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Symmetric alternating sign matrices

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Abstract

In this note we consider completions of $n \times n$ symmetric $(0, -1)$-matrices to symmetric alternating sign matrices by replacing certain 0s with +1s. In particular, we prove that any $n \times n$ symmetric $(0, -1)$-matrix that can be completed to an alternating sign matrix by replacing some 0s with +1s can be completed to a symmetric alternating sign matrix. Similarly, any $n \times n$ symmetric $(0, +1)$-matrix that can be completed to an alternating sign matrix by replacing some 0s with −1s can be completed to a symmetric alternating sign matrix.

1 Introduction

An *alternating sign matrix*, abbreviated ASM, is an $n \times n$ $(0, +1, -1)$-matrix such that, ignoring 0s, in each row and column, the +1s and −1s alternate, beginning and ending with a +1. An ASM cannot contain any −1s in rows 1 and n and columns 1 and n. The book [1] by Bressoud contains a history of the development of ASMs.

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In [2], there is an investigation of the zero-nonzero patterns of ASMs. The paper [3] considers the problem of completing a $(0, -1)$-matrix to an ASM by replacing some 0s with +1s. Each row and column of an ASM contains an odd number of nonzeros with the first and last rows and columns each containing exactly one nonzero and that nonzero is a +1. If an ASM (regarded as a square) is subjected to any of the symmetries of a square (the dihedral group), the result is also an ASM.

The simplest examples of ASMs are the permutation matrices. Other examples of ASMs are

$$\left[\begin{array}{c|c|c|c|c|c|c} & & & +1 & & & \\ \hline & & +1 & -1 & & +1 & \\ \hline & +1 & -1 & & +1 & & \\ \hline +1 & -1 & & +1 & & & \\ \hline & & +1 & & & -1 & +1 \\ \hline & +1 & & & -1 & +1 & \\ \hline & & & & +1 & & \end{array}\right] \text{ and } \left[\begin{array}{c|c|c|c|c|c} & & & +1 & & \\ \hline & & +1 & -1 & +1 & \\ \hline & +1 & -1 & +1 & -1 & +1 \\ \hline +1 & -1 & +1 & -1 & +1 & \\ \hline & +1 & -1 & +1 & & \\ \hline & & +1 & & & \end{array}\right].$$

(For visual clarity, we usually block off rows and columns and then suppress the 0s in $(0, +1, -1)$-matrices.)

Our emphasis in this note is on combinatorial properties of symmetric ASMs, of which the preceding two ASMs are examples. Given an $n \times n$ $(0, -1)$-matrix A, any matrix B obtained from A by replacing some 0s by +1s is a *(+1)-completion* of A; if B is an ASM, then B is called a *(+1)-completion of A to an ASM* or an *ASM (+1)-completion of A*. In [3] ASM (+1)-completions of $(0, -1)$-matrices (called, simply, ASM completions) were investigated with an emphasis on the so-called bordered-permutation $(0, -1)$-matrices. By an *$n \times n$ bordered-permutation $(0, -1)$-matrix A* we mean an $n \times n$ $(0, -1)$-matrix such that the first and last rows and columns contain only zeros, and the submatrix $A[\{2, 3, \ldots, n-1\}|\{2, 3, \ldots, n-1\}]$ obtained by deleting rows and columns 1 and n is $-P$ where P is a permutation matrix. Here we consider (+1)-completions of symmetric $(0, -1)$-matrices to symmetric ASMs.

We also consider here completions of an $n \times n$ $(0, +1)$-matrix A to ASMs by replacing some 0s with −1s. We call these *ASM (−1)-completions* . In order that A has an ASM (−1)-completion, it is necessary that there be at least one +1 in each row and column, only one +1 in the first and last rows and columns, and no consecutive +1s in a row or column.

Example 1 Let A be the symmetric bordered-permutation $(0, -1)$-matrix:

						−1	
					−1		
			−1				
				−1			
		−1					
	−1						

.

Then it is straightforward to check that A has a unique $(+1)$-completion to an ASM and this $(+1)$-completion is symmetric:

						+1	
					+1	−1	+1
			+1		−1	+1	
		+1	−1	+1			
			+1	−1	+1		
	+1	−1		+1			
+1	−1	+1					
	+1						

.

On the other hand, the symmetric $(0, -1)$-matrix

						−1	
			−1		−1		
		−1					
		−1					
	−1						

does not have a $(+1)$-completion to an ASM; it suffices to examine rows 1, 2, and 3.

Example 2 Consider the 7×7 symmetric $(0, +1)$-matrix

$A =$

				+1		
			+1			
		+1			+1	
	+1					+1
+1				+1		
		+1				
			+1			

.

The -1s in any (-1)-completion of A to an ASM must be in the shaded positions. Any (-1)-completion of A must have three -1s. There are three (-1)-completions of A, namely, as given below, the matrix A' and its transpose, and the symmetric matrix A'':

$$A' = \begin{bmatrix} & & & & +1 & & \\ & & & +1 & & & \\ & & +1 & -1 & & +1 & \\ & +1 & & & -1 & & +1 \\ +1 & & -1 & & +1 & & \\ & & +1 & & & & \\ & & & +1 & & & \end{bmatrix}, \quad A'' = \begin{bmatrix} & & & & +1 & & \\ & & & +1 & & & \\ & & +1 & & -1 & +1 & \\ & +1 & & -1 & & & +1 \\ +1 & & -1 & & +1 & & \\ & & +1 & & & & \\ & & & +1 & & & \end{bmatrix}.$$

In [3] it was shown that every bordered-permutation $(0, -1)$-matrix can be $(+1)$-completed to an ASM. We first show that every $n \times n$ symmetric bordered-permutation $(0, -1)$-matrix can be $(+1)$-completed to a symmetric ASM and obtain a bound on the number of such $(+1)$-completions. There is not an analogue of this result for (-1)-completions, since a permutation matrix is already an ASM. Our main results are that (i) if a symmetric $(0, -1)$-matrix has an ASM $(+1)$-completion, then it also has a symmetric ASM $(+1)$-completion, and (ii) if a symmetric $(0, +1)$-matrix has an ASM (-1)-completion, then it also has a symmetric ASM (-1)-completion.

2 Symmetric ASM Completions

Theorem 3 *Let $A = [a_{ij}]$ be an $n \times n$ symmetric bordered-permutation* $(0, -1)$*-matrix. Then A has a* $(+1)$*-completion to a symmetric ASM.*

Proof. This theorem will follow from Theorem 7 and the theorem in [3] that every bordered-permutation $(0, -1)$-matrix can be $(+1)$-completed to an ASM. We give a short independent proof.

We use induction on n. The theorem is trivial if $n = 2$ or 3. Let $n \geq 4$. Let k be such that $a_{2k} = a_{k2} = -1$. Let $A' = A(2, k|2, k)$ be the symmetric matrix obtained from A by deleting rows and columns 2 and k. (This matrix is $(n-1) \times (n-1)$ if $k = 2$ and $(n-2) \times (n-2)$ otherwise.) We use for the indices of the row and columns of A' the same indices they had in A; thus the index set for rows and columns of A' is $\{1, 2, \ldots, n\} \setminus \{2, k\}$. By induction A' has a $(+1)$-completion $B' = [b'_{ij}]$ to a symmetric ASM. Let r be such that $b'_{1r} = b'_{r1} = +1$.

If $r > k$, let s be the first integer such that $b'_{s,k-1} = b'_{k-1,s} = +1$. We then let B be the matrix which has $+1$s in all other positions that B' has $+1$s except for the positions $(1, r), (r, 1), (s, k-1),$ and $(k-1, s)$ and, in addition, has $+1$s in positions $(1, k), (k, 1), (2, k-1), (k-1, 2), (2, r), (r, 2), (s, k), (k, s)$. Then B is a symmetric ASM $(+1)$-completion of A.

If $r < k$, let s be the first integer such that $b'_{s,k+1} = b'_{k+1,s} = +1$. We then let B be the matrix which has $+1$s in all other positions that B' has $+1$s except for the positions $(1, r), (r, 1), (s, k+1)$, and $(k+1, s)$ and, in addition, has $+1$s in positions $(1, k), (k, 1), (2, k + 1), (k + 1, 2), (2, r), (r, 2), (s, k), (k, s)$. Then B is a symmetric ASM $(+1)$-completion of A. □

We give an example illustrating the inductive proof of Theorem 3.

Example 4 In this example, $n = 8$ and with the above notation, $k = 4$, $r = 6$, and $s = 5$. Let

$$A = \begin{bmatrix} & & & & & & & \\ & & & -1 & & & & \\ & & & & & -1 & & \\ & -1 & & & & & & \\ & & & & -1 & & & \\ & & -1 & & & & & \\ & & & & & & -1 & \\ & & & & & & & \end{bmatrix} \to A' = \begin{bmatrix} & & & & & \\ & & & -1 & & \\ & & -1 & & & \\ & -1 & & & & \\ & & & & -1 & \\ & & & & & \end{bmatrix}.$$

Then, where we have included the row and column indices for clarity, we have

$B' =$

	1	3	5	6	7	8
1				+1		
3			+1	−1	+1	
5		+1	−1	+1		
6	+1	−1	+1			
7		+1			−1	+1
8					+1	

→

$B =$

	1	2	3	4	5	6	7	8
1				+1				
2			+1	−1		+1		
3		+1				−1	+1	
4	+1	−1			+1			
5				+1	−1	+1		
6		+1	−1		+1			
7			+1				−1	+1
8							+1	

.

If A is a symmetric $(0, -1)$-matrix, then $\pi_s(A)$ denotes the number of $(+1)$-completions of A to a symmetric ASM. By Theorem 3, if A is also a bordered-permutation $(0, -1)$-matrix, then $\pi_s(A) \geq 1$. We now give an upper bound for $\pi_s(A)$ in general.

Theorem 5 *Let $A = [a_{ij}]$ be an $n \times n$ bordered symmetric $(0, -1)$-matrix such that A has a -1s on the main diagonal and $2b$ -1s off the main diagonal. Then*

$$\pi_s(A) \leq \frac{1}{2^{a+b}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k(n-2k)!k!}. \tag{1}$$

(The number $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k(n-2k)!k!}$ is the number of $n \times n$ symmetric permutation matrices [5, p. 218].)

Proof. Let k be the maximum index of a row of A with a -1, and let l be the maximum index of a column with a -1 in row k. Thus $a_{kl} = a_{lk} = -1$ and we let $B = [b_{ij}]$ be the symmetric matrix obtained from A by replacing a_{kl} and a_{lk} with 0s. We show that $\pi_s(A) \leq \frac{\pi_s(B)}{2}$ by establishing, when $\pi_s(A) \neq 0$, a one-to-two correspondence from the set $C_s(A)$ of symmetric ASM (+1)-completions of A to the set $C_s(B)$ of symmetric ASM (+1)-completions of B. We consider two cases depending on whether $k \neq l$ or $k = l$.

Case 1 ($k \neq l$): Let $A' = [a'_{ij}] \in C_s(A)$. There exists $l' > l$ such that $a'_{kl'} = +1$. We choose l' to be the smallest such integer so that $a'_{kp} = 0$ for all p with $l < p < l'$. There also exists $k' > k$ such that $a'_{k'l} = +1$, and we choose k' to be the smallest such integer so that $a'_{ql} = 0$ for all q with $k < q < k'$. We then define $B' = [b'_{ij}]$ to be the matrix obtained from A' by replacing $a'_{kl}, a'_{k'l}, a'_{kl'}$, and also $a'_{lk}, a'_{lk'}, a'_{l'k}$, with 0s, and replacing $a'_{k'l'}$ and $a'_{l'k'}$ (both of which must equal 0) with +1s. The matrix B' is an ASM (+1)-completion of B, and the map $f : C_s(A) \to C_s(B)$ given by $A' \to B'$ is injective. In a similar way by choosing the first +1 to the left of $a'_{kl} = -1$ we obtain another injective map $g : C_s(A) \to C_s(B)$. We have that $g(C_s(A)) \cap f(C_s(A)) = \emptyset$. Thus, in the case that $k \neq l$, each (+1)-completion of A gives two (+1)-completions of B.

Case 2 ($k = l$): Thus $a_{kk} = -1$ and the principal submatrix $A[k, k+1, \ldots, n|k, k+1, \ldots, n]$ of A determined by rows and columns $k, k+1, \ldots, n$ has a unique -1 and this -1 is in its $(1, 1)$-position. Let $A' = [a'_{ij}] \in C_s(A)$. In A' there is a unique +1 to the right of $a'_{kk} = -1$, say in column r and a unique +1 below it, so in row r. The principal submatrix $A'[k, k+1, \ldots, n|k, k+1, \ldots, n]$ of A is a symmetric permutation matrix with an additional -1 in its $(1, 1)$-position. Let B' be the matrix obtained from A' by replacing $a'_{kk} = -1, a_{kr} = +1, a'_{rk} = +1$ with 0s and replacing $a_{rr} = 0$ with +1. Then B' is an ASM (+1)-completion of B. In a similar way, we determine in A' the largest integer p with $p < k$ such that $a_{kp} = +1$, and thus $a_{pk} = +1$. Let B' be the matrix obtained from A' by replacing $a'_{kk} = -1$ with +1, replacing $a_{kp} = a_{kr} = a_{rk} = a_{pk} = +1$ with 0s, and replacing $a'_{rp} = a'_{pr} = 0$ with +1. Then B' is an ASM (+1)-completion of B. As before we have two injections of $C_s(A)$ into $C_s(B)$ with disjoint images, and thus each (+1)-completion of A gives two (+1)-completions of B.

Iterating the above, we see that $\pi_s(A) \leq \frac{\pi_s(C)}{2^{a+b}}$ where C is the $n \times n$ zero matrix.

The number of ASM (+1)-completions of C is the number of symmetric permutation matrices, and the theorem now follows. □

We note that equality occurs in (1) if $A = O$.

In the proof of the next theorem we shall make use of an idea from [3]. Let A be an $n \times n$ $(0, -1)$-matrix and assume that A can be (+1)-completed to an ASM. Let $\sigma(A)$ equal the number of -1s in A. Let $Z \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ be the set of zero positions of A. The -1s of A partition Z into two families of $(n + \sigma(A))$ sets, the *horizontal partition* $\mathcal{H}(A) = (H_i : 1 \leq i \leq n + \sigma(A))$, consisting of the *horizontal blocks*, and the *vertical partition* $\mathcal{V}(A) = (V_i : 1 \leq i \leq n + \sigma(A))$, consisting of the *vertical blocks*. These are defined as follows: If there are $c_i \geq 0$ -1s in row i of A, then row i determines the $c_i + 1$ horizontal blocks consisting of those positions occupied by the 0s to the left of the first -1, in-between two consecutive -1s, and to the right of the last -1. The vertical blocks are defined in a similar way. Included in $\mathcal{H}(A)$ is the set of n positions in the first row and the set of n positions in the last row. Included in the vertical partition $\mathcal{V}(A)$ is the set of n positions in the first column and the set of n positions in the last column. Each $H_i \in \mathcal{H}(A)$ and each $V_j \in \mathcal{V}(A)$ intersect in at most one element of Z. The bipartite graph $G(A) \subseteq K_{n+\sigma(A),n+\sigma(A)}$ with vertex bipartition $\mathcal{H}(A), \mathcal{V}(A)$ has an edge joining $H_i \in \mathcal{H}(A)$ and $V_j \in \mathcal{V}(A)$ if and only if $H_i \cap V_j \neq \emptyset$ (and thus $|H_i \cap V_j| = 1$). As observed in [3], the matrix A has an ASM (+1)-completion if and only if the bipartite graph $G(A)$ has a perfect matching; more specifically, if $(\{H_i, V_{\theta(i)}\} : 1 \leq i \leq n + \sigma(A))$ is a perfect matching of $G(A)$, where θ is a permutation of $\{1, 2, \ldots, n + \sigma(A)\}$, then a (+1)-completion of A to an ASM is obtained by replacing the 0s in A in the positions $\{H_i \cap V_{\theta(i)} : 1 \leq i \leq n + \sigma(A)\}$ with +1s.

Now suppose that A is an $n \times n$ symmetric $(0, -1)$-matrix. Then there is a bijection between $\mathcal{H}(A)$ and $\mathcal{V}(A)$ defined by $H_i \to V_i$ where $V_i = \{(s, r) : (r, s) \in H_i\}$ $(i = 1, 2, \ldots n + \sigma(A))$. With subscripts for the blocks in $\mathcal{H}(A)$ and $\mathcal{V}(A)$ as in this bijection, we have that $H_i \cap V_j \neq \emptyset$ if and only if $H_j \cap V_i \neq \emptyset$ $(1 \leq i, j \leq n + \sigma(A))$. Thus the $(n + \sigma(A)) \times (n + \sigma(A))$ biadjacency matrix $C = [c_{ij}]$ of the bipartite graph $G(A)$ is symmetric and can be viewed as the adjacency matrix of a *loopy graph* $G^*(A)$ with vertex set $u_1, u_2, \ldots, u_{n+\sigma(A)}$ (u_i corresponds to both H_i and V_i) whose edges are all those pairs $\{u_i, u_j\}$ such that $H_i \cap V_j \neq \emptyset$ (equivalently, $H_j \cap V_i \neq \emptyset$). ($G^*(A)$ may have loops since it is possible that for some i, $H_i \cap V_i \neq \emptyset$ giving a loop at u_i, and thus we use the common term of loopy graph.) A *perfect matching of a loopy graph* is a collection of pairwise disjoint edges (possibly including loops) such that each vertex occurs on exactly one edge. Such a perfect matching corresponds to a symmetric permutation matrix P such that $P \leq C$ (entrywise). A perfect matching determines positions of A in which to put +1s in order to get a (+1)-completion of A to a symmetric ASM.

Example 6 Let $n = 5$ and consider the symmetric $(0, -1)$-matrix

$$A = \begin{bmatrix} & & & & \\ & & -1 & & \\ & -1 & & & \\ & & & -1 & \\ & & & & \end{bmatrix}$$

where $\sigma(A) = 3$. Then the loopy graph $G^*(A)$ has 8 vertices and its adjacency matrix C (as usual, only the 1s are shown) is

$C =$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	1	1		1		1		1
u_2	1	1						
u_3						1		1
u_4	1							
u_5					1	1		1
u_6	1		1		1			
u_7								1
u_8	1		1		1		1	1

.

The loopy graph $G^*(A)$ has a perfect matching (corresponding to the shaded 1s), equivalently, there exists a symmetric permutation matrix $Q \leq C$ (entrywise), and hence there exists a $(+1)$-completion of A to a symmetric ASM, namely

$$\begin{bmatrix} & & +1 & & \\ & +1 & -1 & +1 & \\ +1 & -1 & +1 & & \\ & +1 & & -1 & +1 \\ & & & +1 & \end{bmatrix}.$$

Theorem 7 *Let A be an $n \times n$ symmetric $(0, -1)$-matrix that has a $(+1)$-completion to an ASM. Then A has a $(+1)$-completion to a symmetric ASM.*

Proof. Let the $n \times n$ matrix $B = [b_{ij}]$ be an ASM $(+1)$-completion of A and let

$$q(B) = \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij} - b_{ji}|.$$

Then $q(B)$ is an even integer which counts the number of positions (i, j) with $i \neq j$ such that $b_{ij} + b_{ji} = 1$. If $q(B) = 0$, then B is a symmetric ASM $(+1)$-completion of A. Suppose that $q(B) \neq 0$. Since B is an ASM $(+1)$-completion of A, the $+1$s of B determine a perfect matching M of the bipartite graph $G(A)$. We consider those $q(B)/2$ edges $\{H_i, V_j\}$ of M such that the position (p, q) of A in their intersection $H_i \cap V_j$ contains a $+1$ but there is not a $+1$ in position (q, p) of A in the intersection

$H_j \cap V_i$ (so there is a $+1$ in the unique position in $H_j \cap V_k$ for some $k \neq j$). These $q(B)/2$ edges determine an asymmetric digraph D (an orientation of a graph), whose vertex set is $\{u_1, u_2, \ldots, u_{n+\sigma(A)}\}$, with no loops and at least one edge, such that any vertex with positive indegree also has positive outdegree, and vice-versa. Thus D has a directed cycle

$$\gamma : u_{i_1} \to u_{i_2} \to \cdots \to u_{i_k} \to u_{i_1}$$

of length $k \geq 2$.

First suppose that the length k of γ is even. Then we obtain a new $(+1)$-completion B' of A to an ASM by replacing with 0s, the $+1$s in positions $H_{i_1} \cap V_{i_2}, H_{i_3} \cap V_{i_4}, \ldots, H_{i_{k-1}} \cap V_{i_k}$, and by replacing with $+1$s, the 0s in positions $H_{i_3} \cap V_{i_2}, H_{i_5} \cap V_{i_4}, \ldots, H_{i_{k-1}} \cap V_{i_{k-2}}, H_{i_1} \cap V_{i_k}$. Moreover, $q(B') < q(B)$.

Now suppose that k is odd. Then we claim that there is a vertex u_r of the cycle γ such that $H_r \cap V_r \neq \emptyset$, and thus $H_r \cap V_r = \{(s,s)\}$ for some s. If not, then for each i, H_i consists of positions strictly above the main diagonal or else positions strictly below the main diagonal. A similar conclusion holds for each V_i. This implies that γ has even length, a contradiction. Thus we may assume that $H_{i_1} \cap V_{i_1} = \{(s,s)\}$ and thus that the entry in B in position (s,s) is 0. Then we obtain a new $(+1)$-completion B' of A to an ASM by replacing the 0s in positions $H_{i_1} \cap V_{i_1}, H_{i_3} \cap V_{i_2}, H_{i_5} \cap V_{i_4}, \ldots, H_{i_k} \cap V_{i_{k-1}}$ with $+1$s and replacing the $+1$s in positions $H_{i_1} \cap V_{i_2}, H_{i_3} \cap H_{i_4}, \ldots, H_{i_{k-2}} \cap V_{i_{k-1}}, H_{i_k} \cap V_{i_1}$ with 0s. Again we have that $q(B') < q(B)$. By repeating this argument, after a finite number of steps, we obtain a symmetric $(+1)$-completion of A to an ASM. □

Another way to formulate Theorem 7 is: Let A be an $n \times n$ ASM whose -1s are in a symmetric pattern. Then there is an $n \times n$ symmetric ASM B with -1s exactly where A has -1s.

We now give two examples illustrating the argument in the proof of Theorem 7 in both the even cycle and odd cycle cases.

Example 8 Let A be the 7×7 symmetric $(0,-1)$-matrix and let B be the 7×7 (non-symmetric) $(+1)$-completion of A to an ASM as shown:

$$A = \begin{bmatrix} & & & & & & \\ & -1 & & & & & \\ & & & & & & \\ & & & -1 & & & \\ & & & & & & \\ & & & & & -1 & \\ & & & & & & \end{bmatrix} \quad \text{and } B = \begin{bmatrix} & +1 & & & & & \\ +1 & -1 & & +1 & & 0 & \\ & & & 0 & +1 & & \\ & 0 & +1 & -1 & 0 & +1 & \\ & & 0 & +1 & & & \\ & +1 & & 0 & & -1 & +1 \\ & & & & & +1 & \end{bmatrix}$$

The positions which are not symmetrically occupied are shaded.

Label the sets in $\mathcal{H}(A)$ in the order of the rows and from left to right, and label the sets in $\mathcal{V}(A)$ in order of the columns and from top to bottom. Then the 10×10

biadjacency matrix C of the bipartite graph $G(A)$ is

$C =$

1	1		1	1		1	1		1
1									
		0	1	1		1	1		1
1		1	1	1		1	1		1
1		1	1	0					
					0	1	1		1
1		1	1		1	1	1		1
1		1	1		1	1	0		
									1
1		1	1		1	1		1	1

.

The shaded positions, where we have also shaded the corresponding diagonal elements, determine a directed cycle of even length 6 given by

$$u_3 \to u_5 \to u_4 \to u_7 \to u_6 \to u_8 \to u_3.$$

This directed cycle then determines the symmetric (+1)-completion of A to an ASM given by

$B' =$

	+1					
+1	−1		+1		0	
			0	+1		
	+1	0	−1	0	+1	
		+1	0			
	0		+1		−1	+1
					+1	

.

Example 9 Let A be the 9×9 symmetric $(0, -1)$-matrix

$A =$

				−1		−1		
			−1					
		−1						
							−1	
		−1				−1		
					−1			

.

Then A has a non-symmetric ASM (+1)-completion

$$B = \begin{bmatrix}
 & & & & 0 & & +1 & & \\
 & & & & +1 & & 0 & & \\
 & & & +1 & -1 & +1 & -1 & +1 & \\
 & & +1 & -1 & \mathbf{0} & & \mathbf{+1} & & \\
0 & +1 & -1 & \mathbf{+1} & & & \mathbf{0} & & \\
 & & +1 & & & & & -1 & +1 \\
+1 & 0 & -1 & \mathbf{0} & \mathbf{+1} & & \mathbf{-1} & +1 & \\
 & & +1 & & & -1 & +1 & & \\
 & & & & & +1 & & &
\end{bmatrix},$$

where the non-symmetric +1s and their symmetrically opposite 0s have been shaded. Using the same labeling procedure as in Example 8, the digraph D for this example has the directed cycle of length 3

$$u_7 \to u_{13} \to u_9 \to u_7$$

which, by using the fact that the entry in $H_9 \cap V_9 = \{(5,5)\}$ is a 0, then gives the symmetric ASM (+1)-completion of A

$$B' = \begin{bmatrix}
 & & & & 0 & & +1 & & \\
 & & & & +1 & & 0 & & \\
 & & & +1 & -1 & +1 & -1 & +1 & \\
 & & +1 & -1 & \mathbf{0} & & \mathbf{+1} & & \\
0 & +1 & -1 & \mathbf{0} & \mathbf{+1} & & \mathbf{0} & & \\
 & & +1 & & & & & -1 & +1 \\
+1 & 0 & -1 & \mathbf{+1} & \mathbf{0} & & \mathbf{-1} & +1 & \\
 & & +1 & & & -1 & +1 & & \\
 & & & & & +1 & & &
\end{bmatrix}.$$

We now consider an $n \times n$ $(0,+1)$-matrix A with at least one +1 in each row and column. In this case we consider the *horizontal partition* $\mathcal{H}^+(A) = (H_i^+ : 1 \leq i \leq p)$ where the H_i^+, taken in some order, consist of those positions between two neighboring +1s in a row and, similarly, the *vertical partition* $\mathcal{V}^+(A) = (V_i^+ : 1 \leq i \leq p)$ where the V_i^+, taken in some order, consist of those positions between two neighboring +1s in a column. As indicated, for the following reason, the number of sets p in each of the two partitions is the same: Let the row sum vector of A be $(r_1, r_2, \ldots, r_n)$ and let the column sum vector be $S = (s_1, s_2, \ldots, s_n)$. Then the number of sets in the horizontal partition is

$$\sum_{i=1}^{n}(r_i - 1) = \left(\sum_{i=1}^{n} r_i\right) - n = \left(\sum_{i=1}^{n} s_i\right) - n = \sum_{i=1}^{n}(s_i - 1),$$

the same as the number of sets in the vertical partition. Note that if a row (respectively, column) of A contains only one +1, then none of the positions in that row (respectively, column) are in a set of the horizontal partition (respectively, vertical partition). Let $C = [c_{ij}]$ be the $p \times p$ $(0,1)$-matrix where $c_{ij} = 1$ if and only if $H_i^+ \cap V_j^+ \neq \emptyset$ $(1 \leq i, j \leq p)$. The matrix C is the biadjacency matrix of a bipartite graph $BG(C)$ with vertices bipartitioned as $\{H_i^+ : 1 \leq i \leq p\}$ and $\{V_i^+ : 1 \leq i \leq p\}$ with an edge joining H_i^+ and V_j^+ if and only if $H_i^+ \cap V_j^+ \neq \emptyset$ (and so consists of a single posiition). There will be a (-1)-completion of A to an ASM if and only if $BG(C)$ has a perfect matching, equivalently, if and only if there is a permutation matrix $P \leq C$ (entrywise).

If A is symmetric, then the matrix C is a symmetric $(0,1)$-matrix, possibly with 1s on the main diagonal, and so is the adjacency matrix of a a loopy graph $G(C)$. There is a (-1)-completion of A to a symmetric ASM if and only if $G(C)$ has a perfect matching (that is, a pairwise disjoint collection of edges and loops meeting all the vertices), that is, a symmetric permutation matrix $P \leq C$ (entrywise).

Theorem 10 *Let A be an $n \times n$ symmetric $(0,+1)$-matrix that has a (-1)-completion to an ASM. Then A has a symmetric (-1)-completion to an ASM.*

Proof. The technique of the proof is identical to the technique used in the proof of Theorem 7 and so is omitted. □

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Let A be an $n \times n$ ASM. Then the row sum vector and the column sum vector of A both equal the n-vector $(1,1,\ldots,1)$ of all 1s. Let $\text{patt}(A)$ be the $(0,1)$-matrix obtained from A by replacing each entry with its absolute value. Then $\text{patt}(A)$ is the (*combinatorial*) *pattern* of A. Because of the alternating sign property of ASMs, the pattern of an ASM uniquely determines the ASM. The pattern $\text{patt}(A)$ of A has a row sum vector $R = (r_1, r_2, \ldots, r_n)$ and a column sum vector $S = (s_1, s_2, \ldots, s_n)$ and it is easy to verify [2] that

$$R, S \leq (1,3,5,7,\ldots,7,5,3,1) \quad \text{(entrywise)}. \tag{2}$$

Let $ASM(R,S)$ denote the set of all ASMs whose pattern has row sum vector R and column sum vector S. In a symmetric ASM the row sum vector of its pattern equals its column sum vector. It is an open question to characterize R and S for which $ASM(R,S) \neq \emptyset$; the above conditions (2) are necessary but far from sufficient in general [2]. Let $ASM_{\text{sym}}(R)$ denote the set of all symmetric ASMs whose patterns have row sum vector, and hence column sum vector R. If an ASM A has a symmetric pattern, then A is necessarily a symmetric ASM.

The following question is motivated by a theorem of Fulkerson, Hoffman, and McAndrew (see [4] where their theorem is extended to include 1s on the main diagonal) who proved that if there is a $(0,1)$-matrix with row and column sum vector R, then there is a symmetric $(0, 1)$-matrix with row sum vector, and hence column sum vector, equal to R. We have been unable to answer the following ASM analogue of this theorem.

Question: Let A be an $n \times n$ ASM whose pattern has row and column sum vector equal to R. Is there a symmetric ASM whose pattern has row and column sum vector equal to R?

Let A^+ be the $(0, 1)$-matrix obtained from A by replacing the -1s with 0s. Then A^+ has row and column sum vector R^+ for some R^+. Let A^- be the $(0, -1)$-matrix obtained from A by replacing the $+1$s with 0s. Then A^- has row and column sum vector R^- for some R^-. By the above theorem, there exists a symmetric $(0, 1)$-matrix B with row and column sum vector R^+, and there exists a symmetric $(0, -1)$-matrix C with row and column sum vector R^-. We have $R^+ + R^- = (1, 1, \ldots, 1)$, but B and C need not have disjoint patterns. However, even if they did, $B + C$ need not be an alternating sign matrix.

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