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Totally antimagic total graphs

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Abstract

For a graph G a bijection from the vertex set and the edge set of G to the set $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called a total labeling of G. The edge-weight of an edge is the sum of the label of the edge and the labels of the end vertices of that edge. The vertex-weight of a vertex is the sum of the label of the vertex and the labels of all the edges incident with that vertex. A total labeling is called edge-antimagic total (vertex-antimagic total) if all edge-weights (vertex-weights) are pairwise distinct. If a labeling is simultaneously edge-antimagic total and vertex-antimagic total it is called a totally antimagic total labeling. A graph that admits totally antimagic total labeling is called a totally antimagic total graph.

In this paper we deal with the problem of finding totally antimagic total labeling of some classes of graphs. We prove that paths, cycles, stars, double-stars and wheels are totally antimagic total. We also show that a union of regular totally antimagic total graphs is a totally antimagic total graph.

1 Introduction

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex-set and edge-set of G, respectively. Let $|V(G)| = p$ and $|E(G)| = q$. Let K_n denote the complete graph on n vertices, P_n the path on n vertices, C_n the cycle on n vertices.

A *labeling* of a graph G is any mapping that sends a certain set of graph elements to a certain set of positive integers. If the domain is the vertex-set, or the edge-set, respectively, the labeling is called a *vertex labeling*, or an *edge labeling*, respectively. If the domain is $V(G) \cup E(G)$ then the labeling is called a *total labeling*. More precisely, for a graph G a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ is a *total labeling* of G. Moreover, if the vertices are labeled with the smallest possible numbers, i.e., $f(V(G)) = \{1, 2, \ldots, p\}$, then the total labeling is called *super*.

Under the labeling f, the associated *edge-weight* of an edge uv, $uv \in E(G)$, is defined by

$$wt_f(uv) = f(uv) + f(u) + f(v).$$

The associated *vertex-weight* of a vertex v, $v \in V(G)$, is defined by

$$wt_f(v) = \sum_{u \in N(v)} f(uv) + f(v),$$

where $N(v)$ is the set of the neighbors of v.

In other words, the edge-weight of an edge is the sum of the label of the edge and the labels of the end vertices of that edge, while the vertex-weight of a vertex is the sum of the label of the vertex and the labels of all the edges incident with that vertex.

A labeling f is called *edge-antimagic total* (*vertex-antimagic total*), for short *EAT* (*VAT*), if all edge-weights (vertex-weights) are pairwise distinct. A graph that admits EAT (VAT) labeling is called an *EAT (VAT) graph*. If the edge-weights (vertex-weights) are all the same then the total labeling is called *edge-magic total* (*vertex-magic total*), for short *EMT labeling (VMT labeling)*.

In 1990, Hartsfield and Ringel [18] introduced the concept of an antimagic labeling of a graph, that is, in our terminology, a vertex-antimagic edge labeling, i.e., according to Hartsfield and Ringel, an antimagic labeling of a graph G is an edge labeling, where all the vertex-weights are required to be pairwise distinct. For an edge labeling, a vertex-weight is the sum of the labels of all edges incident with the vertex. Hartsfield and Ringel [18] conjectured that every tree except P_2 admits a vertex-antimagic edge (VAE) labeling and, moreover, every connected graph except P_2 has a VAE labeling. Alon, Kaplan, Lev, Roditty and Yuster [2] showed that this conjecture is true for all graphs having minimum degree $\Omega(\log |V(G)|)$.

If a VAE labeling satisfies the condition that the set of all the vertex-weights is $\{a, a+d, \dots, a+(p-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers, then the labeling is called an (a, d)-*VAE labeling*. The (a, d)-VAE labeling was defined by Bodendiek and Walther [9] as (a, d)-antimagic labeling.

Motivated by results of Bodendiek and Walther (see [9] and [10]), Bača, Bertault, MacDougall, Miller, Simanjuntak and Slamin [3] introduced the concept of an (a, d)-VAT labeling. A VAT labeling is called an (a, d)-*VAT labeling* if the vertex-weights form an arithmetic sequence starting from a and having common difference d, i.e., the set of all the vertex-weights is $\{a, a+d, \dots, a+(p-1)d\}$, for some integers $a > 0$ and $d \geq 0$. The basic properties of an (a, d)-VAT labeling and its relationships to other types of magic-type and antimagic-type labelings are investigated in [3]. In [27], it is shown how to construct super (a, d)-VAT labelings for certain families of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalized Petersen graphs. If $d = 0$ then the labeling is called a *(super) vertex-magic total*, simply (super) VMT. The concept of VMT and super VMT graph was introduced by MacDougall, Miller, Slamin and Wallis in [22]. There are several results known for regular VMT graphs. A VMT labeling for K_n, for odd n, can be found in [21], [22] and [24], and for K_n, with n even, in [13] and [17]. A construction for VMT labeling of complete bipartite graphs $K_{m,m}$ is presented in [22]. In [16], it is completely determined which complete bipartite graphs have VMT labelings. Constructions of VMT labelings of certain regular graphs are given in [14], [15] and [19]. The existence of super (a, d)-VAT labeling for disconnected graphs is examined in [1]. The existence of antimagic labelings for plane graphs is studied in [4] and [8].

An (a, d)-*EAT labeling* of a graph G is a total labeling with the property that the edge-weights form an arithmetic sequence starting from a and having common difference d, where $a > 0$ and $d \geq 0$ are two integers. The notion of (a, d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [26] as a natural extension of *magic valuation* defined by Kotzig and Rosa in [20]. Magic valuation is also very often known as *edge-magic labeling.* Kotzig and Rosa [20] showed that all caterpillars have magic valuations and conjectured that all trees have magic valuations. In [26] Simanjuntak, Bertault and Miller gave constructions of (a, d)-EAT labelings for cycles and paths. Bača, Lin, Miller and Simanjuntak [5] presented some relationships between (a, d)-EAT labeling and other labelings, namely, edge-magic vertex labeling and edge-magic total labeling.

For further results on graph labelings see [6], [12] and [23].

There are known characterizations of all EAT and VAT graphs: In [25] Miller, Phanalasy and Ryan proved

Proposition 1 ([25])**.** *All graphs are (super) EAT.*

Proposition 2 ([25])**.** *All graphs are (super) VAT.*

Since all graphs are EAT and VAT, naturally we can ask whether there exist graphs possessing a labeling that is simultaneously vertex-antimagic total and edge-antimagic total. We will call such a labeling a *totally antimagic total labeling (TAT labeling)* and a graph that admits such a labeling a *totally antimagic total graph (TAT graph).* If, moreover, the vertices are labeled with the smallest possible labels then, as is customary, the labeling is referred to as *super.* The definition of totally antimagic total labeling is a natural extension of the concept of totally magic labeling defined by Exoo, Ling, McSorley, Phillips and Wallis in [11]. They showed that such graphs appear to be rare. They proved that the only connected totally magic graph containing a vertex of degree 1 is P_3, the only totally magic trees are K_1 and P_3, the only totally magic cycle is C_3, the only totally magic complete graphs are K_1 and K_3, and the only totally magic complete bipartite graph is $K_{1,2}$.

In this paper we will deal with the following question. For a graph G does there exist a total labeling that is both edge-antimagic and vertex-antimagic?

2 Join of graphs

We say that a labeling g is *ordered* (*sharp ordered*) if $wt_g(u) \leq wt_g(v)$ ($wt_g(u) < wt_g(v)$) holds for every pair of vertices $u, v \in G$ such that $g(u) < g(v)$. A graph that admits a (sharp) ordered labeling is called a *(sharp) ordered graph.*

Let $G \cup H$ denote the disjoint union of graphs G and H. The join $G \oplus H$ of the disjoint graphs G and H is the graph $G \cup H$ together with all the edges joining vertices of $V(G)$ and vertices of $V(H)$. In this section we will deal with a totally antimagic total labeling of $G \oplus K_1$. According to Proposition 1 we have that every graph G is super EAT. If there exists a super EAT labeling of a graph G satisfying the additional condition that it is also ordered we are able to prove that the join $G \oplus K_1$ is TAT.

Theorem 1. *Let G be an ordered super EAT graph. Then $G \oplus K_1$ is a TAT graph.*

Proof. Let g be an ordered super EAT labeling of G. As g is super, we can denote the vertices of G by the symbols $v_1, v_2, \dots, v_p$ such that

$$g(v_i) = i, \qquad \text{for } i = 1, 2, \dots, p.$$

Since g is ordered then for $i = 1, 2, \dots, p-1$, we have

$$wt_g(v_i) \le wt_g(v_{i+1}).$$

By the symbol u we denote the vertex of $G \oplus K_1$ not belonging to G.

We define a new labeling f of $G \oplus K_1$ such that

$$\begin{aligned} f(x) &= g(x) && x \in V(G) \cup E(G) \\ f(u) &= 2p + q + 1 && \\ f(v_i u) &= p + q + i && i = 1, 2, \dots, p. \end{aligned}$$

It is easy to see that f is a bijection from $V(G \oplus K_1) \cup E(G \oplus K_1)$ to the set $\{1, 2, \dots, 2p + q + 1\}$.

For the vertex-weights under the labeling f we have the following.

$$\begin{aligned} wt_f(u) &= f(u) + \sum_{i=1}^{p} f(uv_i) \\ &= \big(2p + q + 1\big) + \sum_{i=1}^{p} \big(p + q + i\big) \\ &= \sum_{i=1}^{p+1} \big(p + q + i\big) \\ &= \frac{\big(p+1\big)\big(3p + 2q + 2\big)}{2}. \end{aligned}$$

For $i = 1, 2, \dots, p$ we get

$$\begin{aligned} wt_f(v_i) &= f(v_i) + \sum_{v \in N_G(v_i)} f(v_i v) + f(v_i u) \\ &= g(v_i) + \sum_{v \in N_G(v_i)} g(v_i v) + \big(p + q + i\big) \\ &= wt_g(v_i) + p + q + i \\ &\le wt_g(v_{i+1}) + p + q + i \\ &< wt_g(v_{i+1}) + p + q + i + 1 = wt_f(v_{i+1}). \end{aligned}$$

Moreover, as g is a super EAT labeling of G, we get

$$\begin{aligned} wt_f(v_p) &= g(v_p) + \sum_{v \in N_G(v_p)} g(v_p v) + \big(2p+q\big) \\ &\leq p + \sum_{j=1}^{p-1} \big(p+q+1-j\big) + 2p + q \\ &= 3p + q + \frac{\big(p-1\big)\big(p+2q+2\big)}{2} \\ &< wt_f(u). \end{aligned}$$

Thus, the vertex-weights are all different.

The edge-weights of the edges in $E(G)$ under the labeling f are all different as g is an EAT labeling of G. More precisely, we have

$$wt_f(e) = wt_g(e), \qquad \text{for every } e \in E(G).$$

Moreover, as g is super, for the upper bound on the maximum edge-weight of $e \in E(G)$ under the labeling f, we have

$$\begin{aligned} wt_f^{max}(e) = wt_g^{max}(e) &\leq p + \big(p-1\big) + \big(p+q\big) \\ &= 3p + q - 1. \end{aligned}$$

For $i = 1, 2, \ldots, p$, we get

$$\begin{aligned} wt_f(uv_i) &= f(u) + f(uv_i) + f(v_i) \\ &= \big(2p+q+1\big) + \big(p+q+i\big) + g(v_i) \\ &= 3p + 2q + 1 + 2i > 3p + q - 1 \geq wt_f^{max}(e), \end{aligned}$$

where $e \in E(G)$.

It is easy to see that the edge-weights are also all different.

Thus f is a TAT labeling of $G \oplus K_1$. □

The labeling method presented in Theorem 1 allows us to find TAT labeling for $G \oplus K_1$ if we can find an ordered super EAT labeling of G. In some cases, if we consider a given graph G with the corresponding (not ordered) super EAT labeling, after applying the labeling method presented in Theorem 1 the obtained labeling can be also TAT. However, also when the obtained labeling is not TAT it seems to be very easy to modify the obtained labeling by swapping some labels and to get a TAT labeling of $G \oplus K_1$. Thus we state the following conjecture.

Conjecture 1. *Every graph $G \oplus K_1$ is TAT.*

We also formulate the weaker version of the Conjecture 1.

Conjecture 2. *Every complete graph is TAT.*

Let mG denote the disjoint union of m copies of graph G. In the following lemmas we prove that certain families of graphs are totally antimagic total. These include totally disconnected graphs on m vertices, m copies of K_2, paths, and cycles. Furthermore, the totally antimagic total labelings of these graphs have useful extra properties.

Observation 1. *For every positive integer m the graph mK_1 is sharp ordered super TAT.*

Proof. Trivial. □

Lemma 1. *For every positive integer m the graph mK_2 is sharp ordered super TAT.*

Proof. We denote the vertices of mK_2 by the symbols v_i, $i = 1, 2, \dots, 2m$, such that its edge set is

$$E(mK_2) = \{v_1v_2, v_3v_4, \dots, v_{2m-1}v_{2m}\}.$$

Let us consider the labeling g of mK_2 defined in the following way.

$$\begin{aligned} g(v_i) &= i & i &= 1, 2, \dots, 2m \\ g(v_iv_{i+1}) &= 2m + \tfrac{i+1}{2} & i &= 1, 3, \dots, 2m-1. \end{aligned}$$

It is easy to see that g is a super TAT labeling of mK_2, as the edge-weight of the edge v_iv_{i+1}, $i = 1, 3, \dots, 2m-1$, is

$$\begin{aligned} wt_g(v_iv_{i+1}) &= g(v_i) + g(v_iv_{i+1}) + g(v_{i+1}) = i + \big(2m + \tfrac{i+1}{2}\big) + (i+1) \\ &= 2m + \tfrac{5i+3}{2} \end{aligned}$$

and for the vertex-weights we get

$$wt_g(v_i) = \begin{cases} 2m + \frac{3i+1}{2} & i = 1, 3, \dots, 2m-1 \\ 2m + \frac{3i}{2} & i = 2, 4, \dots, 2m. \end{cases}$$

This concludes the proof. □

Lemma 2. *The graph P_n, $n \geq 2$, is sharp ordered super TAT.*

Proof. We denote the vertices of P_n by the symbols v_i, $i = 1, 2, \dots, n$, such that

$$E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}.$$

It is easy to check that the labeling g, $g : V(P_n) \cup E(P_n) \to \{1, 2, \dots, 2n-1\}$ satisfies the above mentioned condition, when

$$g(v_i) = \begin{cases} 2i - 1 & i = 1, 2, \dots, \lceil \frac{n}{2} \rceil \\ 2n + 2 - 2i & i = \lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \dots, n \end{cases}$$

$$g(v_iv_{i+1}) = \begin{cases} n - 1 + 2i & i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor \\ 3n - 2i & i = \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \dots, n-1. \end{cases}$$

□

Lemma 3. *The graph C_n, $n \geq 3$, is sharp ordered super TAT.*

Proof. We denote the vertices of C_n by the symbols v_i, $i = 1, 2, \dots, n$, such that the edge set of C_n is

$$E(C_n) = \{v_1v_2, v_2v_3, \dots, v_nv_1\}.$$

Let us consider the labeling $g : V(C_n) \cup E(C_n) \to \{1, 2, \dots, 2n\}$, defined as

$$g(v_i) = \begin{cases} 1 & i = 1 \\ 2i - 2 & i = 2, 3, \dots, \lfloor \frac{n}{2} \rfloor + 1 \\ 2n + 3 - 2i & i = \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 3, \dots, n \end{cases}$$

$$g(v_iv_{i+1}) = \begin{cases} n - 1 + 2i & i = 1, 2, \dots, \lceil \frac{n}{2} \rceil \\ 3n + 2 - 2i & i = \lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \dots, n - 1 \end{cases}$$

$$g(v_nv_1) = n + 2.$$

It is a simple mathematical exercise to check that g is a sharp ordered super TAT labeling of C_n with the desired properties. □

A *wheel* W_n with n spokes is isomorphic to the graph $C_n \oplus K_1$. The vertex of degree n is called the *central vertex*. The edges not incident to the central vertex are called *rim edges*. Immediately from Lemma 3 and Theorem 1, we get the following result.

Corollary 1. *The wheel W_n is TAT for $n \geq 3$.*

The *friendship graph* $\mathcal{F}_n$ is a graph isomorphic to $(nK_2) \oplus K_1$. Alternatively, the friendship graph $\mathcal{F}_n$ can be obtained from the wheel W_{2n} by removing every second rim edge.

Corollary 2. *The friendship graph $\mathcal{F}_n$ is TAT for $n \geq 1$.*

Proof. The result follows from Lemma 1 and Theorem 1. □

If one rim edge is removed from W_n, the resulting graph is called a *fan*, denoted by F_n. Alternatively, a fan F_n is isomorphic to the graph $P_n \oplus K_1$. From Lemma 2 and Theorem 1, we get

Corollary 3. *The fan F_n is TAT for $n \geq 2$.*

Corollary 4. *The star S_n is TAT for $n \geq 1$.*

Proof. The star S_n is isomorphic to $(nK_1) \oplus K_1$. The result follows from Observation 1 and Theorem 1. □

3 Corona graph

If G has order p, the *corona* of G with a graph H, denoted by $G \odot H$, is the graph obtained by taking one copy of G and p copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H. A cycle of order m with n pendant edges attached at each vertex, i.e., $C_m \odot nK_1$, is called an *n-crown* with cycle of order m.

Theorem 2. *Let G be a regular ordered super EAT graph. Then the graph $G \odot nK_1$ is TAT for every $n \geq 1$.*

Proof. Let G be a regular graph. Let g be an ordered super EAT labeling of G. As g is super, we can denote the vertices of G by the symbols $v_1, v_2, \dots, v_p$, such that

$$g(v_i) = i, \qquad \text{for } i = 1, 2, \dots, p.$$

Since g is ordered, this means that $wt_g(v_i) \leq wt_g(v_{i+1})$, for $i = 1, 2, \dots, p-1$.

We denote the vertices of $G \odot nK_1$ not belonging to G by the symbols $u_{i,j}$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, n$, such that $u_{i,j}v_i \in E(G \odot nK_1)$.

We define a labeling f of $G \odot nK_1$ in the following way.

$$\begin{aligned} f(x) &= g(x) + 2pn && x \in V(G) \cup E(G) \\ f(u_{i,j}) &= (i-1)n + j && i = 1, 2, \dots, p,\ j = 1, 2, \dots, n \\ f(u_{i,j}v_i) &= (i-1)n + j + np && i = 1, 2, \dots, p,\ j = 1, 2, \dots, n. \end{aligned}$$

It is easy to see that f is a bijection from the union of the vertex set and the edge set of $G \odot nK_1$ to the set $\{1, 2, \dots, p+q+2pn\}$. For the edge-weight of the edge v_iv_k, $i = 1, 2, \dots, p$, $k = 1, 2, \dots, p$, $i \neq k$, under the labeling f, we have

$$\begin{aligned} wt_f(v_iv_k) &= f(v_i) + f(v_iv_k) + f(v_k) \\ &= \big(g(v_i) + 2pn\big) + \big(g(v_iv_k) + 2pn\big) + \big(g(v_k) + 2pn\big) \\ &= wt_g(v_iv_k) + 6pn. \end{aligned}$$

As g is an EAT labeling, the weights of all the edges $e \in E(G)$ are different under the labeling f as well.

For the edge $u_{i,j}v_i$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, n$, we obtain

$$\begin{aligned} wt_f(u_{i,j}v_i) &= f(u_{i,j}) + f(u_{i,j}v_i) + f(v_i) \\ &= \big((i-1)n + j\big) + \big((i-1)n + j + np\big) + \big(g(v_i) + 2pn\big) \\ &= 3pn - 2n + i(2n+1) + 2j. \end{aligned}$$

This means that all the edges in $E(G \odot nK_1) \backslash E(G)$ have different edge-weights. Moreover,

$$\begin{aligned} wt_f(u_{i,j}v_i) &= 3pn - 2n + i(2n+1) + 2j \\ &\leq 3pn - 2n + p(2n+1) + 2n = p + 5pn \\ &\leq 6pn, \end{aligned}$$

for $i = 1, 2, \dots, p$, $j = 1, 2, \dots, n$. Thus, for all the edges e_1 and e_2, $e_1 \in E(G)$ and $e_2 \in E(G \odot nK_1) \backslash E(G)$,

$$wt_f(e_1) > wt_f(e_2).$$

Thus the edge-weights of all the edges in $G \odot nK_1$ are different.

We have to check that also the vertex-weights are different. For the vertex $u_{i,j}$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, n$, we get

$$\begin{aligned} wt_f(u_{i,j}) &= f(u_{i,j}) + f(u_{i,j}v_i) = \big((i-1)n + j\big) + \big((i-1)n + j + np\big) \\ &= pn - 2n + 2ni + 2j. \end{aligned}$$

Thus the vertex-weights are all different numbers from the set

$$\{pn + 2, pn + 4, \dots, 3pn\}.$$

For the vertex v_i, $i = 1, 2, \dots, p$, we get

$$\begin{aligned} wt_f(v_i) &= \sum_{j=1}^{n} f(u_{i,j}v_i) + f(v_i) + \sum_{v_iv_k \in E(G)} f(v_iv_k) \\ &= \sum_{j=1}^{n} \big((i-1)n + j + np\big) + \big(g(v_i) + 2pn\big) \\ &\qquad + \sum_{v_iv_k \in E(G)} \big(g(v_iv_k) + 2pn\big) \\ &= n^2(i-1) + \sum_{j=1}^{n} j + n^2p + 2pn \\ &\qquad + \Bigg(g(v_i) + \sum_{v_iv_k \in E(G)} g(v_iv_k)\Bigg) + 2pn \cdot \deg_g(v_i) \\ &= \frac{n(n+1)}{2} + n^2p + 2pn\big(\deg_g(v_i) + 1\big) \\ &\qquad + wt_g(v_i) + n^2(i-1). \end{aligned}$$

As G is a regular graph, we have

$$\deg_g(v_i) = r, \qquad \text{for } i = 1, 2, \dots, p.$$

Thus

$$\begin{aligned} wt_f(v_i) &= \frac{n(n+1)}{2} + n^2p + 2pn\big(r + 1\big) + wt_g(v_i) + n^2(i-1) \\ &> 3pn. \end{aligned}$$

As $wt_g(v_i) \leq wt_g(v_{i+1})$, for $i = 1, 2, \dots, p-1$, this means that also the vertex-weights under the labeling f are all different.

□

Immediately from Theorem 2 and Lemmas 1 and 3, we obtain the following corollaries.

Corollary 5. *The double-star $S_{n,n} \cong K_2 \odot nK_1$ is TAT for every $n \geq 1$.*

Corollary 6. *The n-crown $C_m \odot nK_1$ is TAT for every $n \geq 1$, $m \geq 3$.*

Note that using Theorem 2 and Observation 1, we obtain that the star S_n, $S_n \cong K_1 \odot nK_1$, is TAT for every $n \geq 1$.

In a similar manner as in the proof of Theorem 2 and using Lemma 2, we can also prove the following result.

Corollary 7. *The graph $P_m \odot nK_1$ is TAT for every $n \geq 1$, $m \geq 2$.*

4 Union of graphs

In this section we will deal with disjoint union of regular TAT graphs.

Theorem 3. *Disjoint union of regular TAT graphs is a TAT graph.*

Proof. Let G_i be a r_i-regular graph of order $|V(G_i)|$ and size $|E(G_i)| = \frac{r_i|V(G_i)|}{2}$, $i = 1, 2, \ldots, m$. Let g_i, $i = 1, 2, \ldots, m$, be a TAT labeling of G_i. Thus,

$$g_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, |V(G_i)| + |E(G_i)|\}$$

such that

$$wt_{g_i}(v) \neq wt_{g_i}(u),$$

for all $v, u \in V(G_i)$, $u \neq v$ and

$$wt_{g_i}(e) \neq wt_{g_i}(h),$$

for all $e, h \in E(G_i)$, $e \neq h$.

Without loss of generality we assume that $r_{i+1} \geq r_i$, $i = 1, 2, \ldots, m-1$. We define a labeling f of $\bigcup_{i=1}^m G_i$ such that

$$f(x) = \begin{cases} g_1(x) & x \in V(G_1) \cup E(G_1) \\ g_i(x) + \sum_{j=1}^{i-1} |V(G_j)| + \sum_{j=1}^{i-1} |E(G_j)| & x \in V(G_i) \cup E(G_i), \quad i = 2, 3, \ldots, m. \end{cases}$$

It is easy to see that f is a total labeling of $\bigcup_{i=1}^m G_i$.

For the edge-weights under the labeling f we obtain

$$wt_f(e) = \begin{cases} wt_{g_1}(e) & e \in E(G_1) \\ wt_{g_i}(e) + 3\left(\sum_{j=1}^{i-1} |V(G_j)| + \sum_{j=1}^{i-1} |E(G_j)|\right) & e \in E(G_i), \quad i = 2, 3, \ldots, m. \end{cases}$$

As g_i, $i = 1, 2, \dots, m$ is edge-antimagic labeling, the edge-weights of all edges in $E(G_i)$ under the labeling f are pairwise distinct. Moreover, the maximum edge-weight of an edge $e \in E(G_i)$, $i = 1, 2, \dots, m$, is

$$wt_f^{max}(e) \le \sum_{k=1}^{3}\left(\sum_{j=1}^{i}|V(G_j)| + \sum_{j=1}^{i}|E(G_j)| + 1 - k\right)$$
$$= 3\left(\sum_{j=1}^{i}|V(G_j)| + \sum_{j=1}^{i}|E(G_j)|\right) - 3.$$

Thus f is an edge-antimagic labeling of $\bigcup_{i=1}^{m} G_i$.

For the vertex-weights under the labeling f, we get

$$wt_f(v) = \begin{cases} wt_{g_1}(v) & v \in V(G_1) \\ wt_{g_i}(v) + (r_i + 1)\left(\sum\limits_{j=1}^{i-1}|V(G_j)| + \sum\limits_{j=1}^{i-1}|E(G_j)|\right) & v \in V(G_i), i = 2, 3, \dots, m. \end{cases}$$

As g_i, $i = 1, 2, \dots, m$ is vertex-antimagic labeling, the vertex-weights of all vertices in $V(G_i)$ under the labeling f are pairwise distinct. The maximum vertex-weight of a vertex $v^i \in V(G_i)$ is

$$wt_f^{max}(v^i) \le \sum_{k=1}^{r_i+1}\left(\sum_{j=1}^{i}|V(G_j)| + \sum_{j=1}^{i}|E(G_j)| + 1 - k\right)$$
$$= (r_i + 1)\left(\sum_{j=1}^{i}|V(G_j)| + \sum_{j=1}^{i}|E(G_j)|\right) - \frac{r_i(r_i + 1)}{2}.$$

Moreover, the minimum vertex-weight of a vertex $v^{i+1} \in V(G_{i+1})$, $i = 1, 2, \dots, m-1$ is

$$wt_f^{min}(v^{i+1}) \ge \sum_{k=1}^{r_{i+1}+1}\left(\sum_{j=1}^{i}|V(G_j)| + \sum_{j=1}^{i}|E(G_j)| + k\right)$$
$$= (r_{i+1} + 1)\left(\sum_{j=1}^{i}|V(G_j)| + \sum_{j=1}^{i}|E(G_j)|\right) + \frac{(r_{i+1} + 1)(r_{i+1} + 2)}{2}.$$

As $r_{i+1} \ge r_i$, $i = 1, 2, \dots, m-1$, a labeling f is also a vertex-antimagic labeling of $\bigcup_{i=1}^{m} G_i$. □

Immediately, from the previous theorem, we obtain

Corollary 8. *If G is a regular TAT graph then mG is TAT for every $m \ge 1$.*

5 Conclusion

In this paper we have dealt with the problem of finding totally antimagic total labelings of graphs. Thus we were trying to find total labelings that are simultaneously vertex-antimagic total and edge-antimagic total. We showed the existence of such labelings for some classes of graphs, such as paths, cycles, stars, double-stars, wheels, etc. We also proved that a union of regular totally antimagic total graphs is a totally antimagic total graph.

For further investigation we state the following open problems.

Open problem 1. *Find some necessary and some sufficient conditions for a graph to be totally antimagic total.*

Open problem 2. *Characterize totally antimagic total graphs.*

Another interesting problem related to totally antimagic total labelings is to find total labeling that is simultaneously vertex-magic and edge-antimagic, or simultaneously vertex-antimagic and edge-magic total. Thus we conclude this paper with the following open problems.

Open problem 3. *Characterize the graphs that allow a total labeling that is simultaneously vertex-magic and edge-antimagic.*

Open problem 4. *Characterize the graphs that allow a total labeling that is simultaneously vertex-antimagic and edge-magic.*

Open problem 5. *Characterize the graphs that allow a total labeling in which the vertex-weights and edge-weights are all distinct.*

Note that some results related to Open problems 4 and 5 are presented in [7].

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