

The Erdős-Ko-Rado property for some permutation groups

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Abstract

A subset in a group $G \leq \text{Sym}(n)$ is *intersecting* if for any pair of permutations π, σ in the subset there is an $i \in \{1, 2, \dots, n\}$ such that $\pi(i) = \sigma(i)$. If the stabilizer of a point is the largest intersecting set in a group, we say that the group has the *Erdős-Ko-Rado (EKR) property*. Moreover, the group has the *strict EKR property* if every intersecting set of maximum size in the group is the coset of the stabilizer of a point. In this paper we look at several families of permutation groups and determine if the groups have either the EKR property or the strict EKR property. First, we prove that all cyclic groups have the strict EKR property. Next we show that all dihedral and Frobenius groups have the EKR property and we characterize which ones have the strict EKR property. Further, we show that if all the groups in an external direct sum or an internal direct sum have the EKR (or strict EKR) property, then the product does as well. Finally, we show that the wreath product of two groups with EKR property also has the EKR property.

1 Introduction

The Erdős-Ko-Rado (EKR) theorem [10] describes the largest collection of subsets of size k from the set $\{1, 2, \dots, n\}$ such that any two of the subsets contain a common element. If $n \geq 2k$ then the largest collection has size $\binom{n-1}{k-1}$, moreover, if $n > 2k$, the only collections of this size are the collections of all subsets that contain a fixed element from $\{1, 2, \dots, n\}$. This result can be generalized to many objects other than sets, such as integer sequences [12], vector spaces over a finite field [18], matchings and partitions [23]. In this paper we will consider versions of the EKR theorem for permutation groups.

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Let $\text{Sym}(n)$ denote the symmetric group and $G \leq \text{Sym}(n)$ be a permutation group with the natural action on the set $\{1, 2, \dots, n\}$. Two permutations $\pi, \sigma \in G$ are said to *intersect* if $\pi\sigma^{-1}$ has a fixed point in $\{1, 2, \dots, n\}$. If π and σ intersect, then $\pi(i) = \sigma(i)$ for some $i \in [n] = \{1, 2, \dots, n\}$ and we say that π and σ *agree* at i . Further, if σ and π do not intersect, then $\pi\sigma^{-1}$ is a derangement. A subset $S \subseteq G$ is called *intersecting* if any pair of its elements intersect. Clearly, the stabilizer of a point is an intersecting set in G , as is any coset of the stabilizer of a point.

We say the group G has the *EKR property* if the size of any intersecting subset of G is bounded above by the size of the largest point-stabilizer in G . Further, G is said to have the *strict EKR property* if the only intersecting subsets of maximum size in G are the cosets of the point-stabilizers. It is clear from the definition that if a group has the strict EKR property then it will have the EKR property.

In 1977 Frankl and Deza [11] proved that $\text{Sym}(n)$ has the EKR property and conjectured that it had the strict EKR property. This conjecture caught the attention of several researchers, indeed, it was proved using vastly different methods in each of [7, 15, 22] and [28]. We state this result using our terminology.

Theorem 1.1. *For all integers $n \geq 2$, the group $\text{Sym}(n)$ has the strict EKR property.* \square

Researchers have also worked on finding other subgroups of $\text{Sym}(n)$ that have the strict EKR property. For example in [21] it is shown that $\text{Alt}(n)$ has the strict EKR property, and it was also determined exactly which Young subgroups have the strict EKR property. In [28] it is shown that some Coxeter groups have the EKR property. Further, the projective general linear groups $\text{PGL}(2, q)$ have the strict EKR property [24], while the groups $\text{PGL}(3, q)$ do not [25].

We point out that the group action is essential for the concepts of the EKR and the strict EKR properties. A group can have the (strict) EKR property under one action while it fails to have this property under another action (in Section 7 we give an example of such a group). This is the reason that in this paper, we always consider the “permutation groups” (i.e. the subgroups of $\text{Sym}(n)$ with their natural action on $\{1, 2, \dots, n\}$) rather than general groups. This is in contrast to [6] where they define a group to have the EKR property if the group has the EKR property for every group action.

In this paper we prove that any group with a sharply transitive set has the EKR property and that all cyclic groups have the strict EKR property. We also show that all dihedral groups have the strict EKR property, all Frobenius groups have the EKR property and we characterize which Frobenius groups have the strict EKR property.

Next we consider three group products. Assume that G and H are groups and G acts on Ω and H on Δ . The group elements (g, h) , where $g \in G$ and $h \in H$, act on $\Omega \times \Delta$ by $(\omega, \delta)^{(g, h)} = (\omega^g, \delta^h)$. This is the external direct product of G and H . These same elements also act on the set $\Omega \cup \Delta$. The value of $x^{(g, h)}$ is x^g , if $x \in \Omega$, and x^h if $x \in \Delta$. This is the internal direct product. We show that if all the groups in either an external or an internal direct product have the EKR (or strict EKR) property, then the product does as well. Further, we show that if the external product of a set of groups has the EKR (or strict EKR) property then each of the group in the

product does as well. The wreath product of two groups with EKR property also has the EKR property. We conclude with a section about some groups that do not have the EKR property.

2 The EKR property for cyclic groups

A common technique in proving EKR theorems is to define a graph that has the property that the independent sets in the graph are exactly the intersecting sets. This transforms the problem to determining the size and the structure of the independent sets in this graph. For the EKR theorem on sets, the graph used is the Kneser graph (see [16, Section 7.8] for more details). For a group G we define the *derangement graph*; this graph has the elements of G as its vertices and two vertices are adjacent if and only if they do not intersect. We will denote the derangement graph of a group by Γ_G . In particular, $\sigma, \pi \in G$ are adjacent in Γ_G if and only if $\sigma\pi^{-1}$ is a derangement. The derangement graph is the Cayley graph on G where the set of all derangements of G is the connection set for the graph. Since the set of derangements is a union of conjugacy classes, Γ_G is a *normal* Cayley graph. The group G acts on the vertices of Γ_G by left multiplication and this action is transitive, thus Γ_G is vertex transitive.

The group G has the EKR property if and only if the size of a maximum independent set in Γ_G is equal to the size of the largest stabilizer of a point in G . The group G has the strict EKR property if and only if the cosets of the largest stabilizer of a point are all the only independent sets of maximum size.

Following the standard notation, we will denote the size of the largest clique in a graph X by $\omega(X)$, and $\alpha(X)$ will represent the size of the largest independent set in X . The next result is known as the *clique-coclique bound* and we refer the reader to [7] for a proof.

Lemma 2.1. *If X is a vertex-transitive graph, then $\omega(X)\alpha(X) \leq |V(X)|$. Moreover, if equality holds, then every maximum independent set and every maximum clique intersect.* \square

For any permutation group $G \leq \text{Sym}(n)$, a clique in Γ_G is a set of permutations in which no two permutations agree on a point. Thus a sharply-transitive subset of G is a clique of size n in Γ_G .

Corollary 2.2. *If a group has a sharply-transitive set, then the group has the EKR property.*

Proof. Assume that a group $G \leq \text{Sym}(n)$ has a sharply-transitive set. Then G is a transitive group and the size of the largest stabilizer of a point is $\frac{|G|}{n}$. The sharply transitive subset of G is a clique of size n in Γ_G . So by the clique-coclique bound $\alpha(\Gamma_G) \leq \frac{|G|}{n}$. \square

A simple case of a sharply-transitive set is the subgroup generated by an n -cycle.

Corollary 2.3. *Any permutation group of degree n that contains an n -cycle has the EKR property.* \square

If G is a sharply-transitive group, then Γ_G is a complete graph; in this case G trivially has the strict EKR theorem. For example, any subgroup of $\text{Sym}(n)$ that is generated by an n -cycle has the strict EKR property. Next we show that every cyclic group has the strict EKR property.

Theorem 2.4. *For any permutation $\sigma \in \text{Sym}(n)$, the cyclic group G generated by σ has the strict EKR property.*

Proof. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$, where σ_i are disjoint cyclic permutations. Assume that σ_i has order r_i , and that $1 \leq r_1 \leq \cdots \leq r_k$. Note that

$$C = \{\sigma^0, \sigma^1, \sigma^2, \dots, \sigma^{r_1-1}\}$$

induces a clique of size r_1 in the graph Γ_G . Then by Lemma 2.1

$$\alpha(\Gamma_G) \leq \frac{|G|}{r_1}.$$

Note, in addition, that if $\sigma_1 = (a_1, \dots, a_{r_1})$, then the stabilizer of a_1 in G is

$$G_{a_1} = \{\sigma^{r_1}, \sigma^{2r_1}, \dots, \sigma^{|G|}\};$$

therefore

$$\alpha(\Gamma_G) = |G_{a_1}| = \frac{|G|}{r_1}.$$

It is clear that this is the largest size of a point-stabilizer in G . This proves that G has the EKR property.

To show G has the strict EKR property, note that the clique C and the independent set G_{a_1} together show that the clique-coclique bound (Lemma 2.1) for Γ_G holds with equality. Hence any maximum independent set must intersect with any maximum clique in Γ_G . Let S be any maximum independent set and without loss of generality assume the identity element, id , is in S . We show that S is the stabilizer of a point.

For any $i \geq 0$, set

$$C_i = \{\sigma^i, \sigma^{i+1}, \dots, \sigma^{i+r_1-1}\}.$$

Note that C_i are cliques in Γ_G of maximum size and that $C_0 = C$. Furthermore, for any $0 \leq t \leq |G|/r_1$, we have $C_{tr_1} \setminus C_{(t+1)r_1} = \{\sigma^{tr_1}\}$ and $C_{(t+1)r_1} \setminus C_{tr_1} = \{\sigma^{(t+1)r_1}\}$. Since for any $0 \leq t \leq |G|/r_1$, the independent set S intersects with each of the cliques $C_{(t+1)r_1}$ and C_{tr_1} in exactly one point, and since $\sigma^0 \in S$, we conclude that $C_1 \cap S = \{\sigma^{r_1}\}$. Continue like this for all $t \in \{0, \dots, |G|/r_1\}$ shows that $\sigma^0, \sigma^{r_1}, \dots, \sigma^{|G|-r_1} \in S$; that is, $S = G_{a_1}$. \square

If G is a cyclic group generated by $\sigma \in \text{Sym}(n)$, and r_1 is the length of the shortest cycle in σ , then the largest intersecting set is actually an r_1 -intersecting set; meaning that any two permutations in the set agree on r_1 elements of $\{1, 2, \dots, n\}$. This is quite different from the case for subsets; in fact, it is not hard to show that any maximal collection of intersecting subsets is not 2-intersecting.

3 The EKR property for dihedral and Frobenius groups

Recall that for any $n \geq 3$, the *dihedral group* of degree n , denoted by $D_n \leq \text{Sym}(n)$, is the group of symmetries of a regular n -gon, including both rotations and reflections.

Proposition 3.1. *All the dihedral groups have the strict EKR property.*

Proof. Assume D_n is generated by the permutations σ , the rotation, and π , the reflection through either antipodal points of the n -gon (if n is even) or through a point and an edge (if n is odd). Then σ is of order n , and π is of order 2. Since $\{\text{id}, \pi\}$ is an intersecting set, we have $\alpha(\Gamma_{D_n}) \geq 2$. To prove the proposition, we show that any maximum independent set in Γ_{D_n} is a coset of a point-stabilizer. Assume S is a maximum independent set in Γ_{D_n} and, without loss of generality, assume $\text{id} \in S$. Clearly, $\sigma^i \notin S$, for any $1 \leq i < n$. If $\sigma^i \pi, \sigma^j \pi \in S$, for some $1 \leq j < i < n$, then their division,

$$\sigma^i \pi (\sigma^j \pi)^{-1} = \sigma^{i-j}$$

must have a fixed-point, which is a contradiction. Therefore, $\alpha(\Gamma_{D_n}) = 2$ and $S = \{e, \sigma^i \pi\}$, for some $1 \leq i < n$. Note, finally, that since no pair $\sigma^i \pi, \sigma^j \pi$ has any common fixed-point, S is indeed the stabilizer of any of the points fixed by $\sigma^i \pi$. \square

Note that the dihedral group D_n can be written as $D_n = \mathbb{Z}_n \mathbb{Z}_2$, where $\mathbb{Z}_n \triangleleft D_n$ corresponds to the subgroup generated by the rotations and \mathbb{Z}_2 is the subgroup generated by the two reflections. More precisely, $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ when the non-identity element of \mathbb{Z}_2 acts on \mathbb{Z}_n by inversion. When n is odd, this is a particular case of a Frobenius group. A transitive permutation group $G \leq \text{Sym}(n)$ is called a *Frobenius group* if no non-trivial element fixes more than one point and some non-trivial element fixes a point. An alternative definition is as follows: a group $G \leq \text{Sym}(n)$ is a Frobenius group if it has a non-trivial proper subgroup H with the condition that $H \cap H^g = \{\text{id}\}$, for all $g \in G \setminus H$, where $H^g = g^{-1} H g$. This subgroup is called a *Frobenius complement*. Define the *Frobenius kernel* K of G to be

$$K = \left(G \setminus \bigcup_{g \in G} H^g \right) \cup \{\text{id}\}.$$

In fact, the non-identity elements of K are all the derangement elements of G . Note that if $G = KH \leq \text{Sym}(n)$ is a Frobenius group with kernel K , then $|K| = n$ and $|H|$ must divide $n - 1$; see [9, Section 3.4] for proofs. This implies that the Frobenius groups are relatively small transitive subgroups of $\text{Sym}(n)$. We also observe the following.

Lemma 3.2. *If $G = KH \leq \text{Sym}(n)$ is a Frobenius group with kernel K , then $|G_x| = |H|$, for any $x \in \{1, 2, \dots, n\}$.*

Proof. Let $x \in \{1, 2, \dots, n\}$. By the orbit-stabilizer theorem we have

$$|x^G| = [G : G_x] = \frac{|G|}{|G_x|} = \frac{|K||H|}{|G_x|} = \frac{n|H|}{|G_x|},$$

where x^G is the orbit of x under the action of G on $\{1, 2, \dots, n\}$. Since this action is transitive, $|x^G| = n$. \square

In order to find the maximum intersecting subsets of a Frobenius group, we first describe their derangement graphs. The eigenvalues of normal Cayley graphs can be calculated using the irreducible representations of the group. This follows from Babai’s formula for the eigenvalues of a Cayley graph [5] (we also refer the reader to either the proof by Diaconis and Shahshahani [8] or to [2, Chapter 4] for details). We will simply state the formula for the derangement graphs.

Theorem 3.3. *Let G be a group. The eigenvalues of the graph Γ_G are given by*

$$\eta_\chi = \frac{1}{\chi(\text{id})} \sum_{x \in \mathcal{D}_G} \chi(x),$$

where χ ranges over all irreducible characters of G . Moreover, the multiplicity of the eigenvalue λ is $\sum_\chi \chi(\text{id})^2$, where the sum is taken over all irreducible representations χ with $\eta_\chi = \lambda$.

We will make use of the following classical result (see [19, Theorem 18.7]).

Theorem 3.4. *Let $G = KH$ be a Frobenius group with kernel K . Then the irreducible representations of G are of the following two types:*

- (a) *Any irreducible representation Ψ of H gives an irreducible representation of G using the quotient map $H \cong G/K$. These give the irreducible representation of G with K in their kernel.*
- (b) *If Ξ is any non-trivial irreducible representation of K , then the corresponding induced representation of G is also irreducible. These give the irreducible representations of G with K not in their kernel. \square*

The following fact will be used in the proof of the next theorem; the reader may refer to [26, Section 1.10] for a proof.

Proposition 3.5. *Let $\{\Psi_i\}$ be the set of all irreducible representations of a group G . Then*

$$\sum_i (\dim \Psi_i)^2 = |G|. \quad \square$$

Now we can describe the derangement graph of the Frobenius groups.

Theorem 3.6. *Let $G = KH \leq \text{Sym}(n)$ be a Frobenius group with the kernel K . Then Γ_G is the disjoint union of $|H|$ copies of the complete graph on n vertices.*

Proof. According to Theorem 3.3, the eigenvalues of Γ_G are given by

$$\eta_\chi = \frac{1}{\chi(\text{id})} \sum_{\sigma \in \mathcal{D}_G} \chi(\sigma),$$

where χ runs through the set of all irreducible characters of G . First assume χ is the character of an irreducible representation of G of type (a) in Theorem 3.4. Then we have

$$\eta_\chi = \frac{1}{\chi(\text{id})} \sum_{\sigma \in \mathcal{D}_G} \chi(\sigma) = \frac{1}{\chi(\text{id})} \sum_{\sigma \in \mathcal{D}_G} \chi(\text{id}) = |\mathcal{D}_G| = |K| - 1 = n - 1.$$

According to Theorem 3.3 and Proposition 3.5, the multiplicity of $\eta_\chi = n - 1$ is

$$\sum_{\Psi \in \text{Irr}(H)} (\dim \Psi)^2 = |H|.$$

Furthermore, assume Ξ is an irreducible representation of G of type (b) in Theorem 3.4, whose character is ξ and let χ be the character of the corresponding induced representation of G . If $\sigma \in G \setminus K$, then $\sigma \in H^g$, for some $g \in G$. Thus $x^{-1}\sigma x \in H^{gx}$, for any $x \in G$; hence $x^{-1}\sigma x \notin K$. By the definition of an induced representation (see [13, Section 3.3]), this implies that $\chi(\sigma) = 0$. On the other hand, let χ_1 be the trivial character of G . Since $\chi \neq \chi_1$, the inner product of χ and χ_1 is zero (see [13, Section 2.2]). Since

$$\langle \chi, \chi_1 \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\chi_1(\sigma^{-1}) = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) = \frac{1}{|G|} \left(\chi(\text{id}) + \sum_{\sigma \in \mathcal{D}_G} \chi(\sigma) \right),$$

and the characters χ and χ_1 are orthogonal, we have that

$$\sum_{\sigma \in \mathcal{D}_G} \chi(\sigma) = -\chi(\text{id}).$$

This yields

$$\eta_\chi = \frac{1}{\chi(\text{id})} \sum_{\sigma \in \mathcal{D}_G} \chi(\sigma) = \frac{-\chi(\text{id})}{\chi(\text{id})} = -1.$$

We have, therefore, shown that

$$\text{Spec}(\Gamma_G) = \begin{pmatrix} n - 1 & -1 \\ |H| & |H|(n - 1) \end{pmatrix}.$$

Now the theorem follows from the fact that any graph with these eigenvalues is the union of $|H|$ complete graphs. □

We point out that this proof is not the easiest method to describe the graph Γ_G . In fact, this result can be seen by considering the derangement graph on kernel of the Frobenius group and the cosets of the kernel. It is however a nice example of an application of character theory to graph theory.

Theorem 3.7. *Let $G = KH$ be a Frobenius group with kernel K . Then G has the EKR property. Furthermore, G has the strict EKR property if and only if $|H| = 2$.*

Proof. Using Theorem 3.6, the independence number of Γ_G is $|H|$. This along with Lemma 3.2 shows that G has the EKR property. For the second part of the theorem, first note that if $|H| = 2$ and S is an intersecting subset of G of size two, then S is, trivially, a point stabilizer. To show the converse, we note that the cliques of Γ_G are induced by the $|H|$ cosets of K in G . Now suppose $|H| > 2$ and let S be a maximum intersecting subset of G which is a coset of a point-stabilizer in G . Without loss of generality we may assume $S = \{\text{id} = s_1, s_2, \dots, s_{|H|}\}$ and, hence, $S = G_x$, for some $x \in \{1, 2, \dots, n\}$. Since S is an independent set in Γ_G , no two elements of S are in the same coset of K in G . Note that the only fixed point of any non-identity element of S is x . Let s_3 be in the coset gK . If all the elements of gK fix x , then all the elements of K will have fixed points, which is a contradiction. Hence there is an $s'_3 \in gK$ which does not fix x . Now the maximum intersecting set $S' = (S \setminus \{s_3\}) \cup \{s'_3\}$ is not a point-stabilizer. \square

Note that one can show the second part of Theorem 3.7 by a simple counting argument as follows. There are n^2 cosets of point-stabilizers in G . Since Γ_G is the union of $|H|$ copies of the complete graph on n vertices, the total number of maximum independent sets is $n^{|H|}$. Therefore, in order for all the maximum independent sets of Γ_G to be cosets of point-stabilizers, the necessary and sufficient condition is $|H| = 2$.

Theorem 3.7 provides an alternative proof for the fact that the dihedral group D_n has the strict EKR property, when $n \geq 3$ is odd.

4 The EKR property for external direct products

Given a sequence of permutation groups $G_1 \leq \text{Sym}(n_1), \dots, G_k \leq \text{Sym}(n_k)$, their *external direct product* is defined to be the group $G_1 \times \dots \times G_k$, whose elements are (g_1, \dots, g_k) , where $g_i \in G_i$, for $1 \leq i \leq k$, and the binary operation is, simply, the component-wise multiplication. This group has a natural action on the set $\Omega = [n_1] \times \dots \times [n_k]$ induced by the natural actions of G_i on $[n_i]$; that is, for any tuple $(x_1, \dots, x_k) \in \Omega$ and any element $(g_1, \dots, g_k) \in G_1 \times \dots \times G_k$, we have

$$(x_1, \dots, x_k)^{(g_1, \dots, g_k)} := (x_1^{g_1}, \dots, x_k^{g_k}).$$

Let $G = G_1 \times \dots \times G_k$. Then the derangement graph Γ_G of G , is the graph with vertex set G in which two vertices (g_1, \dots, g_k) and (h_1, \dots, h_k) are adjacent if and only if $g_i h_i^{-1}$ is a derangement, for some $1 \leq i \leq k$.

If a group that is an external direct product, then it is possible to express the derangement graph for the group as a product of derangement graphs. To do this, we need two graph operations. The first is the *complement*. The complement of a graph X is the graph \overline{X} on the same vertex set as X , and the edge set is exactly all the pairs of vertices which are not adjacent in X . The second operation is the *direct product* of two graphs. Let X and Y be two graphs. The direct product of X and Y is the graph $X \times Y$ whose vertex set is

$$V(X \times Y) = V(X) \times V(Y),$$

and in which two vertices (x_1, y_1) and (x_2, y_2) are adjacent if $x_1 \sim x_2$ in X and $y_1 \sim y_2$ in Y . We observe the following.

Lemma 4.1. *Let the group $G = G_1 \times \cdots \times G_k$ be the external direct product of the groups G_1, \dots, G_k , then*

$$\Gamma_G = \overline{\overline{\Gamma_{G_1}} \times \cdots \times \overline{\overline{\Gamma_{G_k}}}}$$

Proof. By the definition of the external direct product, the vertices (g_1, \dots, g_k) and (h_1, \dots, h_k) of $\overline{\overline{\Gamma_G}}$ are adjacent if and only if $g_i h_i^{-1}$ has a fixed point, for each $1 \leq i \leq k$. This is equivalent to the case where g_i is adjacent to h_i in $\overline{\overline{\Gamma_{G_i}}}$, for each $1 \leq i \leq k$. This occurs if and only if (g_1, \dots, g_k) and (h_1, \dots, h_k) are adjacent in $\overline{\overline{\Gamma_{G_1}}} \times \cdots \times \overline{\overline{\Gamma_{G_k}}}$. This completes the proof. \square

In the next lemma, we evaluate the independence number of Γ_G . If $G = G_1 \times \cdots \times G_k$, define p_i to be the projection of G onto the component G_i .

Lemma 4.2. *Let the group $G = G_1 \times \cdots \times G_k$ be the external direct product of the groups G_1, \dots, G_k , then*

$$\alpha(\Gamma_G) = \alpha(\Gamma_{G_1}) \times \cdots \times \alpha(\Gamma_{G_k}).$$

Proof. Let S_i be a maximum independent set in Γ_{G_i} , for any $1 \leq i \leq k$. Then the set $S = S_1 \times \cdots \times S_k$ is an independent set in Γ_G , proving that $\alpha(\Gamma_G) \geq \alpha(\Gamma_{G_1}) \times \cdots \times \alpha(\Gamma_{G_k})$. On the other hand, let S be a maximum independent set in Γ_G , then for any $1 \leq i \leq k$, the set $p_i(S)$ is an independent set in G_i . Thus $|p_i(S)| \leq \alpha(\Gamma_{G_i})$. Since $S \subseteq p_1(S) \times \cdots \times p_k(S)$ the lemma follows. \square

Theorem 4.3. *With the notation above, all the G_i have the (strict) EKR property if and only if G has the (strict) EKR property.*

Proof. Note that the stabilizer of any point $(x_1, \dots, x_k) \in \Omega$ in G is

$$(G_1)_{x_1} \times \cdots \times (G_k)_{x_k}.$$

We first prove the “only if” part of the theorem. If all the groups G_i have the EKR property, according to Lemma 4.2, the maximum size of an independent set in Γ_G will be equal to

$$|(G_1)_{x_1}| \times \cdots \times |(G_k)_{x_k}|,$$

for some $(x_1, \dots, x_k) \in \Omega$; this proves that G has the EKR property. Furthermore, assume all the groups G_i have strict EKR property and let S be a maximum independent set in Γ_G that contains the identity. This implies that for each $1 \leq i \leq k$, the set $p_i(S)$ is a maximum independent set in Γ_{G_i} that contains the identity; hence $p_i(S) = (G_i)_{x_i}$, for some $x_i \in [n_i]$. Therefore, $S = G_{(x_1, \dots, x_k)}$, which shows that G has the strict EKR property.

To prove the “if” part, first assume G has the EKR property. Then the size of any independent set is bounded above by $|G_{(x_1, \dots, x_k)}|$, for some $(x_1, \dots, x_k) \in$

$[n_1] \times \cdots \times [n_k]$. Let S_1 be an independent set in G_1 . Then $S_1 \times (G_2)_{x_2} \times \cdots \times (G_k)_{x_k}$ is an independent set in G . Hence

$$|S_1 \times (G_2)_{x_2} \times \cdots \times (G_k)_{x_k}| \leq |G_{(x_1, \dots, x_k)}| = |(G_1)_{x_1}| \times \cdots \times |(G_k)_{x_k}|,$$

thus $|S_1| \leq |(G_1)_{x_1}|$. This shows that G_1 and, similarly, G_2, \dots, G_k have the EKR property. Finally, assume G has the strict EKR property and let S_1 be a maximum independent set in G_1 which contains the identity. This implies that $S_1 \times (G_2)_{x_2} \times \cdots \times (G_k)_{x_k}$ is a maximum independent set in G ; hence it must stabilize a point, say $(y_1, \dots, y_k) \in [n_1] \times \cdots \times [n_k]$. Therefore, S_1 stabilizes y_1 ; that is, $S_1 \subseteq (G_1)_{y_1}$. Since S_1 has the maximum size, we conclude that $S_1 = (G_1)_{y_1}$, which completes that proof. \square

We conclude this section by noting this theorem shows how to construct infinite families of graphs that do not have (strict) EKR property.

5 The EKR property for internal direct products

The next group product that we consider is the internal direct product. Assume $\Omega_1, \dots, \Omega_k$ are pair-wise disjoint non-empty subsets of $\{1, 2, \dots, n\}$, and let $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$. Consider the sequence $G_1 \leq \text{Sym}(\Omega_1), \dots, G_k \leq \text{Sym}(\Omega_k)$. Then their *internal direct product* is defined to be the group $G_1 \cdot G_2 \cdots G_k$, whose elements are $g_1 g_2 \cdots g_k$, where $g_i \in G_i$, for $1 \leq i \leq k$ and the binary operation is defined as follows: for the elements $g_1 g_2 \cdots g_k$ and $h_1 h_2 \cdots h_k$ in $G_1 \leq \text{Sym}(\Omega_1), \dots, G_k \leq \text{Sym}(\Omega_k)$,

$$g_1 g_2 \cdots g_k \cdot h_1 h_2 \cdots h_k := (g_1 h_1)(g_2 h_2) \cdots (g_k h_k). \tag{1}$$

Note that since the Ω_i do not intersect, any permutation in G_i commutes with any permutation in G_j , for any $1 \leq i \neq j \leq k$; hence the multiplication in Equation (1) is well-defined. This group also has a natural action on the set Ω induced by the natural actions of G_i on Ω_i ; that is, for any $x \in \Omega$ and any element $g_1 g_2 \cdots g_k \in G_1 \cdot G_2 \cdots G_k$, we have

$$x^{g_1 g_2 \cdots g_k} := x^{g_i}, \quad \text{where } x \in \Omega_i.$$

Let $G = G_1 \cdot G_2 \cdots G_k$. Then the derangement graph of G is the graph Γ_G with vertex set G in which two vertices $g_1 g_2 \cdots g_k$ and $h_1 h_2 \cdots h_k$ are adjacent if and only if $g_i h_i^{-1}$ is a derangement, for all $1 \leq i \leq k$. In other words, Γ_G is the direct product of $\Gamma_{G_1}, \dots, \Gamma_{G_k}$; that is

$$\Gamma_G = \Gamma_{G_1} \times \cdots \times \Gamma_{G_k}. \tag{2}$$

It is not difficult to see that

$$\alpha(X \times Y) \geq \max\{\alpha(X)|Y|, \alpha(Y)|X|\}.$$

This inequality can be strict for general graphs (see [20]), but recently Zhang [29] proved that equality holds if both graphs are vertex transitive.

Theorem 5.1. *If X and Y are vertex-transitive graphs, then*

$$\alpha(X \times Y) = \max\{\alpha(X)|Y|, \alpha(Y)|X|\}. \quad \square$$

The following can, then, be easily derived.

Corollary 5.2. *If X_1, \dots, X_k are vertex-transitive graphs, then*

$$\alpha(X_1 \times \dots \times X_k) = \max_i \{ \alpha(X_i) \prod_{\substack{j=1, \dots, n \\ j \neq i}} |V(X_j)| \}. \quad \square$$

Theorem 5.3. *With the notation above, if all the groups G_i have the EKR property, then G also has the EKR property.*

Proof. For any $x \in \Omega$, the stabilizer of x in G is $G_1 \cdots \cdots G_{j-1} \cdot (G_j)_x \cdot G_{j+1} \cdots \cdots G_k$, where $x \in \Omega_j$. Hence

$$|G_x| = |(G_j)_x| \prod_{\substack{i=1, \dots, n \\ i \neq j}} |G_i|.$$

According to Corollary 5.2

$$\alpha(\Gamma_G) = \max_j \{ \alpha(\Gamma_{G_j}) \prod_{\substack{i=1, \dots, n \\ i \neq j}} |G_i| \}.$$

Therefore, if all the groups G_i have the EKR property, then G also has the EKR property. □

We point out that the converse of Theorem 5.3 does not hold, examples of this are given in Section 7. Further, the “strict” version of this result does not hold. In fact, Theorem 5.6 gives examples of groups, which are the internal direct product of groups that all have the strict EKR property, but do not satisfy the strict EKR property.

Using Theorem 2.4 and Theorem 5.3, one can observe the following.

Corollary 5.4. *For any sequence r_1, \dots, r_k of positive integers, the internal direct product $\mathbb{Z}_{r_1} \cdot \mathbb{Z}_{r_2} \cdots \cdots \mathbb{Z}_{r_k}$ has the EKR property.* □

Let $\lambda = [\lambda_1, \dots, \lambda_k]$ be a partition of n ; that is, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$ and $\sum_i \lambda_i = n$. Define a set partition of $\{1, 2, \dots, n\}$ by $\{1, 2, \dots, n\} = \Omega_1 \cup \dots \cup \Omega_k$, where $\Omega_i = \{\lambda_1 + \dots + \lambda_{i-1} + 1, \dots, \lambda_1 + \dots + \lambda_i\}$. Then the internal direct product $\text{Sym}(\Omega_1) \cdot \text{Sym}(\Omega_2) \cdots \cdots \text{Sym}(\Omega_k)$ is called the *Young subgroup* of $\text{Sym}(n)$ corresponding to λ and is denoted by $\text{Sym}(\lambda)$. An easy consequence of Theorem 5.3 and Theorem 1.1 is the following.

Corollary 5.5. *Any Young subgroup has the EKR property.* □

It is not difficult to see that $\Gamma_{\text{Sym}(n)}$ is connected if and only if $n \neq 3$, the graph $\Gamma_{\text{Sym}(3)}$ is the disjoint union of two complete graphs K_3 and $\Gamma_{\text{Sym}([2,2,2])}$ is disconnected. From this we can deduce that if $\lambda = [3, 2, \dots, 2], [3, 3]$ or $[2, 2, 2]$, then $\Gamma_{\text{Sym}(\lambda)}$ will be disconnected and one can find maximum independent sets which do not correspond to cosets of point-stabilizers. More generally, if λ is any partition of n which “ends” with one of these three cases, then $\text{Sym}(\lambda)$ fails to have the strict EKR property. Ku and Wong [21] prove that these are the only Young subgroups which don’t have the strict EKR property. In other words, they have proved the following.

Theorem 5.6. *Let $\lambda = [\lambda_1, \dots, \lambda_k]$ be a partition of n with all parts larger than one. Then $\text{Sym}(\lambda)$ has the strict EKR property unless one of the following hold*

- (a) $\lambda_j = 3$ and $\lambda_{j+1} = \dots = \lambda_k = 2$, for some $1 \leq j < k$;
- (b) $\lambda_k = \lambda_{k-1} = 3$;
- (c) $\lambda_k = \lambda_{k-1} = \lambda_{k-2} = 2$. □

6 The EKR property for wreath products

Let $G \leq \text{Sym}(m)$ and $H \leq \text{Sym}(n)$. Then the wreath product of G and H , denoted by $G \wr H$ is the group whose set of elements is

$$\underbrace{(G \times \dots \times G)}_{n \text{ times}} \times H,$$

and the binary operation is defined as follows:

$$(g_1, \dots, g_n, h) \cdot (g'_1, \dots, g'_n, h') := (g_1 g'_{h(1)}, \dots, g_n g'_{h(n)}, hh').$$

In particular, note that the identity element of $G \wr H$ is $(\text{id}_G, \dots, \text{id}_G, \text{id}_H)$ and for any $(g_1, \dots, g_n, h) \in G \wr H$,

$$(g_1, \dots, g_n, h)^{-1} = (g_{h^{-1}(1)}^{-1}, \dots, g_{h^{-1}(n)}^{-1}, h^{-1}).$$

Note also that the size of $G \wr H$ is $|G|^n |H|$. We point out that $G \wr H$ is in fact the “semi-direct product”

$$\underbrace{(G \times \dots \times G)}_{n \text{ times}} \rtimes H,$$

and the action of H on $G \times \dots \times G$ is simply permuting the positions of copies of G (see [9, Section 2.5] for a more detailed discussion on semi-direct products). It is not hard to see that this group is the stabilizer in $\text{Sym}(nm)$ of a partition of the set $\{1, 2, \dots, nm\}$ into n parts each of size m .

Similar to what we have done in the previous sections, we will describe the derangement graph $\Gamma_{G \wr H}$ as a subgraph of a graph product of Γ_G and Γ_H . In this case, we consider the *lexicographic product* of graphs. Let X and Y be two graphs. Then

the *lexicographical product* $X[Y]$ is a graph with vertex set $V(X) \times V(Y)$ in which two vertices $(x_1, y_1), (x_2, y_2)$ are adjacent if and only if $x_1 \sim x_2$ in X or $x_1 = x_2$ and $y_1 \sim y_2$ in Y . An easy interpretation of $X[Y]$ is as follows: to construct $X[Y]$, replace any vertex of X with a copy of Y , and if two vertices x_1 and x_2 in X are adjacent, then in $X[Y]$ all the vertices which replaced the vertex x_1 will be adjacent to all the vertices which replaced the vertex x_2 .

Note that if S_X and S_Y are independent sets in X and Y , respectively, then the $S_X[S_Y]$ is an independent set of vertices of $X[Y]$. It is straight-forward to determine the size of the maximum independent sets in a lexicographic product, see [14] for details.

Proposition 6.1. *Let X and Y be graphs. Then*

$$\alpha(X[Y]) = \alpha(X)\alpha(Y). \quad \square$$

For any $x \in V(X)$, let $Y_x = \{x\} \times Y$ (this is the copy of Y that replaces x in X). If S is a subset of the vertices of $X[Y]$, define the projection of S to X as

$$\text{proj}_X(S) = \{x \in X \mid (x, y) \in S, \text{ for some } y \in Y\}.$$

Similarly, for any $x \in V(X)$ we define the projection of S to Y_x as

$$\text{proj}_{Y_x}(S) = \{y \in Y \mid (x, y) \in S\}.$$

We can, then, observe the following.

Proposition 6.2. *Let X and Y be graphs. If S is an independent set in $X[Y]$ of size $\alpha(X)\alpha(Y)$, then $\text{proj}_X(S)$ is a maximum independent set in X and, for any $x \in V(X)$, the set $\text{proj}_{Y_x}(S)$ is a maximum independent set in Y_x . \square*

Assume $\Omega = \{1, \dots, m\} \times \{1, \dots, n\}$, then the group $G \wr H$ acts on Ω in the following fashion:

$$(x, j)^{(g_1, \dots, g_n, h)} := (x^{g_j}, j^h) = (g_j(x), h(j)), \tag{3}$$

for any $(x, j) \in \Omega$ and $(g_1, \dots, g_n, h) \in G \wr H$. If (g_1, \dots, g_n, h) has a fixed point (x, j) , then $h(j) = j$ and $g_j(x) = x$. Thus, it is not difficult to verify the following.

Lemma 6.3. *For any pair $(x, j) \in \Omega$, the stabilizer of (x, j) in $G \wr H$ is*

$$\left(G \times \cdots \times \underset{j\text{th position}}{(G)_x} \times \cdots \times G \right) \times H_j. \quad \square$$

Theorem 6.4. *If $G \leq \text{Sym}(m)$ and $H \leq \text{Sym}(n)$ have the EKR property, then $G \wr H$ also has the EKR property.*

Proof. For convenience we let $W := G \wr H$ and $P = G \times \cdots \times G$. Note that by the definition of the wreath product, P is in fact the internal direct product of G_1, \dots, G_n , where $G_i \cong G$ and $G_i \leq \text{Sym}(\{1, 2, \dots, m\} \times \{i\})$, for any $1 \leq i \leq n$. Hence according to (2), we have

$$\Gamma_P = \underbrace{\Gamma_G \times \cdots \times \Gamma_G}_{n \text{ times}}.$$

Consider the lexicographic product $\Gamma = \Gamma_H[\Gamma_P]$. Define the map $f : \Gamma \rightarrow \Gamma_W$ by

$$f(h, (g_1, \dots, g_n)) = (g_1, \dots, g_n, h).$$

We claim that f is a graph homomorphism. To prove this, assume that $(h, (g_1, \dots, g_n))$ and $(h', (g'_1, \dots, g'_n))$ are adjacent in Γ . We should show that

$$(g'_1, \dots, g'_n, h') \cdot (g_1, \dots, g_n, h)^{-1} = (g'_1 g_{h'h^{-1}(1)}^{-1}, \dots, g'_n g_{h'h^{-1}(n)}^{-1}, h'h^{-1}) \quad (4)$$

has no fixed point. By the definition of the lexicographic product, either $h \sim h'$ in Γ_H or $h = h'$ and $(g_1, \dots, g_n) \sim (g'_1, \dots, g'_n)$ in P . In the first case, $h'h^{-1}$ has no fixed point. Thus $(g'_1, \dots, g'_n, h') \cdot (g_1, \dots, g_n, h)^{-1}$ cannot have a fixed point. In the latter case, $(g'_1, \dots, g'_n)(g_1, \dots, g_n)^{-1}$ has no fixed point; thus, according to (4),

$$(g'_1, \dots, g'_n, h') \cdot (g_1, \dots, g_n, h)^{-1} = (g'_1 g_1^{-1}, \dots, g'_n g_n^{-1}, \text{id}_H)$$

cannot have a fixed point. Thus the claim is proved.

We can, therefore, apply the no-homomorphism lemma to get

$$\frac{|V(\Gamma)|}{\alpha(\Gamma)} \leq \frac{|V(\Gamma_W)|}{\alpha(\Gamma_W)}.$$

Therefore, using Proposition 6.1, we have

$$\alpha(\Gamma_W) \leq \alpha(\Gamma_P)\alpha(\Gamma_H). \quad (5)$$

But since G has the EKR property, according to Theorem 5.3, P has the EKR property; this means that there is a point $x \in \{1, 2, \dots, m\}$ such that

$$\alpha(\Gamma_P) = |P_x|.$$

Similarly, since H has the EKR property, there exists a $j \in \{1, 2, \dots, n\}$ such that

$$\alpha(\Gamma_H) = |H_j|.$$

This, along with Lemma 6.3, implies that the stabilizer of a point is the largest independent set in Γ_W . \square

The map $f : \Gamma \rightarrow \Gamma_W$ is one-to-one, so can be considered as an embedding. It is not the case that Γ and Γ_W are isomorphic, since there are edges in Γ_W that are not in Γ .

In the case of symmetric groups, we can say more.

Proposition 6.5. *The group $\text{Sym}(m) \wr \text{Sym}(n)$ has the strict EKR property.*

Proof. The first part follows from Theorem 6.4. For the second part, as in the proof of Theorem 6.4, we let $W = \text{Sym}(m) \wr \text{Sym}(n)$ and define the internal direct sum

$$P = \text{Sym}([m] \times \{1\}) \times \cdots \times \text{Sym}([m] \times \{n\}).$$

Let S be an intersecting subset of W of maximum size, i.e. S has the size of a point-stabilizer in W . Without loss of generality we assume that S contains the identity element of W . Consider the homomorphism $f : \Gamma_{\text{Sym}(n)}[\Gamma_P] \rightarrow \Gamma_W$ defined in the proof of Theorem 6.4. The function f is an injection; hence there is a copy of $\Gamma_{\text{Sym}(n)}[\Gamma_P]$ in Γ_W . This implies that S is an independent set in $\Gamma_{\text{Sym}(n)}[\Gamma_P]$ of size $\alpha(\Gamma_{\text{Sym}(n)})\alpha(\Gamma_P)$.

For $x \in \text{Sym}(n)$, let $(\Gamma_P)_x$ denote the copy of Γ_P that replaces x from $\Gamma_{\text{Sym}(n)}$ in the graph $\Gamma_{\text{Sym}(n)}[\Gamma_P]$. Then, according to Proposition 6.2 and the fact that P has the strict EKR property (see Theorem 5.6), we have that the projection of S to $(\Gamma_P)_x$ is a point-stabilizer in P . Denote this point stabilizer by $P_{(y_x, j_x)}$ where $y_x \in \{1, \dots, m\}$ and $j_x \in \{1, \dots, n\}$. To show that S is the stabilizer of a point, we need to show that the pair (y_x, j_x) is the same for all $x \in \text{Sym}(n)$.

Assume that there are two permutation x_1 and x_2 in $\text{Sym}(n)$ with $(y_{x_1}, j_{x_1}) \neq (y_{x_2}, j_{x_2})$. Let σ_{x_1} be a permutation whose projection to $(\Gamma_P)_{x_1}$ is the identity if $x = x_1$, and a derangement otherwise. Let σ_{x_2} be the permutation whose projection to $(\Gamma_P)_{x_2}$ is a derangement if $x = x_2$ and the identity otherwise. Then $\sigma_{x_1} \in P_{(y_{x_1}, j_{x_1})}$, and $\sigma_{x_2} \in P_{(y_{x_2}, j_{x_2})}$ and both are in S . But this is a contradiction since these permutations are not intersecting, thus $j_{x_1} = j_{x_2} = j_x$.

Similarly, if $m > 2$ we can show that $y_{x_1} = y_{x_2}$ by finding permutations that are in $P_{(y_{x_1}, j_x)}$ and $P_{(y_{x_2}, j_x)}$, but are not intersecting. If $m = 2$, then $P_{(1, j_x)} = P_{(2, j_x)}$, so again the point stabilizers must be the same.

The projection of S to each $(\Gamma_P)_x$ is the stabilizer of the point (y_x, j_x) , thus S is the stabilizer of (y_x, j_x) in $\text{Sym}(m) \wr \text{Sym}(n)$. \square

7 Groups that do not have the EKR property

Our first example of a group that does not have the EKR property is the Mathieu group M_{20} (this group is the stabilizer of two points in the Mathieu group M_{22}). The stabilizer of a point in M_{20} has size 48 but there is an independent set in $\Gamma_{M_{20}}$ of size 64.

The group M_{20} acts on the set $\{1, 2, \dots, 20\}$ and, under this action, there is a system of imprimitivity that is comprised of 5 blocks, each of size 4. Label these blocks by B_i . Consider the following four possible ways that the first four blocks could be moved:

$$B_1 \rightarrow B_1, \quad B_2 \rightarrow B_2, \quad B_3 \rightarrow B_3, \quad B_4 \rightarrow B_5.$$

Then the set of all permutations that move the blocks in at least three of these four ways forms an independent set of size 64. There are many other sets of four ways to

move that blocks that will similarly produce independent sets of size 64. This shows that M_{20} does not have the EKR property. We conjecture that these are the largest independent sets in $\Gamma_{M_{20}}$, and we leave it as an open problem whether or not there are other ways to construct maximum independent sets in this graph.

The original EKR theorem for sets considered collections of sets in which any two sets have at least t elements in common for some integer $t \geq 1$. Similarly, two permutations π and σ are t -intersecting if $\pi\sigma^{-1}$ has at least t fixed points. We can then ask what is the size of the largest t -intersecting set of permutations from a group?

The largest set of t -intersecting permutations of a group $G \leq \text{Sym}(n)$ is also the largest set of intersecting permutations in G when we consider a different group action. Namely, if $G \leq \text{Sym}(n)$, then G has a natural action on the ordered t -sets from $\{1, \dots, n\}$; so G can also be considered as a subgroup of $\text{Sym}(n \cdot (n - 1) \cdot \dots \cdot (n - t + 1))$ with this action. Two permutations are intersecting under this action if and only if they are t -intersecting under the usual action on $\{1, \dots, n\}$. Following the construction of t -intersecting sets of maximum size, we can produce groups that do not have the EKR property.

For example, consider 4-intersecting permutations in the group $\text{Sym}(8)$ with the natural action on $\{1, 2, \dots, 8\}$. The stabilizer of 4 points has size $4! = 24$. But the set of all permutations from $\text{Sym}(8)$ that fix at least 5 points from the set $\{1, 2, 3, 4, 5, 6\}$ is also 4-intersecting. The size of this set 26 (there are 2 permutations that fix all 6 of $\{1, 2, 3, 4, 5, 6\}$, and $\binom{6}{5}(4) = 24$ that fix exactly 5 of them). If we consider $\text{Sym}(8)$ with its action on the ordered sets of size 4 from $[8]$ then we have a subgroup of $\text{Sym}(1680)$ isomorphic to $\text{Sym}(8)$ that does not have the EKR property.

This can be generalized for any t to give more examples of groups that do not have the EKR property.

Lemma 7.1. *For $t \geq 4$ the size of a t -intersecting set in $\text{Sym}(2t)$ is at least*

$$(t^2 + t - 1)(t - 2)!.$$

Proof. The set of all permutations that fix at least $t + 1$ points from $\{1, 2, \dots, t + 2\}$ is a t -intersecting set of size

$$\binom{t + 2}{t + 2}(t - 2)! + \binom{t + 2}{t + 1}(t - 1)(t - 2)! \quad \square$$

Note for all $t > 0$ that $(t^2 + t - 1)(t - 2)! \geq t!$ so these are all examples of groups that do not have the EKR property. □

Finally, we note that it is possible to take the internal direct product of a group that does not have the EKR property with one that does and have the result be a group that does have the EKR property. To see this, let $G_1 = M_{20}$ and $G_2 = \text{Sym}(n)$ and let G be the internal direct product of G_1 and G_2 . Then for n sufficiently large we have that

$$\alpha(\Gamma_G) = \max\{\alpha(\Gamma_{G_1})|G_2|, \alpha(\Gamma_{G_2})|G_1|\} = \alpha(\Gamma_{G_1})|(G_2)_n| = (n - 1)!|G_1|,$$

which is the size of a stabilizer of a point in G . This shows that $G = G_1 \times G_2$ does have the EKR property while G_1 does not.

8 Further Work

There is clearly much more work to be done to determine which groups that have the EKR property. We would like to find more conditions on groups to determine if they have either the EKR property or the strict EKR property.

We would also like to investigate groups that do not have either the EKR property or the strict EKR property. In this paper we show that the Frobenius groups do not have the strict EKR property, but this example is not the most interesting since the derangement graph is the union of complete graphs. The Mathieu group M_{20} is a more interesting example since it has non-trivial intersecting sets and these sets have an interesting structure. The next step is to find more examples of groups that do not have the EKR property. The hope would be to determine properties of the group that predicts when this happens and to determine if the maximum independent sets have an interesting structure. For example, the groups $\text{PGL}(3, q)$ never have the strict EKR property since the stabilizer of a hyperplane forms an intersecting set that has the same size as the stabilizer of a point (in [25] it is shown that these are all the intersecting sets of maximum size).

Another direction to consider is based on the EKR theorem for sets. The EKR theorem for t -intersecting k -subsets from $\{1, 2, \dots, n\}$ has been completely solved for all values of n, k and t [1]. Depending on the size of n , relative to k and t , the largest t -intersecting k -subsets are all the subsets that contain $t + i$ elements from a set of $t + 2i$ elements (where i is an integer that depends on n, k and t). Similarly, for permutations, we can define the set of all permutations that fix at least $t + i$ elements from a set of $t + 2i$ elements. These sets are natural candidates for intersecting sets of maximum size for groups in which the EKR theorem does not hold. We plan to check these sets for some small groups.

References

- [1] R. Ahlswede and L.H. Khachatrian, The complete intersection theorem for systems of finite sets, *European J. Combin.*, 18 (1997), 125–136.
- [2] B. Ahmadi, *Maximum Intersecting Families of Permutations*, University of Regina, (2013), Ph.D. Thesis.
- [3] B. Ahmadi and K. Meagher, The Erdős-Ko-Rado property for some 2-transitive groups, *Ann. Comb.* (2014) (to appear).
- [4] B. Ahmadi and K. Meagher, A new proof for the Erdős-Ko-Rado Theorem for the alternating group, *Discrete Math.* 324 (2014), 28–40.
- [5] L. Babai, Spectra of Cayley graphs, *J. Combin. Theory Ser. B* 27 (1979) 180–189.
- [6] M. Bardestani and K. Mallahi-Karai, On the Erdős-Ko-Rado property for finite groups, *ArXiv e-prints*, Oct. 2013.

- [7] P. J. Cameron and C. Y. Ku, Intersecting families of permutations, *European J. Combin.* 24 (2003), 881–890.
- [8] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, *Z. Wahrsch. Verw. Gebiete* 57 (1981), 159–179.
- [9] J. D. Dixon and B. Mortimer, *Permutation Groups*, Graduate Texts in Mathematics Series, Springer London, Limited, 2012.
- [10] P. Erdős, Chao Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math.* 12 (1961), 313–320.
- [11] P. Frankl and M. Deza, On the maximum number of permutations with given maximal or minimal distance, *J. Combin. Theory Ser. A* 22 (1977), 352–360.
- [12] P. Frankl and N. Tokushige, The Erdős-Ko-Rado theorem for integer sequences, *Combinatorica* 19 (1999), 55–63.
- [13] W. Fulton and J. Harris, *Representation Theory*, Vol. 129 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1991.
- [14] D. Geller and S. Stahl, The chromatic number and other functions of the lexicographic product, *J. Combin. Theory Ser. B* 19 (1975), 87–95.
- [15] C. Godsil and K. Meagher, A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations, *European J. Combin.* 30 (2009), 404–414.
- [16] C. Godsil and G. Royle, *Algebraic Graph Theory*, Vol. 207 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001.
- [17] P. Hell and J. Nešetřil, *Graphs and homomorphisms*, Vol. 28 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2004.
- [18] W. N. Hsieh, Intersection theorems for systems of finite vector spaces, *Discrete Math.* 12 (1975), 1–16.
- [19] B. Huppert, *Character Theory of Finite Groups*, De Gruyter expositions in mathematics, Walter de Gruyter, 1998.
- [20] P. K. Jha and S. Klavzar, Independence in direct-product graphs, *Ars Combin.* 50 (1998), 53–63.
- [21] C. Y. Ku and T. W. H. Wong, Intersecting families in the alternating group and direct product of symmetric groups, *Electron. J. Combin.* 14, (2007).
- [22] B. Larose and C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, *European J. Combin.* 25 (2004), 657–673.

- [23] K. Meagher and L. Moura, Erdős-Ko-Rado theorems for uniform set-partition systems, *Electron. J. Combin.* 12, (2005), 12 pp.
- [24] K. Meagher and P. Spiga, An Erdős-Ko-Rado theorem for the derangement graph of $PGL(2, q)$ acting on the projective line, *J. Combin. Theory, Ser. A* 118 (2011), 532–544.
- [25] K. Meagher and P. Spiga, An Erdős-Ko-Rado-type theorem for $PGL_3(q)$, *SIAM J. Discrete Math.* 28 (2014), 918–941.
- [26] B. E. Sagan, *The Symmetric Group*, Vol. 203 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, second ed., 2001.
- [27] C. Tardif, Graph products and the chromatic difference sequence of vertex-transitive graphs, *Discrete Math.* 185 (1998), 193–200.
- [28] J. Wang and S. J. Zhang, An Erdős-Ko-Rado-type theorem in Coxeter groups, *European J. Combin.* 29 (2008), 1112–1115.
- [29] H. Zhang, Independent sets in direct products of vertex-transitive graphs, *J. Combin. Theory Ser. B* 102 (2012), 832–838.

(Received 5 Nov 2013; revised 22 Oct 2014)