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Cycles in 3-anti-circulant digraphs*

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Abstract

A digraph D is a 3-anti-circulant digraph, if for any four distinct vertices x_1, x_2, x_3, $x_4 \in V(D)$, $x_1 \to x_2 \leftarrow x_3 \to x_4$ implies $x_4 \to x_1$. In this paper, we characterize the structure of 3-anti-circulant digraphs containing a cycle factor and show that the structure is very close to semicomplete and semicomplete bipartite digraphs. Laborde et al. conjectured that every digraph has an independent set intersecting every longest path. It has been shown that the conjecture is true for 3-anti-circulant digraphs. In this paper, we generalize the result to the longest cycle and prove that there exists an independent set intersecting every longest cycle for 3-anti-circulant digraphs.

1 Introduction and terminology

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops and multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $x \to y$ if $xy \in A(D)$. For disjoint subsets X and Y of $V(D)$ or subdigraphs of D, we use the notation $X \to Y$ to denote that every vertex of X dominates every vertex of Y, the notation $X \Rightarrow Y$ to denote that there is no arc from Y to X, and $X \mapsto Y$ to denote that both of $X \to Y$ and $X \Rightarrow Y$ hold.

For a vertex x in D, its *out-neighborhood* is $N^+(x) = \{y \in V(D) : xy \in A(D)\}$ and its *in-neighborhood* is $N^-(x) = \{y \in V(D) : yx \in A(D)\}$. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called the out-degree and the in-degree

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of the vertex x, respectively. For a pair X, Y of vertex sets of D, define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$.

A *path* is a finite sequence of distinct vertices $P = x_0x_1 \ldots x_n$ such that $x_{i-1} \to x_i$ for every $1 \le i \le n$; and its length is n. A *cycle* is a finite sequence of distinct vertices $C = x_0x_1 \ldots x_nx_0$ such that $x_{i-1} \to x_i$ for every $1 \le i \le n$ and $x_n \to x_0$; its length is $n+1$. The *predecessor* of x_i on C is the vertex x_{i-1} and is also denoted by x_i^-; the *successor* of x_i on C is the vertex x_{i+1} and is also denoted by x_i^+. A *k-cycle factor* (also cycle factor) of a digraph D is a set of k vertex-disjoint cycles which span the vertex set of D. In this paper, we only consider the cycle factor $C_1 \cup C_2 \cup \ldots \cup C_k$ such that the length of C_i is at least 3 for $i \in \{1, 2, \ldots, k\}$. A cycle of D with order $|V(D)|$ is called a Hamiltonian cycle and D is called a *Hamiltonian digraph.* A cycle is *anti-directed* if the orientation of each arc on the cycle is opposite to that of its predecessor. An anti-directed cycle of order k is called a k-anti-directed cycle.

A digraph D is *complete* if for every pair x, y of distinct vertices of D, both of xy and yx are in D. A digraph D is *semicomplete* if for every pair x, y of distinct vertices of D, vertices x and y are adjacent. A *semicomplete bipartite digraph* is a digraph whose underlying undirected graph is a complete bipartite graph.

Classes of digraphs characterized by forbidding families of induced digraphs play an important role in graph theory. Given a family of digraphs $\mathcal{F}$, we say that D is an orientedly $\mathcal{F}$-free digraph if there is no digraph in $\mathcal{F}$ isomorphic to any induced subdigraph of any orientation of D (an orientation of a digraph D is a spanning subdigraph of D in which we choose only one arc between any two adjacent vertices of D). If $\mathcal{F} = \{F\}$, we say orientedly F-free instead of orientedly $\mathcal{F}$-free.

There are four different possible orientations of the 3-path; see Figure 1 (also see Figure 2 in [2]). In H_1, H_2, H_3 and H_4, any arc between the two vertices with a dotted edge between them is forbidden. For $i \in \{1, 2, 3, 4\}$, orientedly H_i-free digraphs were introduced by Bang-Jensen in [2] as a common generalization of both semicomplete digraphs and semicomplete bipartite digraphs.

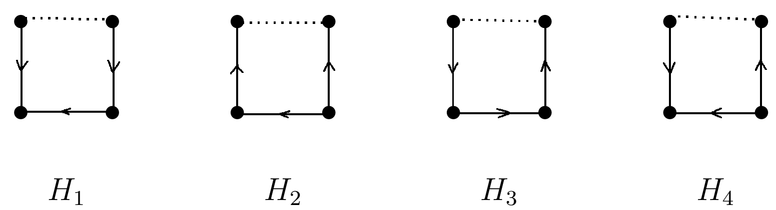

Figure 1.

Orientedly $\{H_1, H_2\}$-free digraphs are called arc-locally semicomplete digraphs, and were characterized by Galeana-Sánchez and Goldfeder in [5].

Orientedly $H_1(H_2)$-free digraphs are called arc-locally in (out)-semicomplete digraphs. In [10], the structure of strong arc-locally in (out)-semicomplete digraphs is characterized; this is very close to semicomplete digraphs, semicomplete bipartite digraphs and an extension of cycles.

Orientedly H_3-free digraphs are called 3-quasi-transitive digraphs. In [4], the structure of strong 3-quasi-transitive digraphs is characterized; this is very close to semicomplete and semicomplete bipartite digraphs.

Orientedly H_4-free digraphs are called 3-anti-quasi-transitive digraphs in [11]. The results on 3-anti-quasi-transitive digraphs are still very few and this class of digraphs seems to be difficult to work. A digraph D is a line digraph if and only if for any four distinct vertices $x, y, z, w \in V(D)$, $x \to y \leftarrow z \to w$ implies $x \to w$ (see [1]). Clearly, line digraphs are a subclass of 3-anti-quasi-transitive digraphs. Line digraphs have been extensively studied; see for example [1]. Here, we define another subclass of 3-anti-quasi-transitive digraphs. A digraph D is a 3-anti-circulant digraph if for any four distinct vertices $x, y, z, w \in V(D)$, $x \to y \leftarrow z \to w$ implies $w \to x$. Clearly, a 3-anti-circulant digraph must be a 3-anti-quasi-transitive digraph.

The following characterization was obtained independently by Gutin [7] and by Häggkvist and Manoussakis [8].

Theorem 1.1. *[7, 8] A semicomplete bipartite digraph is Hamiltonian if and only if it is strong and has a cycle factor.*

The hamiltonicity characterization can be generalized to arc-locally semicomplete digraphs [2], arc-locally in (out)-semicomplete digraphs [10], 3-quasi-transitive digraphs [4] and 3-anti-quasi-transitive digraphs [6, 11]. Because a 3-anti-circulant digraph must be a 3-anti-quasi-transitive digraph, we obtain the following result.

Theorem 1.2. *Let D be a 3-anti-circulant digraph. Then D is Hamiltonian if and only if D is strong and has a cycle factor.*

The subdivision of an arc uv of a digraph D consists of replacing uv by two arcs ux, xv, where x is a new vertex. Let us denote by D^* the digraph obtained by subdividing every arc of a given digraph D. Note that D^* is a 3-anti-circulant digraph. Hence to characterize the structure for general 3-anti-circulant digraphs seems a hard problem. In this paper, we shall characterize the structure of strong 3-anti-circulant digraphs containing a cycle factor.

Laborde, Payan and Xuong [9] proposed the following conjecture, which is still open: every digraph has an independent set intersecting every longest path. In recent years the conjecture has attracted quite a bit of attention and a number of results have been obtained in support of the conjecture. For example, in [3], Galeana-Sánchez and Gómez showed that the conjecture is true for 3-anti-quasi-transitive digraphs. Because a 3-anti-circulant digraph must be a 3-anti-quasi-transitive digraph, the conjecture is also true for 3-anti-circulant digraphs. However, the general conjecture appears to be quite difficult to settle. In this context it seems quite natural to ask what can be said about the obvious cycle analogue of the conjecture. To the knowledge of the author, this problem has not previously been addressed in the literature. In this paper, we prove that there exists an independent set intersecting every longest cycle for strong 3-anti-circulant digraphs.

2 Longest cycles in 3-anti-circulant digraphs

The following easy facts will be very useful in our proofs.

Lemma 2.1. *Let D be a strong 3-anti-circulant digraph and D' be a strong subdigraph of D. If for any $x \in V(D')$, $d^-_{D'}(x) \geq 2$ and $d^+_{D'}(x) \geq 2$, then for any $s \in V(D) - V(D')$, we have $(V(D'), s) \neq \emptyset$ and $(s, V(D')) \neq \emptyset$.*

Proof. Since the converse of a 3-anti-circulant digraph is still a 3-anti-circulant digraph, it suffices to prove $(V(D'), s) \neq \emptyset$. Since D is strong, there exists a path from s to D'. Let $P = sy_1 \dots y_k$ be a shortest path from s to D', where $y_k \in V(D')$. We prove $(V(D'), s) \neq \emptyset$ by induction on the length k of P. If $k = 1$, then, by the hypothesis of the lemma, there exist $z, w \in V(D')$ such that $z \rightarrow y_1$ and $z \rightarrow w$. Then $w \rightarrow s$ because $s \rightarrow y_1 \leftarrow z \rightarrow w$ and D is a 3-anti-circulant digraph. For any $k \geq 2$, we suppose that the assertion holds for $k - 1$. Note that $y_1 y_2 \dots y_k$ is a path of length $k - 1$. By the induction hypothesis, there exists a vertex $u \in V(D')$ such that $u \rightarrow y_1$. Since $d^+_{D'}(u) \geq 2$, there exists a vertex $v \in V(D')$ such that $u \rightarrow v$. Then $v \rightarrow s$ because $s \rightarrow y_1 \leftarrow u \rightarrow v$. □

In the following three lemmas, all the subscripts of x_i are taken modulo m and all the subscripts of y_i are taken modulo n.

Lemma 2.2. *Let D be a 3-anti-circulant digraph, $C_1 = x_0 x_1 \dots x_{m-1} x_0$ and $C_2 = y_0 y_1 \dots y_{n-1} y_0$ be two vertex-disjoint cycles of D. For any $x_i \in V(C_1)$ and $y_j \in V(C_2)$, if $x_i \rightarrow y_j$, then $x_{i+k} \rightarrow y_{j-k}$ and $x_{i-k} \rightarrow y_{j+k}$, for any integer k.*

Proof. We will show $x_{i+k} \rightarrow y_{j-k}$ by induction on k. The base case $k = 0$ is immediate. Assume that the result holds by induction for $k - 1$. Then $y_{j-k} \rightarrow y_{j-(k-1)} \leftarrow x_{i+(k-1)} \rightarrow x_{i+k}$ implies $x_{i+k} \rightarrow y_{j-k}$. From this with the arbitrariness of k, we have $x_{i+(am-k)} \rightarrow y_{j-(am-k)}$, where $ma = nb$. As a result, $x_{i-k} \rightarrow y_{j+k}$. □

The following useful fact is an easy consequence of Lemma 2.2.

Lemma 2.3. *Let D be a 3-anti-circulant digraph, $C_1 = x_0 x_1 \dots x_{m-1} x_0$ and $C_2 = y_0 y_1 \dots y_{n-1} y_0$ be two vertex-disjoint cycles of D. For any $x_i \in V(C_1)$ and $y_j \in V(C_2)$, if x_i and y_j are adjacent, then x_{i+k} and y_{j-k} are adjacent as well x_{i-k} and y_{j+k}, for any integer k.*

Lemma 2.2 also implies the following. Below $\gcd(m, n)$ means the greatest common divisor of m and n. For example, $\gcd(12, 8) = 4$.

Lemma 2.4. *Let D be a 3-anti-circulant digraph, $C_1 = x_0 x_1 \dots x_{m-1} x_0$ and $C_2 = y_0 y_1 \dots y_{n-1} y_0$ be two vertex-disjoint cycles of D. For any $x_i \in V(C_1)$ and $y_j \in V(C_2)$, if $x_i \rightarrow y_j$, then $x_i \rightarrow y_{j+d}$ and $x_i \rightarrow y_{j-d}$, where $d = \gcd(m, n)$.*

Proof. For convenience, we, without loss of generality, assume that $i = j = 0$. From $x_0 \to y_0$ and Lemma 2.2, we conclude that $x_0 \to y_{km}$, for $k = 0, 1, 2, \cdots$. Let $G = \{km \in Z_n : k \in Z\}$. It is easy to show that $G = \{km \in Z_n : k \in Z\} = \{kd : k = 0, 1, \ldots, \frac{n}{d} - 1\}$. Hence $x_0 \to y_{kd}$, for $k = 0, 1, \ldots, \frac{n}{d} - 1$. In particular, $x_0 \to y_d$ and $x_0 \to y_{n-d}$. □

For a strong 3-anti-circulant digraph containing a 4-anti-directed cycle, there is the following nice structural characterization.

Lemma 2.5. *Let D be a strong 3-anti-circulant digraph. If D contains a 4-anti-directed cycle, then D is either a complete digraph or a semicomplete bipartite digraph.*

Proof. Let $C = x_0 \to x_1 \leftarrow x_2 \to x_3 \leftarrow x_0$ be a 4-anti-directed cycle of D. Since $x_0 \to x_1 \leftarrow x_2 \to x_3$, we have $x_3 \to x_0$. Similarly, we can obtain that $x_1 \to x_0$, $x_1 \to x_2$ and $x_3 \to x_2$. Hence every arc of $A(C)$ is contained in a 2-cycle.

Claim 1. For any $x \in V(D) - V(C)$, if $x_i \to x$ then $x \to x_{i+2}$; if $x \to x_i$ then $x_{i+2} \to x$.

If $x_i \to x$, then since $x_{i+2} \to x_{i+1} \leftarrow x_i \to x$, we have $x \to x_{i+2}$. If $x \to x_i$, then since $x \to x_i \leftarrow x_{i+1} \to x_{i+2}$, we have $x_{i+2} \to x$.

Claim 2. If x_i and x_{i+2} are adjacent for $i = 0$ or $i = 1$, then D is a complete digraph.

Assume, without loss of generality, that x_0 and x_2 are adjacent and $x_0 \to x_2$. Since $x_1 \to x_2 \leftarrow x_0 \to x_3$, we have that $x_3 \to x_1$. By $x_3 \to x_2 \leftarrow x_0 \to x_1$, we have that $x_1 \to x_3$. By $x_0 \to x_3 \leftarrow x_1 \to x_2$, we have that $x_2 \to x_0$. Hence C is a complete digraph.

For any $x \in V(D) - V(C)$, by Lemma 2.1, $(V(C), x) \neq \emptyset$. Assume, without loss of generality, that $x_0 \to x$. Since $x_2 \to x_1 \leftarrow x_0 \to x$, $x_1 \to x_3 \leftarrow x_0 \to x$ and $x_3 \to x_2 \leftarrow x_0 \to x$, we have $x \to x_2$, $x \to x_1$ and $x \to x_3$, respectively. By $x \to x_1 \leftarrow x_2 \to x_3$, $x \to x_3 \leftarrow x_1 \to x_2$, $x \to x_3 \leftarrow x_2 \to x_1$, we have that $x_3 \to x$, $x_2 \to x$ and $x_1 \to x$, respectively. By $x_0 \to x_3 \leftarrow x_2 \to x$, we have that $x \to x_0$. Hence $V(C) \to x \to V(C)$. For any two vertices $x, y \in V(D) - V(C)$, by the above argument, $V(C) \to \{x, y\} \to V(C)$. By $x \to x_0 \leftarrow x_1 \to y$ and $y \to x_1 \leftarrow x_0 \to x$, we have that $y \to x \to y$. So D is a complete digraph. The proof of Claim 2 is complete.

By Lemma 2.1 and Claim 1, for any $x \in V(D) - V(C)$, x either is adjacent to exactly two vertices of $V(C)$ or is adjacent to every vertex of $V(C)$. First, suppose that there exists a vertex $x \in V(D) - V(C)$ such that x is adjacent to every vertex of $V(C)$. By Lemma 2.1 and Claim 1, assume, without loss of generality, that $x_0 \to x$ and $x_1 \to x$. By $x_0 \to x \leftarrow x_1 \to x_2$, we have $x_2 \to x_0$. By Claim 2, D is a complete digraph. Next assume that for any $x \in V(D) - V(C)$, x is adjacent to exactly two vertices of $V(C)$. Hence we divide $V(D) - V(C)$ into two sets X and Y, where $X = \{x \in V(D) - V(C) : x$ is adjacent to x_1 and x_3; x is not adjacent to x_0 and $x_2\}$

and $Y = \{x \in V(D) - V(C) : x$ is adjacent to x_0 and x_2; x is not adjacent to x_1 and $x_3\}$.

Claim 3. Every vertex of $X \cup \{x_0, x_2\}$ is adjacent to every vertex of $Y \cup \{x_1, x_3\}$.

By the definitions of X and Y, we only need to prove that every vertex of X is adjacent to every vertex of Y. For any $x \in X$ and $y \in Y$, by Claim 1, assume, without loss of generality, that $x_1 \to x, x \to x_3$ and $x_0 \to y, y \to x_2$. Since $y \to x_2 \leftarrow x_1 \to x$, we have $x \to y$. The proof of Claim 3 is complete.

If X and Y are both independent sets, then, by Claim 3, D is a semicomplete bipartite digraph; if not, then one of X and Y is not an independent set, say X. Hence there exist two vertices $x', x'' \in X$ such that x' and x'' are adjacent and without loss of generality, assume that $x' \to x''$. By the definition of X and Claim 1, assume, without loss of generality, that $x_1 \to x''$ and $x'' \to x_3$. By $x' \to x'' \leftarrow x_1 \to x_2$, we have that $x_2 \to x'$, a contradiction to the definition of X. The proof of the lemma is complete. $\square$

Now we consider the strong 3-anti-circulant digraph containing a cycle factor. First define two digraphs F_6 and F_8, see Figure 2. It is not difficult to check that both of the digraphs F_6 and F_8 are 3-anti-circulant digraphs and each of F_6 and F_8 contains a cycle factor.

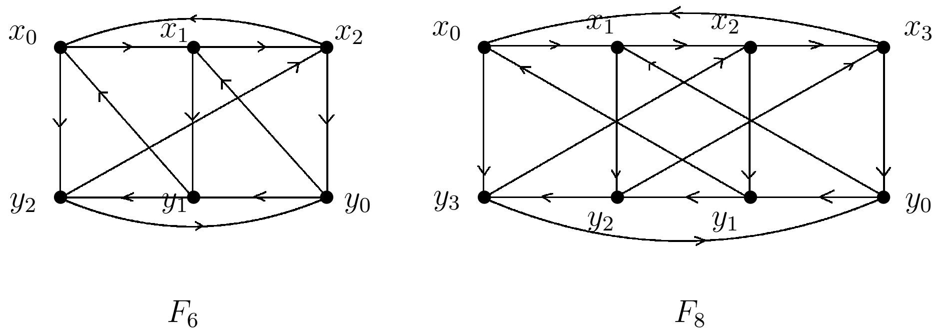

Figure 2.

Lemma 2.6. *Let D be a strong 3-anti-circulant digraph containing a cycle factor $C_1 \cup C_2$ such that the lengths of C_1 and C_2 are equal, denoted by n. Suppose that D contains no 4-anti-directed cycle. Then D is either a semicomplete digraph or isomorphic to F_6 or F_8.*

Proof. Let $C_1 = x_0x_1 \dots x_{n-1}x_0$ and $C_2 = y_0y_1 \dots y_{n-1}y_0$. From now on, all subscripts appearing in this proof are taken modulo n. Since D is strong, we have $(V(C_1), V(C_2)) \neq \emptyset$ and $(V(C_2), V(C_1)) \neq \emptyset$. If there exist $x_i \in V(C_1)$ and $y_j \in V(C_2)$ such that $x_i \to y_j \to x_i$, then, by Lemma 2.2, $y_{j-1} \to x_{i+1}$. Note that $x_i \to y_j \leftarrow y_{j-1} \to x_{i+1} \leftarrow x_i$ is a 4-anti-directed cycle, a contradiction. Now assume that the arc between C_1 and C_2 is not contained in any 2-cycle, that is to say, for any $u \in V(C_i)$ and $v \in V(C_{3-i})$ with $i = 1$ or 2, $u \to v$ implies $u \mapsto v$. Since $(V(C_1), V(C_2)) \neq \emptyset$, we assume, without loss of generality, $x_0 \to y_{n-1}$ and furthermore $x_0 \mapsto y_{n-1}$. By Lemma 2.2, $x_i \to y_{n-1-i}$ and furthermore $x_i \mapsto y_{n-1-i}$, for $i = 0, 1, \ldots, n-1$. According to $(V(C_2), V(C_1)) \neq \emptyset$ and Lemma 2.2, we can deduce that there is an arc from $V(C_2)$ to every vertex of $V(C_1)$, in particular, there is an arc from $V(C_2)$ to x_0.

Suppose $n = 3$. If x_0 and every vertex of $V(C_2)$ are adjacent, then by Lemma 2.3, every vertex of $V(C_1)$ is adjacent to every vertex of $V(C_2)$. Hence D is a semicomplete digraph. Assume that x_0 and one of y_0 and y_1 is not adjacent. Since $(V(C_2), x_0) \neq \emptyset$ and $x_0 \mapsto y_2$, we assume, without loss of generality, that $y_0 \mapsto x_0$ and x_0 is not adjacent to y_1. From Lemma 2.2, we have that $y_1 \mapsto x_2, y_2 \mapsto x_1$ and $x_1(x_2)$ is not adjacent to $y_0(y_2$ respectively). It is not difficult to deduce that there are no other arcs in D. Define a mapping $\theta : V(D) \to V(F_6)$ such that $\theta(y_2) = x_0, \theta(y_1) = x_2, \theta(y_0) = x_1, \theta(x_0) = y_1, \theta(x_1) = y_2, \theta(x_2) = y_0$. It is easy to check that the mapping θ is an isomorphism. Hence D is isomorphic to F_6. Next assume that $n \geq 4$.

Claim 1. For any $2 \leq j \leq n-2$, if $y_j \to x_0$, then $y_{j-2} \to x_0$, $x_2 \to y_{n-1}$ and $y_{n-3} \to x_0$.

By Lemma 2.2 and $y_j \to x_0$, we know that $y_{j-1} \to x_1$ and $y_{j-2} \to x_2$. Then $y_{j-1} \to x_1 \leftarrow x_0 \to y_{n-1}$ implies that $y_{n-1} \to y_{j-1}$. By $x_0 \to y_{n-1}$ and Lemma 2.2, we have that $x_{n-j+1} \to y_{j-2}$ and $x_{n-j} \to y_{j-1}$. Then $y_{n-1} \to y_{j-1} \leftarrow x_{n-j} \to x_{n-j+1}$ implies that $x_{n-j+1} \to y_{n-1}$. By $x_0 \to y_{n-1} \leftarrow x_{n-j+1} \to y_{j-2}$, we have that $y_{j-2} \to x_0$. By $y_{n-1} \to y_{j-1} \leftarrow y_{j-2} \to x_2$, we have that $x_2 \to y_{n-1}$. Then $x_0 \to y_{n-1} \leftarrow x_2 \to y_{n-3}$ implies that $y_{n-3} \to x_0$.

Claim 2. For any $u \in V(C_i)$, there exists no vertex $v \in V(C_{3-i})$ such that both of $u \to v$ and $v \to u^+$ hold simultaneously, where $i = 1$ or 2.

By contradiction. Without loss of generality, assume that $y_{n-2} \to x_0 \to y_{n-1}$. By $n \geq 4$ and Claim 1, we have that $y_{n-3} \to x_0$ and $x_2 \to y_{n-1}$. Recalling that we have assumed that $x_2 \mapsto y_{n-1}$. By $y_{n-2} \to x_0$, $y_{n-3} \to x_0$ and repeated application of Claim 1, we can obtain $y_1 \to x_0$. This together with Lemma 2.2, we have $y_{n-1} \to x_2$, a contradiction to $x_2 \mapsto y_{n-1}$. The proof of the claim is complete.

Using Claim 2, we know that y_{n-2} does not dominate x_0. From Claim 2, we can also know that y_{n-1} does not dominate x_1, which implies that y_0 does not dominate x_0 from Lemma 2.2. Hence we have that y_{n-2} and y_0 do not dominate x_0. Similarly, we may assume y_{n-2-i} and y_{n-i} do not dominate x_i, for $i = 1, 2, \ldots, n-1$.

If $n = 4$, then it must be $y_1 \to x_0$. In this case, it is easy to see that D is isomorphic to F_8. Now assume $n \geq 5$ and $y_j \to x_0$ for $1 \leq j \leq n-3$.

Claim 3. For any $y_i \in V(C_2)$, y_i dominates neither y_{i+2} nor y_{i+3}.

Without loss of generality, assume that $i = 0$. By contradiction. If $y_0 \to y_2$, then, by $x_{n-3} \to y_2 \leftarrow y_0 \to y_1$, we have that $y_1 \to x_{n-3}$, a contradiction to the fact that y_1 does not dominate x_{n-3}. If $y_0 \to y_3$, then $y_0 \to y_3 \leftarrow x_{n-4} \to x_{n-3}$ implies that $x_{n-3} \to y_0$ and $x_{n-3} \mapsto y_0$. By $x_{n-4} \to y_3 \leftarrow y_0 \to y_1$, we have that $y_1 \to x_{n-4}$. From this with Lemma 2.2, $y_0 \to x_{n-3}$, a contradiction to $x_{n-3} \mapsto y_0$. The proof of the claim is complete.

Now we claim that there exist two vertices $y_i, y_k \in V(C_2)$ such that $y_k \to y_i$ where $i - k \geq 2$. If $y_1 \to x_0$, then by Lemma 2.2, $y_{n-2} \to x_3$. By $x_0 \to y_{n-1} \leftarrow y_{n-2} \to x_3$, we have that $x_3 \to x_0$. By $x_3 \to x_0 \leftarrow x_{n-1} \to y_0$, we have that $y_0 \to x_3$. By $y_{n-2} \to x_3 \leftarrow y_0 \to y_1$, we have that $y_1 \to y_{n-2}$. By $n \geq 5$, we have $(n-2) - 1 = n - 3 \geq 2$. If $y_j \to x_0$, where $2 \leq j \leq n-3$, then by $y_j \to x_0 \leftarrow x_{n-1} \to y_0$, we have that $y_0 \to y_j$. Hence the claim holds. If $i - k = 2$ or $i - k = 3$, then it is a contradiction according to Claim 3. If $i - k \geq 4$, then by $y_{i-1} \to y_i \leftarrow y_k \to y_{k+1}$, we have that $y_{k+1} \to y_{i-1}$. Continuing in this way, there exist two vertices y_t, y_s such that $y_t \to y_s$ with $s - t = 2$ or $s - t = 3$, which is also a contradiction according to Claim 3. $\square$

Lemma 2.7. *Let D be a connected 3-anti-circulant digraph containing a cycle factor $C_1 \cup C_2$ such that the lengths of C_1 and C_2 are not equal. Then D is strong and has a 4-anti-directed cycle.*

Proof. Let $C_1 = x_0x_1 \ldots x_{m-1}x_0$ and $C_2 = y_0y_1 \ldots y_{n-1}y_0$. From now on, all the subscripts of x_i appearing in this proof are taken modulo m and all the subscripts of y_j appearing in this proof are taken modulo n. Since D is connected, there exists at least an arc between C_1 and C_2. Assume, without loss of generality, that there is one arc from C_1 to C_2 and $x_0 \to y_{n-1}$. If $d = \gcd(m, n) = 1$, then by Lemma 2.4, $V(C_1) \to V(C_2)$. By $x_1 \to y_{n-2} \leftarrow x_0 \to y_{n-1}$, we have that $y_{n-1} \to x_1$. Hence D is strong. Note that $x_0 \to y_{n-1} \leftarrow x_1 \to y_{n-2} \leftarrow x_0$ is a 4-anti-directed cycle. Next assume that $d = \gcd(m, n) \geq 2$.

First we show that there is an arc from C_2 to C_1. By $x_0 \to y_{n-1}$ and Lemmas 2.2 and 2.4, $x_0 \to y_{n-1-d}$ and $x_d \to y_{n-1-d}$. By $x_d \to y_{n-1-d} \leftarrow x_0 \to y_{n-1}$, we have that $y_{n-1} \to x_d$. Note that $y_{n-1}x_d$ is an arc from C_2 to C_1. Hence D is strong.

Now we show that D contains a 4-anti-directed cycle. We may assume, without loss of generality, that $m < n$. Since $x_0 \to y_{n-1}$, by Lemma 2.2, $x_0 \to y_{n-m-1}$ and $x_1 \to y_{n-2}$. Noting that $(n-1) - (n-m-1) = m \geq 3$, we have $n - 4 \geq n - m - 1$. By $y_{n-2} \to y_{n-1} \leftarrow x_0 \to y_{n-m-1}$, we have that $y_{n-m-1} \to y_{n-2}$. By $x_1 \to y_{n-2} \leftarrow y_{n-m-1} \to y_{n-m}$, we have that $y_{n-m} \to x_1$. By $y_{n-m-1} \to y_{n-2} \leftarrow x_1 \to x_2$, we have that $x_2 \to y_{n-m-1}$. Combining this with Lemma 2.2, we have $x_1 \to y_{n-m}$. Note that $x_1 \to y_{n-2} \leftarrow y_{n-m-1} \to y_{n-m} \leftarrow x_1$ is a 4-anti-directed cycle. $\square$

Theorem 2.8. *Let D be a strong 3-anti-circulant digraph. If D has a cycle factor $C_1 \cup C_2 \cup \ldots \cup C_t$ with $t \geq 2$, then D is either a semicomplete digraph or a semicomplete bipartite digraph or isomorphic to F_6.*

Proof. Let $F = C_1 \cup C_2 \cup \ldots \cup C_t$. If D has a 4-anti-directed cycle, then by Lemma 2.5, D is either a complete digraph or a semicomplete bipartite digraph. Now assume that D has no 4-anti-directed cycle. By Lemma 2.7, we know that the lengths of all cycles are equal, say n.

First we claim that D contains a cycle factor consisting of exactly two cycles. We show the claim by induction on t. If $t = 2$, it is nothing to prove. Suppose $t \geq 3$. If there exist two cycles C_i and C_j of F such that $(V(C_i), V(C_j)) \neq \emptyset$ and $(V(C_j), V(C_i)) \neq \emptyset$, then by Theorem 1.2, D contains a cycle covering the vertices of $V(C_i) \cup V(C_j)$. By the induction hypothesis, the claim holds. Hence we assume that for any two cycles C_i and C_j of F, we have $C_i \Rightarrow C_j$ or $C_j \Rightarrow C_i$. Since D is strong, it must exist three cycles C_i, C_j and C_k such that there is at least one arc from C_i to C_j and from C_j to C_k. Let $C_i = x_0x_1 \dots x_{n-1}x_0$, $C_j = y_0y_1 \dots y_{n-1}y_0$ and $C_k = z_0z_1 \dots z_{n-1}z_0$. Without loss of generality, assume that $x_0 \to y_{n-1} \to z_0$. Combining this with Lemma 2.2, we have $y_{n-2} \to z_1$, $x_{n-1} \to y_0$ and $y_0 \to z_{n-1}$. By $x_0 \to y_{n-1} \leftarrow y_{n-2} \to z_1$, we have that $z_1 \to x_0$. By $z_1 \to x_0 \leftarrow x_{n-1} \to y_0$, we have that $y_0 \to z_1$. By $y_{n-2} \to z_1 \leftarrow y_0 \to z_{n-1}$, we have that $z_{n-1} \to y_{n-2}$. Hence $V(C_j) \cup V(C_k)$ induces a strong digraph, a contradiction.

By the above claim, D has a cycle factor $C'_1 \cup C'_2$. As we have assumed that D has no 4-anti-directed cycle. This together with Lemma 2.7 implies that the lengths of C'_1 and C'_2 are same. Note that F_8 is a semicomplete bipartite digraph. By Lemma 2.6, D is either a semicomplete digraph or a semicomplete bipartite digraph or isomorphic to F_6. □

Lemma 2.9. *Let D be a 3-anti-circulant digraph. If C is an anti-directed cycle of length k in D, then D contains a cycle of length k covering the vertices of $V(C)$.*

Proof. Let $C = (x_0, x_1, \dots, x_{k-1}, x_0)$ be an anti-directed cycle of length k such that $x_0 \to x_1$. By the definition of anti-directed cycles, k must be even. For any even $i \in \{0, 2, \dots, k-2\}$, $x_i \to x_{i+1} \leftarrow x_{i+2} \to x_{i+3}$ implies $x_{i+3} \to x_i$. Note that $x_0x_1x_{k-2}x_{k-1}x_{k-4}x_{k-3} \dots x_4x_5x_2x_3x_0$ is a cycle of length k covering the vertices of $V(C)$. □

Lemma 2.9 immediately implies the following result.

Theorem 2.10. *Let D be a 3-anti-circulant digraph of order even. If D has an anti-directed cycle of length $|V(D)|$, then D is Hamiltonian.*

Conjecture 2.11. *[9] Every digraph has an independent set intersecting every longest path.*

In [3], Galeana-Sánchez and Gómez showed that Conjecture 2.11 is true for 3-anti-quasi-transitive digraphs.

Theorem 2.12. *[3] Let D be a 3-anti-quasi-transitive digraph. There exists an independent set in D intersecting every longest path in D.*

Noting that a 3-anti-circulant digraph is also a 3-anti-quasi-transitive digraph, Theorem 2.12 immediately implies the following.

Corollary 2.13. *Let D be a 3-anti-circulant digraph. There exists an independent set in D intersecting every longest path in D.*

Now we generalize the above result to the longest cycle and give the following theorem.

Theorem 2.14. *Let D be a strong 3-anti-circulant digraph. There exists an independent set intersecting every longest cycle.*

Proof. Let $C = x_0x_1 \dots x_{n-1}x_0$ be a longest cycle in D and let F be a maximal independent set. If $V(C) \cap F \neq \emptyset$, then we are done. Now suppose $V(C) \cap F = \emptyset$. First show that $F \Rightarrow V(C)$ or $V(C) \Rightarrow F$ holds. Since F is a maximal independent set, for any $x \in V(C)$, there exists a vertex $y \in F$ such that x and y are adjacent. If $(V(C), F) \neq \emptyset$ and $(F, V(C)) \neq \emptyset$, then there exist $x_i \in V(C)$ and $u, v \in F$ such that $u \rightarrow x_i$ and $x_{i-1} \rightarrow v$. Then $u = v$ because $u \rightarrow x_i \leftarrow x_{i-1} \rightarrow v$ and F is an independent set. But $x_{i-1}uC[x_i, x_{i-1}]$ is a longer cycle than C in D contradicting that C is a longest cycle. Hence $F \Rightarrow C$ or $C \Rightarrow F$. Since the converse of a 3-anti-circulant digraph is still a 3-anti-circulant digraph, we assume that $F \Rightarrow C$. Now divide the vertices of $V(D) - V(C)$ into four sets I, O, B, Y such that

$$\begin{array}{rcl} I &=& \{x \in V(D) - V(C) : (x, V(C)) \neq \emptyset,\ x \Rightarrow C\}, \\ O &=& \{x \in V(D) - V(C) : (V(C), x) \neq \emptyset,\ C \Rightarrow x\}, \\ B &=& \{x \in V(D) - V(C) : (x, V(C)) \neq \emptyset,\ (V(C), x) \neq \emptyset\} \\ \text{and} \quad Y &=& V(D) - I - O - B. \end{array}$$

By the hypothesis, we can see that $F \subset I$. For any $y \in O \cup B$, by the definition of $O \cup B$, there exists $x \in V(C)$ such that $x \rightarrow y$. By the definition of F, there exists $u \in F$ such that $u \rightarrow x^+$. Then $y \rightarrow u$ from $u \rightarrow x^+ \leftarrow x \rightarrow y$. But $xyuC[x^+, x]$ is a longer cycle than C, a contradiction. Hence $O \cup B = \emptyset$, which is a contradiction to the fact that D is strong. □

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