w) \quad &\leq \quad \begin{cases} 26(c + k + c_1 - 1) + 27 + 2 \cdot 28 + \cdots + d - 1 \\ +d\left(n - c - c_1 - k - \frac{3}{2}d\right) \qquad \text{if } d \text{ is even,} \\ \\ 26(c + k + c_1 - 1) + 27 + 2 \cdot 28 + \cdots + 2(d - 1) \\ +d\left(n - c - c_1 - k - \frac{3}{2}d\right) \qquad \text{if } d \text{ is odd,} \end{cases} \\ &= \quad d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n). \end{aligned}$$

Thus, if $\{a, b\}$ is a pair in $A_1 \cup B_1$, then

$$\begin{aligned} D'(a) + D'(b) \quad &\leq \quad \deg a\left(d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n)\right) \\ &\quad + \quad \deg b\left(d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n)\right) \\ &= \quad (\deg a + \deg b)\left(d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2)\right). \end{aligned}$$

By Fact 2, $\deg a + \deg b \leq n - \frac{3}{2}d + O(1)$. Therefore,

$$\begin{aligned} D'(a) + D'(b) \quad &\leq \quad \left(n - \frac{3}{2}d + O(1)\right)\left(d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2)\right) \\ &= \quad d\left(n - \frac{3}{2}d\right)\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2), \end{aligned}$$

and Claim 5 is proven.

Now consider pairs $\{a, b\}$ of vertices in $\mathcal{C}$ which are not in $A_1 \cup B_1$.

**Claim 6** *Let $\{a, b\} \in \mathcal{C}$. If $d(a, b) \geq 10$, i.e., if $\{a, b\} \in (A - A_1) \cup (B - B_1)$, then*

$$D'(a) + D'(b) \leq d(c + k)\left(n - c - c_1 - k - \frac{3}{4}d\right) + cd\left(n - \frac{3}{4}d - c\right) + O(n^2).$$

*Proof of Claim 6:* We consider vertices from $A - A_1$ and from $B - B_1$ separately. Let $a \in A - A_1$. Then as in Claim 5, all the $c + k - 1$ vertices in $A \cup K$ lie at a distance of 8 from $a$ and all the $c_1$ vertices in $B_1$ lie within a distance of $9 + 8 = 17$ from $a$. Thus, as in Proposition 2,

$$\begin{aligned} D(a) &\leq \begin{cases} 8(c+k-1) + 17c_1 + 18 + 2\cdot 19 + 20 + 2\cdot 21 + \cdots + d - 1 \\ +d\left(n - c - c_1 - k - \frac{3}{2}d\right) \qquad \text{if } d \text{ is odd,} \\ \\ 8(c+k-1) + 17c_1 + 18 + 2\cdot 19 + 20 + 2\cdot 21 + \cdots + 2(d-1) \\ +d\left(n - c - c_1 - k - \frac{3}{2}d\right) \qquad \text{if } d \text{ is even,} \end{cases} \\ &= d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n). \end{aligned}$$

We now find a bound on the degree of $a$. By definition of $\mathcal{C}$, $a$ cannot be adjacent to both $w$ and $u$, where $w \in A$ is a partner of $u \in B$ since $d(w, u) \geq 3$. Hence $a$ is adjacent to at most $c - 1$ vertices in $A \cup B$. Further, $a$ is adjacent to at most 9 vertices in $N$ and has at most $k$ neighbours in $K$. Thus,

$$\deg a \leq c - 1 + 9 + k = c + k + 8.$$

It follows that

$$\begin{aligned} D'(a) &= \deg a D(a) \\ &\leq (c + k + 8)\left(d\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n)\right) \\ &= d(c+k)\left(n - c - c_1 - k - \frac{3}{4}d\right) + O(n^2). \end{aligned} \tag{4}$$

Now let $b \in B - B_1$. By Proposition 2, we have

$$D(b) \leq d\left(n - \frac{3}{4}d - \deg b\right) + O(n),$$

and so

$$D'(b) \leq \deg b\left(d\left(n - \frac{3}{4}d - \deg b\right)\right) + O(n^2). \tag{5}$$

We first maximize $\deg b\left(d\left(n - \frac{3}{4}d - \deg b\right)\right)$ with respect to $\deg b$. Let

$$f(x) := x\left(d\left(n - \frac{3}{4}d - x\right)\right),$$

where $x = \deg b$. A simple differentiation shows that $f$ is increasing for $x \leq \frac{1}{2}\left(n - \frac{3}{4}d\right)$. We find an upper bound on $x$, i.e., on $\deg b$. Note that as above, $b$ can be adjacent to at most $c-1$ vertices in $A \cup B$, and has at most 9 neighbours in $N$. We show that $b$ cannot be adjacent to any vertex in $K$. Suppose to the contrary that $y \in K$ and $d(b, y) = 1$. Recall that $a$ is the partner of $b$ and $d(a, b) \geq 10$. By

Claim 3, $d(a, y) \leq 8$. Hence $10 \leq d(a, b) \leq d(b, y) + d(y, a) \leq 1 + 8$, a contradiction. Thus, $b$ cannot be adjacent to any vertex in $K$. We conclude that

$$\deg b \leq c - 1 + 9 = c + 8.$$

We look at two cases separately. First assume that $\deg b = c + 8$. Then

$$\begin{aligned} f(\deg b) &= f(c+8) \\ &= (c+8)\left(d\left(n - \frac{3}{4}d - (c+8)\right)\right) \\ &= cd\left(n - \frac{3}{4}d - c\right) + O(n^2). \end{aligned} \tag{6}$$

Second, assume that $\deg b \leq c$. From (3) and the fact that $k \geq 2$, we have

$$c \leq \frac{1}{2}\left(n - \frac{3}{2}d - \frac{3}{2} - k\right) + O(1) \leq \frac{1}{2}\left(n - \frac{3}{2}d - \frac{7}{2}\right).$$

Notice that

$$\frac{1}{2}\left(n - \frac{3}{2}d - \frac{7}{2}\right) \leq \frac{1}{2}\left(n - \frac{3}{2}d\right),$$

and so $f$ is increasing in $[1, c]$. Therefore,

$$f(\deg b) \leq f(c) = cd\left(n - \frac{3}{4}d - c\right),$$

for this case. Comparing this with (6), we get that

$$f(\deg b) \leq cd\left(n - \frac{3}{4}d - c\right) + O(n^2).$$

Thus, from (5), we have

$$D'(b) \leq cd\left(n - \frac{3}{4}d - c\right) + O(n^2).$$

Combining this with (4), we get

$$D'(a) + D'(b) \leq d(c+k)\left(n - c - c_1 - k - \frac{3}{4}d\right) + cd\left(n - \frac{3}{4}d - c\right) + O(n^2),$$

and Claim 6 is proven.

Using Claims 1, 4, 5, and 6 we have

$$\begin{aligned} D'(G) &= \sum_{u \in N} D'(u) + \sum_{x \in K} D'(x) + \sum_{\{a,b\} \in \mathcal{C}} (D'(a) + D'(b)) \\ &\leq dk\left(n - \frac{3}{2}d - c\right)\left(n - c - c_1 - k - \frac{3}{4}d\right) \\ &\quad + c_1\left(d\left(n - \frac{3}{2}d\right)\left(n - c - c_1 - k - \frac{3}{4}d\right)\right) \end{aligned}$$

$$
\begin{aligned}
& + \ (c-c_1)\left(d(c+k)\left(n-c-c_1-k-\frac{3}{4}d\right)+cd\left(n-\frac{3}{4}d-c\right)\right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +O(n^3) \\
= \ & dk\left(n-\frac{3}{2}d-c\right)\left(n-c-c_1-k-\frac{3}{4}d\right) \\
& + \ c_1\left(d\left(n-\frac{3}{2}d\right)\left(n-c-c_1-k-\frac{3}{4}d\right)\right) \\
& + \ d(c-c_1)\left((c+k)\left(n-c-k-\frac{3}{4}d\right)-c_1(c+k)+c\left(n-\frac{3}{4}d-c\right)\right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +O(n^3).
\end{aligned}
$$

For easy calculation in maximizing this term, we note that $c-c_1 \geq 0$, and that by (3), $n-c-k-\frac{3}{4}d \geq 0$. Hence the last term in the previous inequalities

$$
d(c-c_1)\left((c+k)\left(n-c-k-\frac{3}{4}d\right)-c_1(c+k)+c\left(n-\frac{3}{4}d-c\right)\right)
$$

is at most

$$
d(c-c_1)\left((c+k+4)\left(n-c-k-\frac{3}{4}d\right)-c_1(c+k)+c\left(n-\frac{3}{4}d-c\right)\right).
$$

It follows that

$$
\begin{aligned}
D'(G) \ \leq \ & dk\left(n-\frac{3}{2}d-c\right)\left(n-c-c_1-k-\frac{3}{4}d\right) \\
& + \ c_1\left(d\left(n-\frac{3}{2}d\right)\left(n-c-c_1-k-\frac{3}{4}d\right)\right) \\
& + \ d(c-c_1)\left((c+k+4)\left(n-c-k-\frac{3}{4}d\right)-c_1(c+k)\right. \\
& + \ \left. c\left(n-\frac{3}{4}d-c\right)\right)+O(n^3).
\end{aligned}
$$

Let $g(n,d,c,c_1)$ be the function

$$
\begin{aligned}
g(n,d,c,c_1) \ := \ & dk\left(n-\frac{3}{2}d-c\right)\left(n-c-c_1-k-\frac{3}{4}d\right) \\
& + \ c_1\left(d\left(n-\frac{3}{2}d\right)\left(n-c-c_1-k-\frac{3}{4}d\right)\right) \\
& + \ d(c-c_1)\left((c+k+4)\left(n-c-k-\frac{3}{4}d\right)-c_1(c+k)\right. \\
& + \ \left. c\left(n-\frac{3}{4}d-c\right)\right).
\end{aligned}
$$

We first maximize $g$ subject to $c_1$, keeping the other variables fixed. We show that the derivative of $g$ with respect to $c_1$ is negative. Note that the derivative is

$$
\begin{aligned}
\frac{dg}{dc_1} &= -dk\left(n-\frac{3}{2}d-c\right)+d\left(n-\frac{3}{2}d\right)\left(n-c-c_1-k-\frac{3}{4}d\right)\\
&\quad -c_1d\left(n-\frac{3}{2}d\right)\\
&\quad -d\left[(c+k+4)\left(n-c-k-\frac{3}{4}d\right)-c_1(c+k)+c\left(n-\frac{3}{4}d-c\right)\right]\\
&\quad -d(c-c_1)(c+k)\\
&= -dk\left(n-\frac{3}{2}d-c\right)-c_1d\left[n-\frac{3}{2}d-c-k\right]\\
&\quad -d\left[(c+k+4)\left(n-c-k-\frac{3}{4}d\right)+c\left(n-\frac{3}{4}d-c\right)\right]\\
&\quad -d(c-c_1)(c+k)\\
&\quad +d\left(n-\frac{3}{2}d\right)\left(n-c-c_1-k-\frac{3}{4}d\right)\\
&= -dk\left(n-\frac{3}{2}d-c\right)-c_1d\left[n-\frac{3}{2}d-c-k\right]\\
&\quad -dc\left(n-\frac{3}{4}d-c-k\right)-dck\\
&\quad -d(c-c_1)(c+k)-c_1d\left(n-\frac{3}{2}d\right)\\
&\quad +d\left(n-c-k-\frac{3}{4}d\right)\left[n-\frac{3}{2}d-c-k-4\right]\\
&= -dk\left(n-\frac{3}{2}d-c\right)-c_1d\left[n-\frac{3}{2}d-c-k\right]\\
&\quad -dck\\
&\quad -d(c-c_1)(c+k)-c_1d\left(n-\frac{3}{2}d\right)\\
&\quad +d\left(n-c-k-\frac{3}{4}d\right)\left[n-\frac{3}{2}d-2c-k-4\right].
\end{aligned}
$$

From (3), $n-\frac{3}{2}d-2c-k\leq 3$. Thus, since $n-c-k-\frac{3}{4}d\geq 0$, the last term above is negative. From (3), $n-\frac{3}{2}d-2c-k\geq\frac{3}{2}$, and so it follows that the terms

$$n-\frac{3}{2}d-c,\quad n-\frac{3}{2}d-c-k,\quad \text{and } n-\frac{3}{2}d,$$

are all positive. Further, $c-c_1\geq 0$.

It follows that the derivative

$$
\begin{aligned}
\frac{dg}{dc_1} &= -dk\left(n-\frac{3}{2}d-c\right)-c_1d\left[n-\frac{3}{2}d-c-k\right]\\
&\quad -dck\\
&\quad -d(c-c_1)(c+k)-c_1d\left(n-\frac{3}{2}d\right)\\
&\quad +d\left(n-c-k-\frac{3}{4}d\right)\left[n-\frac{3}{2}d-2c-k-4\right]
\end{aligned}
$$

is negative. Therefore, $g$ is decreasing in $c_1$. Thus, in conjunction with (3), we have

$$\begin{aligned} g(n,d,c,c_1) &\leq g(n,d,c,0) \\ &= dk\left(n-\frac{3}{2}d-c\right)\left(n-c-k-\frac{3}{4}d\right) \\ &\quad + dc\left((c+k+4)\left(n-c-k-\frac{3}{4}d\right)+c\left(n-\frac{3}{4}d-c\right)\right) \\ &= d\left(\left(n-\frac{3}{2}d-c\right)^2\left(c+\frac{3}{4}d\right)+c^2\left(n-\frac{3}{4}d-c\right)\right)+O(n^3). \end{aligned}$$

A simple differentiation with respect to $c$ shows that the function

$$\begin{aligned} \phi(c) &:= \left(n-\frac{3}{2}d-c\right)^2\left(c+\frac{3}{4}d\right)+c^2\left(n-\frac{3}{4}d-c\right) \\ &= (3d-n)c^2+\left(n-\frac{3}{2}d\right)(n-3d)c+\frac{3}{4}d\left(n-\frac{3}{2}\right)^2, \end{aligned}$$

has a critical point at $c=\frac{1}{2}\left(n-\frac{3}{2}d\right)$. Recall that $k\geq 2$ and from (3),

$$c=\frac{1}{2}(n-\frac{3}{2}d-k)+O(1)\leq\frac{1}{2}(n-\frac{3}{2}d)-\frac{3}{2}=c^*.$$

Hence, we obtain the domain of $c$, $0\leq c\leq c^*$. Now we look at two cases.

SUBCASE A: For $d<\frac{n}{3}$, the function $\phi$ is increasing for $c\leq\frac{1}{2}\left(n-\frac{3}{2}d\right)$ and so

$$\phi\leq\phi\left(\frac{1}{2}\left(n-\frac{3}{2}d\right)-\frac{3}{2}\right)=\frac{n}{4}\left(n-\frac{3}{2}d\right)^2+O(n^2)$$

and so

$$D'(G)\leq\frac{1}{4}dn(n-\frac{3}{2}d)^2+O(n^3).$$

SUBCASE B: If $d\geq\frac{n}{3}$, then $\phi$ is decreasing over the domain of $c$ so it is maximised at $c=0$, and hence $\phi(c)\leq\phi(0)=\dfrac{3}{4}d\left(n-\dfrac{3}{2}d\right)$. It follows that

$$D'(G)\leq\frac{3}{4}d^2\left(n-\frac{3}{2}d\right)^2+O(n^3),$$

and Theorem 1 is proven.

To see that the bound is asymptotically sharp, when $d<\frac{n}{3}$ and for $\lambda=2$, consider the graph $G_{n,d,\lambda}=G_0+G_1+\cdots+G_d$ where $G_0=G_d=K_{\lceil\frac{1}{2}(n-\frac{3}{2}d)\rceil}$ and for $i=1,2,3,\ldots,d-1$,

$$G_i=\begin{cases} K_1 & \text{if } i \text{ is odd,} \\ K_2 & \text{if } i \text{ is even.} \end{cases}$$

Then $G_{n,d,2}$ is 2-edge-connected and has diameter $d$ and degree distance at least $\frac{1}{4}dn\left(n-\frac{3}{2}d\right)^2$. For $d \geq \frac{n}{3}$, consider the graph $G_{n,d,2} = G_0 + G_1 + \cdots + G_d$ where $G_d = K_{\lceil (n-\frac{3}{2}d)\rceil}$ and for $i = 0, 1, 2, \ldots, d-1$,

$$G_i = \begin{cases} K_1 & \text{if } i \text{ is even,} \\ K_2 & \text{if } i \text{ is odd.} \end{cases}$$

□

**Corollary 1** *Let $G$ be a 2-edge-connected graph of order $n$. Then*

$$D'(G) \leq \frac{2n^4}{81} + O(n^3).$$

*Moreover, this inequality is asymptotically sharp.*

**Proof:** Let $d$ be the diameter of $G$. By the theorem above,

$$D'(G) \leq \begin{cases} \frac{1}{4}dn(n-\frac{3}{2}d)^2 + O(n^3) & \text{if } d < \frac{n}{3}, \\ \\ \frac{3}{4}d^2(n-\frac{3}{2}d)^2 + O(n^3) & \text{if } d \geq \frac{n}{3}. \end{cases}$$

The term $\frac{1}{4}dn\left(n-\frac{3}{2}d\right)^2$ is maximized, with respect to $d$, for $d = \frac{2n}{9}$, to give

$$\frac{1}{4}dn\left(n-\frac{3}{2}d\right)^2 \leq \frac{2n^4}{81}.$$

Hence,

$$D'(G) \leq \frac{2n^4}{81} + O(n^3).$$

The term $\frac{3}{4}d^2\left(n-\frac{3}{2}d\right)^2$ is maximized, with respect to $d$, for $d = \frac{n}{3}$, to give

$$\frac{3}{4}d^2\left(n-\frac{3}{2}d\right)^2 \leq \frac{n^4}{48} < \frac{2n^4}{81}.$$

Therefore, in both cases

$$D'(G) = \frac{2n^4}{81} + O(n^3),$$

as desired.

To see that the bound is asymptotically best possible, consider the graph $G_{n,d,\lambda}$ constructed above with $d = \frac{2n}{9}$. Note that

$$D'(G_{n,\frac{2n}{9},\lambda}) = \frac{2n^4}{81} + O(n^3),$$

as claimed. □

Using similar proofs as for Theorem 1 we obtain the following results.

**Theorem 2** *Let $G$ be a 3-and 4-edge-connected graph of order $n$ and diameter $d$. Then*

$$D'(G) \leq \begin{cases} \frac{1}{4}dn(n-2d)^2 + O(n^3) & \text{if } d < \frac{n}{4}, \\ \\ d^2(n-2d)^2 + O(n^3) & \text{if } d \geq \frac{n}{4}. \end{cases}$$

*Moreover, this inequality is asymptotically sharp.*

To see that the bound is asymptotically sharp, for $d < \frac{n}{4}$ and for $\lambda = 3, 4$ consider the graph $G_{n,d,\lambda} = G_0 + G_1 + \cdots + G_d$ where $G_0 = G_d = K_{\lceil \frac{1}{2}(n-2d) \rceil}$ and $G_i = K_2$ for $i = 1, 2, \ldots, d-1$. For $d \geq \frac{n}{4}$, and when $\lambda = 3$, consider the graph $G_{n,d,3} = G_0 + G_1 + \cdots + G_d$ where $G_d = K_{\lceil (n-2d) \rceil}$, $G_0 = K_1$, $G_1 = K_3$ and $G_i = K_2$ for $i = 2, 3, \ldots, d-1$. For $\lambda = 4$ consider the graph $G_{n,d,4} = G_0 + G_1 + \cdots + G_d$ where $G_d = K_{\lceil (n-2d-1) \rceil}$, $G_0 = K_1$, $G_1 = K_4$ and $G_i = K_2$ for $i = 2, 3, \ldots, d-1$.

**Corollary 2** *Let $G$ be a 3-and 4-edge-connected graph of order $n$. Then*

$$D'(G) \leq \frac{n^4}{54} + O(n^3).$$

*Moreover, this inequality is asymptotically sharp.*

To see that the bound is asymptotically best possible, consider the graph $G_{n,d,\lambda}$ constructed above with $d = \frac{n}{6}$. Note that

$$D'(G_{n,\frac{n}{6},\lambda}) = \frac{n^4}{54} + O(n^3),$$

as claimed. □

**Theorem 3** *Let $G$ be a 5-and 6-edge-connected graph of order $n$ and diameter $d$. Then*

$$D'(G) \leq \begin{cases} \frac{1}{4}dn(n-\frac{5}{2}d)^2 + O(n^3) & \text{if } d < \frac{n}{5}, \\ \\ \frac{5}{4}d^2(n-\frac{5}{2}d)^2 + O(n^3) & \text{if } d \geq \frac{n}{5}. \end{cases}$$

*Moreover, this inequality is asymptotically sharp.*

To see that the bound is asymptotically sharp, for $d < \frac{n}{5}$ and for $\lambda = 5, 6$ consider the graph $G_{n,d,\lambda} = G_0 + G_1 + \cdots + G_d$ where $G_0 = G_d = K_{\lceil \frac{1}{2}(n-\frac{5}{2}d) \rceil}$ and for $i = 1, 2, \ldots, d-1$,

$$G_i = \begin{cases} K_3 & \text{if } i \text{ is odd,} \\ K_2 & \text{if } i \text{ is even.} \end{cases}$$

For $d \geq \frac{n}{5}$ and for $\lambda = 5$ consider the graph $G_{n,d,5} = G_0 + G_1 + \cdots + G_d$ where $G_d = K_{\lceil (n-\frac{5}{2}d) \rceil}$, $G_0 = K_1$, $G_1 = K_5$ and for $i = 2, 3, \ldots, d-1$,

$$G_i = \begin{cases} K_3 & \text{if } i \text{ is odd,} \\ K_2 & \text{if } i \text{ is even.} \end{cases}$$

For $\lambda = 6$ consider the graph $G_{n,d,6} = G_0 + G_1 + \cdots + G_d$ where $G_d = K_{\lceil (n-\frac{5}{2}d-1)\rceil}$, $G_0 = K_1$, $G_1 = K_6$ and for $i = 2, 3, \ldots, d-1$,

$$G_i = \begin{cases} K_3 & \text{if } i \text{ is odd,} \\ K_2 & \text{if } i \text{ is even.} \end{cases}$$

**Corollary 3** *Let $G$ be a 5-and 6-edge-connected graph of order $n$. Then*

$$D'(G) \leq \frac{2n^4}{135} + O(n^3).$$

*Moreover, this inequality is asymptotically sharp.*

To see that the bound is asymptotically best possible, consider the graph $G_{n,d,\lambda}$ constructed above with $d = \frac{2n}{15}$. Note that

$$D'(G_{n,\frac{2n}{15},\lambda}) = \frac{2n^4}{135} + O(n^3),$$

as claimed. □

**Theorem 4** *Let $G$ be a 7-edge-connected graph of order $n$ and diameter $d$. Then*

$$D'(G) \leq \begin{cases} \frac{1}{4}dn(n-3d)^2 + O(n^3) & \textit{if } d < \frac{n}{6}, \\ \\ \frac{3}{2}d^2(n-3d)^2 + O(n^3) & \textit{if } d \geq \frac{n}{6}. \end{cases}$$

*Moreover, this inequality is asymptotically sharp.*

To see that the bound is asymptotically sharp, for $d < \frac{n}{6}$ and for $\lambda = 7$ consider the graph $G_{n,d,\lambda} = G_0 + G_1 + \cdots + G_d$ where $G_0 = G_d = K_{\lceil \frac{1}{2}(n-3d)\rceil}$ and $G_i = K_3$, for $i = 1, 2, \ldots, d-1$. For $d \geq \frac{n}{6}$ and for $\lambda = 7$ consider the graph $G_{n,d,7} = G_0 + G_1 + \cdots + G_d$ where $G_d = K_{\lceil (n-3d-2)\rceil}$, $G_0 = K_1$, $G_1 = K_7$ and $G_i = K_3$, for $i = 2, 3, \ldots, d-1$.

**Corollary 4** *Let $G$ be a 7-edge-connected graph of order $n$. Then*

$$D'(G) \leq \frac{n^4}{81} + O(n^3).$$

*Moreover, this inequality is asymptotically sharp.*

To see that the bound is asymptotically best possible, consider the graph $G_{n,d,\lambda}$ constructed above with $d = \frac{n}{9}$. Note that

$$D'(G_{n,\frac{n}{9},\lambda}) = \frac{n^4}{81} + O(n^3),$$

as claimed. □

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