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New families of graphs whose independence polynomials have only real zeros

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Abstract

We describe an inductive means of constructing infinite families of graphs, every one of whose members G has independence polynomial $I(G;x)$ having only real zeros. Consequently, such independence polynomials are logarithmically concave and unimodal.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Recall that an *independent set* S in G is a set of pairwise non-adjacent vertices. The *independence number* of G, $\alpha(G)$, is the cardinality of a largest independent set in G. The *independence* *polynomial* of G, denoted $I(G;x)$, is defined by

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + s_2 x^2 + \cdots + s_{\alpha(G)} x^{\alpha(G)},$$

where s_k is the number of independent sets with cardinality k. Independence polynomials were introduced in the 1970s and have been heavily studied since then. (See [7] for a recent comprehensive survey, and see [4], [5], [9], [10], [11], [13], and [14] for applications.)

A polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ is called *logarithmically concave* (or *log-concave*) if for all i, $1 \leq i \leq n-1$, $a_i^2 \geq a_{i-1}a_{i+1}$. A polynomial is called *unimodal* if the sequence of its coefficients satisfies $a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$ for some j, $0 \leq j \leq n$. It can be shown that if a polynomial is log-concave then it is unimodal. For a fixed natural number n the binomial coefficients $(\binom{n}{i})_{i=0}^{n}$ give what is probably the best-known example of a log-concave sequence.

A classical result due to Newton (see [6]) states that if a polynomial $p(x)$ has non-negative coefficients and only real zeros, then $p(x)$ is log-concave. This fact, of which we make frequent use in this note, has been the basis for proving the log-concavity and unimodality of many independence polynomials.

Our construction follows that of Wang and Zhu, described in [12]. (See also [1], [2], and [15] for similar constructions.) There the *n-concatenation* of a graph G on the vertex $v \in V(G)$, denoted $G_n^-(v)$, is defined as follows: begin with the disjoint union of n copies of G, and identify each copy of v with one of the vertices of the path P_n on n vertices. For example, a "k-regular" caterpillar with k edges pendant from each vertex of its central path can be obtained by concatenating the star S_k with k leaves on its central vertex.

Our main result, proven in much the same manner as the main theorem from [1] (which in turn is based on the "zero-interlacing" technique described in [8]), is the following. Here $N[v]$ denotes the closed neighborhood of v, $N[v] = \{u \in V(G) \mid uv \in E(G)\} \cup \{v\}$.

Theorem 1.1 *Let G be a graph and let $v \in V(G)$ such that*

1. *$I(G-v;x) = f(x)b(x)$ and $I(G-N[v];x) = f(x)c(x)$ for some polynomials $b(x), c(x) \in \mathbb{Z}[x]$, $f(x) = \gcd(I(G-v;x), I(G-N[v];x))$,*
2. *the zeros of $f(x)$ are all real, and*
3. *$m = \deg(b) = \deg(c)+1$ and the zeros $\{\gamma_1, ..., \gamma_m\}$ of $b(x)$ and the zeros $\{\delta_1, ..., \delta_m\}$ of $xc(x)$ are all real and satisfy*

$$\gamma_1 < \delta_1 < \gamma_2 < \cdots < \delta_{m-1} < \gamma_m < \delta_m = 0.$$

Then the graph $I(G_n^-(v);x)$ has only real zeros and as a consequence is log concave and unimodal.

In fact, our proof of Theorem 1.1 will show more. If we let v_0 be the copy of v corresponding to either terminal vertex of P_n in the concatenation construction, $G_n^-(v)$ itself, relative to the vertex v_0, satisfies the hypothesis of Theorem 1.1. This fact enables us to use the graph resulting from an application of the theorem's construction as the *input* for another such application. We use this fact in Section 3 to generate some interesting specific examples, some of which are very far from being "path-like." The variety of these examples is quite great.

First, let us turn our attention to proving the main theorem.

2 Proof of the main theorem

We need a couple of simple technical lemmas to begin our proof. The following fact is well-known, and can be found, for example, in [7].

Lemma 2.1 *Let G be any graph, and let $w \in V(G)$. Then $I(G;x) = I(G-w;x) + xI(G-N[w];x)$.*

We frequently use this lemma without explicit mention. We make similar use of the following fact, whose proof is trivial:

Lemma 2.2 *Let $G \cup G'$ be the disjoint union of the graphs G and G'. Then $I(G \cup G';x) = I(G;x)I(G';x)$.*

Until further notice let G and $v \in V(G)$ satisfy the hypotheses of Theorem 1.1, and let us first assume that $b(x) = I(G-v;x)$ and $c(x) = I(G-N[v];x)$ (that is, that $f(x) = 1$). Let $m = \deg(b) = \deg(xc)$. We may use Lemma 2.1 to compute $I(G_n^-(v);x)$ for various values of n, always selecting w to be a terminal copy of v in $G_n^-(v)$. Let $p_n(x) = I(G_n^-(v);x)$. For $n = 1$ we easily obtain $p_1(x) = b(x) + xc(x)$. If we formally let $p_0(x) = 1$, we obtain an easy recurrence relation for p_n, valid for all $n \geq 2$:

$$p_n(x) = b(x)p_{n-1}(x) + xc(x)b(x)p_{n-2}(x) = b(x)\Big(p_{n-1}(x) + xc(x)p_{n-2}(x)\Big).$$

Using this fact, the following facts are easily proven inductively:

Lemma 2.3 *Let G and $v \in V(G)$ be as above. Then*

1. $\deg(p_n) = mn$ *and*
2. $b(x)$ *evenly divides* $p_n(x)$ *in* $\mathbb{Z}[x]$ *exactly* $\lfloor \frac{n}{2} \rfloor$ *times.*

Define $q_n(x) = \frac{p_n(x)}{(b(x))^{\lfloor n/2 \rfloor}}$. Lemma 2.3 implies that $\deg(q_n) = m\lceil \frac{n}{2} \rceil$. Our recursive formula for p_n is then easily modified into a similar formula for q_n:

Lemma 2.4 *Let $q_n(x)$ be defined as above. Then $q_0 = 1$, $q_1(x) = b(x) + xc(x)$, and*

$$q_n(x) = \begin{cases} q_{n-1} + xc(x)q_{n-2}(x) & \text{for } n \geq 2 \text{ even, and} \\ b(x)q_{n-1}(x) + xc(x)q_{n-2}(x) & \text{for } n \geq 3 \text{ odd.} \end{cases}$$

We note that $\deg(q_{n-1}) = \deg(xcq_{n-2}) = \frac{mn}{2}$ when n is even and $\deg(bq_{n-1}) = \deg(xcq_{n-2}) = \frac{m(n+1)}{2}$ when n is odd. We now prove inductively that the zeros of these respective pairs of polynomials are interwoven in the same way that those of b and xc are assumed to be. More specifically, we prove the following lemma:

Lemma 2.5 *Let $n \geq 1$ be fixed.*

1. *If n is even, let $t = \frac{m(n+2)}{2}$. Then the zeros $\{\alpha_1, ..., \alpha_t\}$ and $\{\beta_1, ..., \beta_t\}$ of q_{n+1} and xcq_n, respectively, are all real and satisfy*

$$\alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{t-1} < \alpha_t < \beta_t = 0.$$

2. *If n is odd, let $t = \frac{m(n+3)}{2}$. Then the zeros $\{\alpha_1, ..., \alpha_t\}$ and $\{\beta_1, ..., \beta_t\}$ of bq_{n+1} and xcq_n, respectively, are all real and satisfy*

$$\alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{t-1} < \alpha_t < \beta_t = 0.$$

Proof: Consider first the base case $n = 0$. We must show that the zeros of $q_1 = b + xc$ and the zeros of $xcq_0 = xc$ are all real and are related as in (1). By the hypothesis of Theorem 1.1, the zeros γ_j and δ_j of the i polynomials b and xc give rise to intervals

$$(-\infty, \gamma_1), (\gamma_1, \delta_1), ..., (\delta_{m-1}, \gamma_m), (\gamma_m, 0).$$

The polynomials b and xc have the same sign on the interval $(-\infty, \gamma_1)$ and on any interval of the form (δ_j, γ_{j+1}), $j \in \{1, ..., m-1\}$. Therefore the sum $q_1 = b + xc$ has no zero on any such interval. However, since b and xc have opposite signs on the m intervals of the form (γ_j, δ_j), $j \in \{1, ..., m\}$, q_1 does have a zero on each such interval. Since $\deg(q_1) = m$, these zeros account for all of q_1's zeros, which are therefore all real. Moreover, these zeros are interspersed among those of $xcq_0 = xc$ as needed.

The base case $n = 1$ is proven analogously.

Now suppose that $n \geq 2$ is even and that we have proven the lemma for $k \leq n-2$. Applying our inductive hypothesis in the case $k = n-2$, we see that the $mn/2$ zeros $\{\alpha_1, ..., \alpha_{mn/2}\}$ of q_{n-1} and the $mn/2$ zeros $\{\beta_1, ..., \beta_{mn/2}\}$ of xcq_{n-2} are all real and define the intervals

$$(\infty, \alpha_1), (\alpha_1, \beta_1), ..., (\beta_{mn/2-1}, \alpha_{mn/2}), (\alpha_{mn/2}, 0).$$

The polynomials q_{n-1} and xcq_{n-2} have the same sign on the interval $(-\infty, \alpha_1)$ and on any interval of the form (β_j, α_{j+1}), $j \in \{1, ..., \frac{mn}{2} - 1\}$. Therefore $q_n = q_{n-1} + xcq_{n-2}$ has no zero on any such interval. However, since q_{n-1} and xcq_{n-2} have opposite signs on the $mn/2$ intervals of the form (α_j, β_j), $1 \leq j \leq mn/2$, q_n does have a zero on each such interval. Since $\deg(q_n) = mn/2$, these zeros account for all of q_n's zeros. Moreover, relative to the zeros of xcq_{n-1}, these zeros (along with those of $b(x)$) satisfy the condition in (2) above, proving this condition for $n-1$ odd.

In case $n \geq 3$ is odd, an analogous proof establishes that q_n has all real zeros and that these zeros, relative to those of xcq_{n-1}, satisfy the condition in (1) above, proving the condition for $n-1$ even. $\square$

We are now able to complete our proof of Theorem 1.1 in case $f(x) = 1$. In fact, the zeros of $p_n(x) = I(G_n^-(v); x)$ are those of $b(x)$ and $q_n(x)$. The former zeros are assumed real in our hypotheses, and Lemma 2.5 shows that the latter are real as well. As a consequence, $p_n(x)$ is log-concave, and therefore unimodal.

If $f(x) \neq 1$, we need only be a bit more careful in our induction. More generally, letting $p_n(x) = I(G_n^-(v); x)$ once more, we have

$$p_1 = f(b + xc), p_2 = fb(p_1 + xfc), \text{ and } p_n = fb(p_{n-1} + xfcp_{n-2}) \text{ for } n \geq 3.$$

Arguing as above we can show that, after factoring out the appropriate power of $f(x)b(x)$, the remaining zeros of p_{n-1} and $xf(x)c(x)p_{n-2}$ alternate as do those of q_{n-1} and xcq_{n-2} in Lemma 2.5. Consequently, every p_n has only real zeros, and our theorem is again proven.

3 Specific examples

We now develop a number of specific families of graphs to which Theorem 1.1 applies. For our first construction, recall that a graph of order $2m$ is said to be *very well-covered* if $\alpha(G) = m$ and any maximal independent set has this cardinality. It is easy to show that any connected subgraph of $K_{m,m}$ of order $2m$ is very well-covered. It is also not hard to show that for any vertex v in such a graph, $\alpha(G - v) = m$ and $\alpha(G - N[v]) = m - 1$. The following corollary is therefore plausible, and easily proven:

Corollary 3.1 *Let G be a connected subgraph graph of $K_{m,m}$ of order $2m$. Let $v \in V(G)$ such that $I(G - v; x) = f(x)b(x)$, and $I(G - N[v]; x) = f(x)c(x)$ for some $f(x) \in \mathbb{Z}[x]$, and the zeros $\{\gamma_1, ..., \gamma_r\}$ of $b(x)$ and the zeros $\{\delta_1, ..., \delta_r\}$ of $xc(x)$ are all real and satisfy*

$$\gamma_1 < \delta_1 < \gamma_2 < \cdots < \delta_{r-1} < \gamma_r < \delta_r = 0.$$

Then the graph $G_n^-(v)$ is a very well-covered bipartite graph (in fact, a subgraph of $K_{mn,mn}$) whose independence polynomial has mn distinct real zeros. As a consequence, $I(G_n^-(v); x)$ is log concave and unimodal.

This construction leads to a few easy examples:

Corollary 3.2 *$I(G_n^-(v); x)$ has only real zeros (and is therefore log concave and unimodal) for each of the following choices of (G, v):*

1. *$G = P_{2m}$, the path on $2m$ vertices, and v either of the path's terminal vertices; and*

2. *$G = C_{2m}$, the cycle on $2m$ vertices, and v arbitrary.*

Proof: Note that both P_{2m} and C_{2m} are very well-covered subgraphs of $K_{m,m}$. Easy inductions establish the desired conditions on $I(G - v; x)$ and $I(G - N[v]; x)$ in either case. □

Now recall, as indicated in the introduction, that our proof of Theorem 1.1 gives the following meta-algorithm:

Corollary 3.3 *Let $G_n^-(v)$ be any graph obtained through applying Theorem 1.1. Then choosing w to be the vertex v in either of the terminal copies of G in the path P_n from the concatenation construction, $I(G-w;x)$ and $xI(G-N[w];x)$ satisfy the hypotheses of Theorem 1.1, with $f(x) = (b(x))^{\lfloor n/2 \rfloor}$. Thus we may apply the concatenation construction recursively.*

An interesting family of graphs is obtained if we begin this recursive procedure with $G = P_{2n}$, a path on $2n$ vertices for some $n \geq 1$, with v one of G's terminal vertices. The "paths on paths" resulting from repeated applications of the theorem yield trees that are very far from being "path-like," distinguishing them strongly from the graphs considered in [1], [2], [12], and [15]. For instance, consider the tree shown in Figure 1, resulting from just two iterations of the construction.

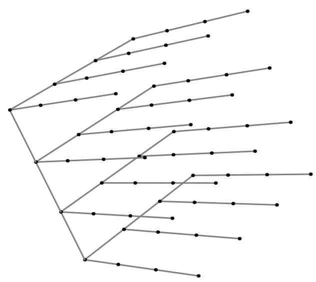

Figure 1: A graph obtained through two iterations of the construction, beginning with P_4

There are other standard families of graphs to which our procedure applies. For instance, for $s \geq 2$, consider the *cocktail party graph* (also known as the *hyperoctahedral graph*) $\mathrm{CP}(s)$, the graph of order $2s$ obtained by removing s disjoint edges from K_{2s}. See [3] for more information on these graphs; the graph $\mathrm{CP}(4)$ is shown in Figure 2.

It is not hard to show that if v is any vertex in V, then $b(x) = I(\mathrm{CP}(s)-v;x) = (s-1)x^2 + (2s-1)x + 1$ and $c(x) = I(\mathrm{CP}(s)-N[v];x) = x+1$. The polynomials $b(x)$ and $xc(x)$ have sets of zeros $\{\frac{1-2s\pm\sqrt{4s^2-8s+5}}{2s-2}\}$ and $\{-1,0\}$ respectively, so that Theorem 1.1 applies and gives us the following result:

Corollary 3.4 *Let $s \geq 2$. Then for any $n \geq 1$ and for any $v \in V(\mathrm{CP}(s))$, $I(\mathrm{CP}(s)_n^-(v);x)$ has all real zeros and is therefore log-concave and unimodal.*

Figure 3 shows the graph $\mathrm{CP}(4)_4^-(v)$.

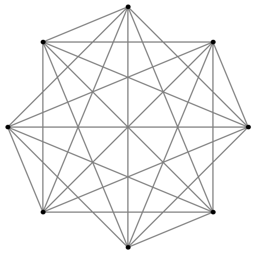

Figure 2: The cocktail party graph CP(4)

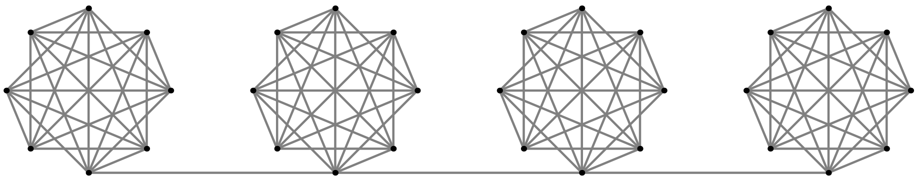

Figure 3: The graph $\mathrm{CP}(4)_4^-(v)$

References

[1] P. Bahls, On the independence polynomials of path-like graphs, *Australas. J. Combin.* **53** (2012), 3–18.

[2] P. Bahls and N. Salazar, Symmetry and unimodality of independence polynomials of generalized paths, *Australas. J. Combin.* **47** (2010), 165–176.

[3] N. Biggs, *Algebraic Graph Theory*, 2nd Ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.

[4] I. Gutman, Some relations for the independence and matching polynomials and their chemical applications, *Bull. Acad. Serbe Sci. Arts* **105** (1992), 39–49.

[5] R. Haggkvist, D. Andren, P.H. Lundow and K. Markstrom, *Graph theory and statistical physics, case study: discrete mathematics*, Department of Mathematics, UMEA University (1999-2004).

[6] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[7] V.E. Levit and E. Mandrescu, The independence polynomial of a graph—a survey, *Proc. 1st Int. Conf. Algebraic Informatics*, Thessaloniki, October 20-23, 2005 (ed. S. Bozapalidis, A. Kalampakas and G. Rahonis), Thessaloniki, Greece: Aristotle University (2005), 233–254.

[8] L.L. Liu and Y. Wang, A unified approach to polynomial sequences with only real zeros, *Adv. Appl. Math.* **35** (2007), 542–560.

[9] J.A. Makowsky, Algorthmic uses of the Feferman-Vaught theorem, *Ann. Pure Appl. Logic* **126** no.,1-2 (2004), 159–213.

[10] A.D. Scott and A.D. Sokal, The repulsive lattice gas, the independent set polynomial, and the Lovasz local lemma, *J. Stat. Phys.* **118** no. 5-6 (2005), 1151–1261.

[11] R.P Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Ann. New York Acad. Sci.* **576** (1989), 500–534.

[12] Y. Wang and B.-X. Zhu, On the unimodality of independence polynomials of some graphs, *European J. Combin.* **32** no. 1 (2011), 10–20.

[13] Y. Zeng and F. Zhang, Extremal polyomino chains on k-matchings and k-independent sets, *J. Math. Chem.* **42** no. 2 (2007), 125–140.

[14] F. Zhang and L. Zhang, Extremal hexagonal chains concerning k-matchings and k-independent sets, *J. Math. Chem.* **27** no. 4 (2000), 319–329.

[15] Z.-F. Zhu, The unimodality of independence polynomials of some graphs, *Australas. J. Combin.* **38** (2007) 27–33.

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