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Left-right arrangements, set partitions and pattern avoidance

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Abstract

We show structural properties of the system of ordered partitions of $[n] := \{1,\ldots,n\}$ all of whose left-to-right minima occur in odd locations, called *left-to-right arrangements*. Our main objectives are (i) to show that the set of all finite left-to-right arrangements is a projective system under a natural choice of restriction operation, (ii) to establish a non-trivial embedding of set partitions of $[n]$ into the set of left-to-right arrangements of $[n]$, and (iii) to illustrate how this embedding can be used to easily enumerate certain sets of pattern-avoiding set partitions.

1 Introduction

A *set partition* of $[n] := \{1,\ldots,n\}$ is a collection π of non-empty disjoint subsets $B_1,\ldots,B_k$ (called *blocks*) such that $\bigcup_{j=1}^k B_j = [n]$. The blocks are unordered, so we adopt the convention of listing them in ascending order of their smallest element; we write $\pi := B_1/\cdots/B_k$, where

$$\min B_1 < \cdots < \min B_k.$$

Alternatively, a partition π can be represented by its *restricted growth function* $\rho(\pi) := \rho_1\cdots\rho_n$, where ρ_i is the index of the block containing element i. For example, the partition $\pi = 156/28/349/7$ corresponds to $\rho(\pi) = 123311423$. Throughout the paper, we write $\mathcal{P}_n$ to denote the collection of set partitions of $[n]$.

An *ordered partition* of $[n]$ is an ordered collection $(B_1,\ldots,B_k)$ of non-empty, disjoint classes for which $\bigcup_{j=1}^k B_j = [n]$. As ordered partitions, $(13,24,5),(24,5,13)$, and $(5,24,13)$ are different objects, though their classes determine the same set partition $13/24/5$.

We highlight some little known properties of *ordered partitions whose left-to-right minima occur at odd locations*, shortened to *left-to-right arrangements* or *left-right arrangements*, and illustrate how these properties relate to certain structural properties of set partitions and permutations. Formally, an ordered partition $\alpha := (\alpha_1, \ldots, \alpha_k)$ is a *left-to-right arrangement* if, for each $1 \leq j \leq k$, the minimum of $\alpha_1 \cup \cdots \cup \alpha_j$ occurs in α_i, where i is an odd index between 1 and j. For the ordered partitions above, $(13, 24, 5)$ and $(24, 5, 13)$ are left-right arrangements and $(5, 24, 13)$ is not, because the minimum of the first two classes occurs in the second class. We write $\mathcal{A}_n$ to denote the set of left-right arrangements of $[n]$. The sets $\{\mathcal{A}_n\}_{n\geq 1}$ are enumerated by the exponential generating function

$$A(x) := \sum_{n\geq 0} \# \mathcal{A}_n \, x^n/n! = \sqrt{\frac{e^x}{2-e^x}},$$

with the convention $\#\mathcal{A}_0 = 1$; see [14]:A014307 and [6, 12].

Left-to-right arrangements possesses nice combinatorial structure, which has not been widely studied. In particular, $\{\mathcal{A}_n\}_{n\geq 1}$ is a projective system under a restriction operation that combines aspects of more familiar operations for set partitions and permutations. Furthermore, there is a natural correspondence between partitions of $[n]$ and the subset of contiguous, inversion-free left-to-right arrangements of $[n]$. These two observations are the subject of Section 2. In Section 3, we use these structural relationships to study occurrences of certain patterns in set partitions and left-right arrangements. Using our correspondence between partitions and left-right arrangements, we give an easy alternative proof for the number of partitions in the Wilf equivalence class of the pattern 12312. We also use the combinatorial structure of $\{\mathcal{P}_n\}_{n\geq 1}$ to derive a family of enumerative triangles $\{T_k(n, m)\}_{n\geq 1}$ connected to $12 \cdots k(k-1)$-avoiding partitions. In Section 4, we list parts of these triangles for small values of k.

2 Projective structure of the space of left-to-right arrangements

2.1 Projective structure of set partitions and permutations

The collection $\{\mathcal{P}_n\}_{n\geq 1}$ of finite set partitions enjoys a projective structure under the *deletion* operation defined as follows. We project $\pi \in \mathcal{P}_n$ into $\mathcal{P}_{n-1}$ by deleting element n from its block and keeping the rest of π unchanged; if $\{n\}$ appears in π as a singleton, the resulting empty set is removed. Formally, we define the deletion operation by $\mathbf{D}_{m,n} : \mathcal{P}_n \to \mathcal{P}_m$, for each $m \leq n$, where

$$\mathbf{D}_{m,n}\, \pi := \{B_1 \cap [m], \ldots, B_k \cap [m]\} \setminus \{\emptyset\}, \quad \pi \in \mathcal{P}_n .$$

For example, with $\pi = 145/23/678$, we have $\mathbf{D}_{7,8}\pi = 145/23/67$ and $\mathbf{D}_{4,8}\pi = 14/23$. When representing $\pi \in \mathcal{P}_n$ by its restricted growth function $\rho(\pi) = \rho_1 \cdots \rho_n$, restriction to $\mathcal{P}_m$ is obtained by removing the last $n - m$ elements of $\rho(\pi)$, i.e., $\rho(\mathbf{D}_{m,n}\, \pi) = \rho_1 \cdots \rho_m$.

A *permutation* of $[n]$ is a one-to-one and onto mapping $\sigma : [n] \to [n]$. We write $\mathcal{S}_n$ to denote the collection of permutations of $[n]$. We can represent $\sigma \in \mathcal{S}_n$ as either a list $\sigma_1 \cdots \sigma_n$, where $\sigma_i := \sigma^{-1}(i)$ is the element of $[n]$ assigned to location $i = 1, \ldots, n$, or as a product of cycles $\sigma := c_1 \cdots c_k$, where

$$c_j := (i_j \sigma(i_j) \sigma^2(i_j) \cdots \sigma^{k_j - 1}(i_j))$$

denotes the jth cycle of σ, which begins with the minimum element not appearing in cycles $c_1, \ldots, c_{j-1}$ and is obtained by iterating σ $(k_j - 1)$-times, where k_j is the smallest integer for which $\sigma^{k_j}(i_j) = i_j$. For example, the permutation 231564 is written as (132)(465) in cycle notation.

Restriction of a permutation $\sigma \in \mathcal{S}_n$ to $\mathcal{S}_{n-1}$ can be defined in at least two inequivalent ways. In this paper, we call attention to the *delete-and-repair* definition of restriction, which is suitable to the cycle representation. For $n \geq 1$, we define $\mathbf{R}_{n-1,n} : \mathcal{S}_n \to \mathcal{S}_{n-1}$ by $\sigma' := \mathbf{R}_{n-1,n}\,\sigma$, where

$$\sigma'(i) := \begin{cases} \sigma(n), & \sigma(i) = n \\ \sigma(i), & \text{otherwise.} \end{cases}$$

In cycle notation, the delete-and-repair operation amounts to deleting element n from its cycle and leaving the rest of σ unchanged, e.g., the restriction of $\sigma = (132)(465)$ to $\mathcal{S}_5$ is obtained by removal of element 6, $\mathbf{R}_{5,6}\sigma = (132)(45)$.

2.2 The system of left-to-right arrangements

For the collection $\{\mathcal{A}_n\}_{n\geq 1}$ of left-to-right arrangements, we define *restriction* by combining the deletion and delete-and-repair operations for set partitions and permutations, respectively. Each $\alpha \in \mathcal{A}_n$ has attributes of both a set partition and a permutation, and care is needed to ensure that the restriction of α satisfies the left-to-right minimum condition. We divide $\mathcal{A}_n$ into the four cases (I)-(IV) below and describe the restriction rule in each case separately.

In words, the cases are: (I) element n occurs in a class with at least one other element; (II) the singleton $\{n\}$ occurs to the right of the class containing 1 and is between classes appearing in ascending order of their minima; (III) the singleton $\{n\}$ occurs to the left of the class containing 1 and is between classes appearing in ascending order of their minima; (IV) the singleton $\{n\}$ is between classes appearing in descending order of their minima—in this case, we say n is part of an *inversion.*

Formally, let $\alpha := (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}_n$. We define the restriction of α to $\mathbf{A}_{n-1,n}\alpha = \alpha' := (\alpha'_1, \ldots, \alpha'_{k'}) \in \mathcal{A}_{n-1}$ as follows. For $j \in [n]$, let I_j denote the index of the class of α containing j and, in particular, let $I^* := I_1$ denote the index of the class of α containing 1. Furthermore, let $m_j := \min \alpha_j$, $j = 1, \ldots, k$, denote the minimum element of class j. Then either

(I) α_{I_n} is not a singleton,

(II) $I_n > I^*$, α_{I_n} is a singleton, and $m_{I_n - 1} < m_{I_n + 1}$,

(III) $I_n < I^*$, α_{I_n} is a singleton, and $m_{I_n - 1} < m_{I_n + 1}$, or

(IV) α_{I_n} is a singleton and $m_{I_n - 1} > m_{I_n + 1}$.

In each case, we obtain $\alpha' := \mathbf{A}_{n-1,n}\alpha$ as follows.

(I) We put $\alpha'_j := \alpha_j \cap [n-1]$ for $j = 1, \dots, k$.

(II) We put $\alpha'_j := \alpha_j$ for $j < I_n$ and $\alpha'_j := \alpha_{j+1}$ for $j \geq I_n$.

(III) – (IV) We put $\alpha'_j := \alpha_j$ for $j < I_n - 1$, $\alpha'_{I_n-1} := \alpha_{I_n-1} \cup \alpha_{I_n+1}$, and $\alpha'_j := \alpha_{j+2}$ for $j \geq I_n$.

Cases (I) and (II) correspond to the usual *deletion* operation for set partitions, while cases (III) and (IV) correspond to a *delete-and-repair*-type operation. In cases (III)-(IV), either element n is part of an inversion (see Definition 2.3) or appears as a singleton to the *left* of element 1. In either situation, simple deletion can result in a shift of left-to-right minima by 1 index to the left, which would result in a minimum occurring in an even location. To avoid this, we *repair* such a removal by merging the classes on either side of $\{n\}$. Note that in case (IV), when $I_n > I^*$, simple deletion would not result in a violation of the left-to-right minima condition; however, this step is fundamental to the structure of $\{\mathcal{A}_n\}_{n\geq 1}$ because it deals with occurrences of inversions. The following example illustrates the restriction operation in each of the above cases.

Example 2.1. *Each of the following left-right arrangements of* $[7]$ *restricts to* $(23, 4, 1, 56)$ *under operations (I)-(IV).*

(I) $\alpha_I = (23, 47, 1, 56)$*:* $\{7\}$ *is not a singleton, so we apply the usual deletion rule for set partitions;*

(II) $\alpha_{II} = (23, 4, 1, 7, 56)$*:* $\{7\}$ *occurs as a singleton to the right of element 1 and* $\min\{1\} < \min\{5, 6\}$*;*

(III) $\alpha_{III} = (2, 7, 3, 4, 1, 56)$*:* $\{7\}$ *occurs as a singleton to the left of element 1;*

(IV) $\alpha_{IV} = (23, 4, 1, 6, 7, 5)$*:* $\{7\}$ *occurs as a singleton and* $\min\{6\} > \min\{5\}$*, i.e., 7 is part of the inversion* $(6, 7, 5)$*.*

Table 1 gives the restriction for all left-to-right arrangements of $\{1, 2, 3, 4\}$. Note that there is no instance of case (III) in $\mathcal{A}_4$. The first instances of case (III) are the left-right arrangements $(2, 3, 5, 4, 1), (2, 5, 3, 4, 1)$, and $(2, 5, 4, 3, 1)$. The restrictions of these to $\mathcal{A}_4$ are $(2, 34, 1), (23, 4, 1)$, and $(24, 3, 1)$, respectively.

Since cases (I)-(IV) exhaust all possibilities, it is clear that $\{\mathcal{A}_n\}_{n\geq 1}$ has projective structure under the above restriction operation. For $m \leq n$, we define $\mathbf{A}_{m,n} : \mathcal{A}_n \to \mathcal{A}_m$ by composition, $\mathbf{A}_{m,n} := \mathbf{A}_{m,m+1} \circ \cdots \circ \mathbf{A}_{n-1,n}$.

Theorem 2.2. *The collection* $\{\mathcal{A}_n\}_{n\geq 1}$ *of left-to-right arrangements is a projective system under the deletion scheme given in (I)-(IV).*

Proof. We need only show that for each $m \leq n$ there is a well-defined projection $\mathbf{A}_{m,n} : \mathcal{A}_n \to \mathcal{A}_m$ such that $\mathbf{A}_{l,m} \circ \mathbf{A}_{m,n} = \mathbf{A}_{l,n}$ whenever $l \leq m \leq n$. But this is obvious since we have defined $\mathbf{A}_{m,n} := \mathbf{A}_{m,m+1} \circ \cdots \circ \mathbf{A}_{n-1,n}$. Since we have chosen cases (I)-(IV) so that α' remains a left-right arrangement, each $\alpha \in \mathcal{A}_n$ corresponds to a unique element $\alpha' \in \mathcal{A}_{n-1}$ such that $\alpha' = \mathbf{A}_{n-1,n}\alpha$. $\square$

$\alpha^* \in \mathcal{A}_3$	$\{\alpha \in \mathcal{A}_4 : \mathbf{A}_{3,4}\alpha = \alpha^*\}$					
123	1234	123,4	23,4,1	2,4,13	3,4,12	
12,3	124,3	12,4,3	12,34	12,3,4	2,4,1,3	
1,23	14,23	1,4,23	1,234	1,23,4	1,3,4,2	
13,2	134,2	13,4,2	13,24	13,2,4	3,4,1,2	
1,2,3	14,2,3	1,4,2,3	1,24,3	1,2,4,3	1,2,34	1,2,3,4
1,3,2	14,3,2	1,4,3,2	1,34,2	1,3,24	1,3,2,4	
2,3,1	24,3,1	2,34,1	2,3,14	2,3,1,4		

Table 1: Table showing the 7 left-right arrangements of $\{1, 2, 3\}$ in the leftmost column. Within each row is the set of left-right arrangements of $\{1, 2, 3, 4\}$ that restricts to the corresponding element in the leftmost column.

2.3 Representing set partitions by left-to-right arrangements

We have already described two equivalent ways to represent a set partition, as a collection of disjoint blocks and by its restricted growth function. There are several other ways to represent set partitions, e.g., by labeling points $1, \ldots, n$ consecutively on a circle and drawing a line between labels that appear consecutively within a block and a line from the largest to the smallest element within each block. Likewise, we can label n points in a horizontal line and draw an arc between consecutive elements within the same block. Yet another representation is by a *rook placement*, which is an arrangement of points on an $(n-1) \times (n-1)$ lower triangular grid so that no two points appear in the same row or column.

We now describe another representation in terms of left-to-right arrangements with certain properties. As we show, this representation is natural in that it respects the projective structure of both $\{\mathcal{P}_n\}_{n\geq 1}$ and $\{\mathcal{A}_n\}_{n\geq 1}$. We later use this representation to give an alternative enumeration of 12312-avoiding partitions.

Definition 2.3. *For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}_n$, let $m(\alpha)$ be the list of minima of α, i.e., $m(\alpha) := m_1 \cdots m_k$, where $m_j := \min \alpha_j$ for each $j = 1, \ldots, k$. An* inversion *in α is a triple $(\alpha_{i-1}, \alpha_i, \alpha_{i+1})$, $i = 2, \ldots, k-1$, for which $m_{i+1} < m_{i-1} < m_i$. Let $(\alpha^{[1]}, \ldots, \alpha^{[n]})$ be the sequence of restrictions of α under operations (I)-(IV). We call α* inversion-free *if none of its restrictions contains an inversion, that is, there is no pair $j = 1, \ldots, n$ and $i = 2, \ldots, k-1$ for which $(\alpha^{[j]}_{i-1}, \alpha^{[j]}_i, \alpha^{[j]}_{i+1})$ is an inversion.*

A characteristic of inversion-free left-right arrangements is that element 1 appears in the first class; however, this criterion does not determine the collection of inversion-free left-right arrangements. For example, $(1, 4, 3, 2)$ is inversion-free and $(1, 3, 4, 2)$ is not. Note also that α might contain no inversions but fail to be inversion-free. For example, $\alpha = (2, 5, 4, 3, 1)$ has no inversions, but $\alpha^{[4]} = (24, 3, 1)$ does; therefore, α is not inversion-free.

Definition 2.4. *For any finite subset $A \subset \mathbb{N}$, let $m := \min A$ and $M := \max A$. We call A* contiguous *if $A - m + 1 := \{a - m + 1 : a \in A\} = \{1, \ldots, M - m + 1\}$. In other words, there is some $m \in \mathbb{N}$ such that $A := \{m, m+1, \ldots, M\}$ consists of* *consecutive integers from m to M. A left-right arrangement is called* contiguous *if each of its classes is contiguous.*

We establish a bijection between partitions of $[n]$ and left-right arrangements of $[n]$ that are both contiguous and inversion-free. To initialize, we put the partition 1 into correspondence with the left-right arrangement (1). Now, for $\pi' \in \mathcal{P}_n$, let $\pi := \mathbf{D}_{m,n}\,\pi'$ be its restriction to $\mathcal{P}_m$, $m < n$, and let $\alpha \in \mathcal{A}_m$ be the left-right arrangement associated to π. We obtain $\alpha^* \in \mathcal{A}_{m+1}$ corresponding to $\pi^* := \mathbf{D}_{m+1,n}\pi'$ as follows. We write $\pi^* := B_1/\cdots/B_k$ and $\alpha := (\alpha_1, \ldots, \alpha_r)$. We also let $m_1, \ldots, m_r$ denote the minima of $\alpha_1, \ldots, \alpha_r$, respectively, $\alpha_M \in \alpha$ denote the class containing element m, and $i_1 < \cdots < i_{k-1}$ be the indices for which $m_{i_j} := \min B_j$, for each $j = 1, \ldots, k-1$. We insert $m+1$ into α to obtain α^* as follows.

(a) If m and $m+1$ are in the same block of π^*, we insert $m+1$ into α_M;

(b) if $m+1$ is a singleton in π^*, we insert $\{m+1\}$ as a new class at the end of α, i.e., $\alpha \mapsto \alpha^* := (\alpha_1, \ldots, \alpha_r, \{m+1\})$;

(c) if $m+1$ is in B_k (the last block of π^*), $\{m+1\}$ is not a singleton of π^*, and $m \notin B_k$, then we insert $\{m+1\}$ as a new class to the immediate left of α_M (the class containing m);

(d) otherwise, let m' be the minimum element of the block containing $m+1$ in π^*; we insert $\{m+1\}$ as a singleton class immediately to the right of α_I, where I is the index of the class containing m' in α.

The following example illustrates the above procedure for a partition of nine elements.

Example 2.5. *Consider the partition* $\pi := 1345/268/7$*, which corresponds to* $\alpha = 1, 6, 345, 2, 8, 7$ *by (a)-(d). According to the above procedure, we can obtain a left-right arrangement of* $[9]$ *by inserting the element 9 in one of four places.*

- *For* $\pi' := 13459/268/7$*, we are in case (d) above and we place* $\{9\}$ *to the immediate right of the class containing* $\min\{1, 3, 4, 5, 9\} = 1$ *to obtain* $\alpha^* := (1, 9, 6, 345, 2, 8, 7)$.
- *For* $\pi' := 1345/2689/7$*, we are in case (a) and we insert* 9 *in the same class as 8 to obtain* $\alpha^* := (1, 6, 345, 2, 89, 7)$.
- *For* $\pi' := 1345/268/79$*, we are in case (c) and we place* $\{9\}$ *to the left of the class containing* 8*, i.e.,* $\alpha^* := (1, 6, 345, 2, 9, 8, 7)$.
- *For* $\pi' := 1345/268/7/9$*, we are in case (b) and we obtain* $\alpha^* := (1, 6, 345, 2, 8, 7, 9)$.

Table 2 shows this correspondence for partitions of $\{1, 2, 3, 4\}$.

partition	left-right arrangement
1234	1234
123/4	123,4
124/3	12,4,3
134/2	1,34,2
1/234	1,234
12/34	12,34
13/24	1,4,3,2
14/23	1,4,23
1/2/34	1,2,34
1/23/4	1,23,4
1/24/3	1,2,4,3
12/3/4	12,3,4
13/2/4	1,3,2,4
14/2/3	1,4,2,3
1/2/3/4	1,2,3,4

Table 2: Correspondence between partitions and left-right arrangements of $\{1, 2, 3, 4\}$. Note that partition 13/24 is an occurrence of case (c) above, for which we obtain the left-right arrangement 1,4,3,2.

In the following proposition, let $\mathcal{A}_n^*$ be the subset of contiguous, inversion-free left-right arrangements.

Proposition 2.6. *Items (a)-(d) above establish a natural correspondence between $\{\mathcal{P}_n\}_{n\geq 1}$ and $\{\mathcal{A}_n^*\}_{n\geq 1}$. That is, for $\pi \in \mathcal{P}_n$, let $\alpha = \alpha(\pi)$ be its corresponding left-right arrangement according to (a)-(d). Then α is uniquely determined by π and, for each $m \leq n$, $\mathbf{A}_{m,n}\alpha(\pi) = \alpha(\mathbf{D}_{m,n}\pi)$.*

Proof. Fix $\pi \in \mathcal{P}_n$ and let $\alpha = \alpha(\pi)$ be the left-right arrangement obtained by applying (a)-(d) above. Then α is clearly an element of $\mathcal{A}_n$, because its first class contains 1 and there is no concern about left-to-right minima. Furthermore, new classes always start to the immediate right of a class containing a right-to-left minimum and so α is inversion-free since an inversion requires a consecutive 2-3-1 pattern in class minima. That α is contiguous is plain since, for every $m \geq 1$, m and $m+1$ are either in the same class of α, as in case (a), or $\{m+1\}$ is inserted as a singleton class in α.

Conversely, let $\alpha := (\alpha_1, \ldots, \alpha_r)$ be a contiguous, inversion-free left-right arrangement. Then we associate it to a partition by inverting (a)-(d) above. In particular, let $\alpha_{(1)}, \ldots, \alpha_{(r)}$ be the *ordered* classes of α so that $\min \alpha_{(1)} < \min \alpha_{(2)} < \cdots < \min \alpha_{(r)}$. We recursively associate α to $\pi \in \mathcal{P}_n$ as follows. For $j < r$, let $\pi^{(j)} := B_1^{(j)}/\cdots/B_{k_j}^{(j)}$ be a partition of $\alpha_{(1)} \cup \cdots \cup \alpha_{(j)}$. In α, the next class $\alpha_{(j+1)}$ must occur either between two classes $\alpha_{(i)}$ and $\alpha_{(i')}$, for $1 \leq i \neq i' \leq r$, or $\alpha_{(j+1)}$ must be to the right of each $\alpha_{(i)}$, $i = 1, \ldots, j$.

(a') If $\alpha_{(j+1)}$ is to the right of every $\alpha_{(1)}, \ldots, \alpha_{(j)}$, then we append the set $\alpha_{(j+1)}$ to $\pi^{(j)}$ as its own block, $\pi^{(j)} \mapsto \pi^{(j+1)} := B_1^{(j)}/\cdots/B_{k_j}^{(j)}/\alpha_{(j+1)}$.

(b') If $\alpha_{(j+1)}$ is to the immediate left of the class b containing $\min \alpha_{(j+1)} - 1$ and b is not part of the last block of $\pi^{(j)}$, then $\alpha_{(j+1)}$ is combined with the last block of $\pi^{(j)}$, $\pi^{(j)} \mapsto \pi^{(j+1)} := B_1^{(j)}/\cdots/B_{k_j}^{(j)} \cup \alpha_{(j+1)}$.

(c') Otherwise, we combine $\alpha_{(j+1)}$ with the block of $\pi^{(j)}$ containing the rightmost class to the left of $\alpha_{(j+1)}$ in α.

Under this bijection, the minimal elements of the blocks of π correspond to the right-to-left minima of the left-right arrangement. Furthermore, in case (c'), the class immediately to the left of $\alpha_{(j+1)}$ must contain a right-to-left minimum. It is clear that the maps (a)-(d) and (a')-(c') are inverse to one another, establishing the desired bijection.

That the restriction operation on $\{\mathcal{A}_n\}_{n\geq 1}$ commutes with deletion on $\{\mathcal{P}_n\}_{n\geq 1}$ should be clear by our construction: since each $\pi \in \mathcal{P}_n$ corresponds to a left-right arrangement that is inversion-free, we are always in cases (I)-(II) of the restriction scheme, which are compatible with deletion for set partitions. □

Example 2.7. *As an illustration of (a)-(d) and the inverse map (a')-(c'), consider* $\alpha = (1, 5, 4, 3, 2)$. *To construct the corresponding partition* $\pi = \pi(\alpha)$, *we take the following steps. Since all classes are singletons, we have* $\alpha_{(i)} = \{i\}$ *for* $i = 1, 2, 3, 4, 5$ *and proceed as follows.*

(1) We begin with $\pi^{(1)} = 1$.

(2) Apply (a') to get $\pi^{(2)} = 1/2$.

(3) Apply (c') to get $\pi^{(3)} = 13/2$.

(4) Since 3 *is not in the last block of* $\pi^{(3)}$, *we then apply (b') to get* $\pi^{(4)} = 13/24$.

(5) Since 4 *is in the last block of* $\pi^{(4)}$, *we apply (c') to get* $\pi = \pi^{(5)} = 135/24$.

In reverse, we start with $\pi = 135/24$ *and construct* $\alpha = \alpha(\pi)$ *recursively.*

(1) We begin with $\alpha^{(1)} = (1)$.

(2) Since 2 *is a singleton in* $\pi^{(2)} = 1/2$, *we apply (b) to get* $\alpha^{(2)} = (1, 2)$.

(3) The minimum element of the block containing 3 *is 1, so we apply (d) to get* $\alpha^{(3)} = (1, 3, 2)$.

(4) Since 4 *appears in the last block of* $\pi^{(4)} = 13/24$, *is not a singleton, and* 3 *is not in the last block of* $\pi^{(4)}$, *we apply (c) to get* $\alpha^{(4)} = (1, 4, 3, 2)$.

(5) We apply (d) again to get $\alpha = \alpha^{(5)} = (1, 5, 4, 3, 2)$.

3 Partition patterns

We conclude the paper with a discussion of patterns for set partitions and left-right arrangements. We preface this section with a discussion of stack-sorting for partitions and left-to-right arrangements.

3.1 Sorting left-to-right arrangements

Given a list $l = l_1 \cdots l_n$ of (not necessarily distinct) positive integers, we define the *reduction* of l as the sequence $\text{REDUCE}(l_1 \cdots l_n) = i_1 \cdots i_n$, where $i_j \in [k]$, $k := \#\{l_1, \ldots, l_n\}$, and, for each pair j, j',

$$i_j \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} i_{j'} \quad \Longleftrightarrow \quad l_j \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} l_{j'}.$$

For example, the sequence $l = 4824385$ contains 5 distinct integers $2 < 3 < 4 < 5 < 8$ and $\text{REDUCE}(4824385) = 3513254$.

Let $\tau := \tau_1 \cdots \tau_k$ be a permutation of $[k]$, called a *permutation pattern*, and let $\sigma \in \mathcal{S}_n$. We say σ *contains* τ, denoted $\sigma \sim \tau$, if there exist indices $i_1 < i_2 < \cdots < i_k$ such that $\text{REDUCE}(\sigma_{i_1} \cdots \sigma_{i_k}) = \tau$. For example, the permutation 425136 contains the pattern 1324 since $\text{REDUCE}(2536) = 1324$. If σ does not contain τ, we say that σ *avoids* τ. The above permutation avoids the pattern 1432.

In a similar way, we define pattern avoidance for left-right arrangements through its list of class minima. That is, for $\alpha := (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}_n$, let $m_\alpha := m(\alpha) = (m_1, \ldots, m_k)$ be its ordered list of minima, where $m_i := \min \alpha_i$. Given a permutation τ, we say α *avoids* τ if $\text{REDUCE}(m_\alpha)$ avoids τ as a permutation.

A classical result in permutation patterns involves sorting permutations using a single stack, see Chapter 8.2 of Bóna [1]. Specifically, a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is *stack-sortable* if it can be rearranged into the sequence $12 \cdots n$ by only a single pass through the following algorithm.

- Beginning with $\sigma^0 = \sigma_1 \cdots \sigma_n$, an empty ordered list $\text{STACK}^0 := ()$, and an empty ordered list $\sigma'^0 := ()$, we move the leftmost element of σ^0, in this case σ_1, to the front of the stack, $\text{STACK}^0 \mapsto \text{STACK}^1 := (\sigma_1)$, and update $\sigma'^0 \mapsto \sigma'^1 := ()$, and $\sigma^0 \mapsto \sigma^1 := \sigma_2 \cdots \sigma_n$.

- At step j, given $\sigma^j = \sigma_{j'} \cdots \sigma_n$, $\text{STACK}^j = (s_1, s_2, \ldots, s_{j''})$, and $\sigma'^j = \sigma'_1 \cdots \sigma'_{j^*}$, we choose either

  - to move $\sigma_{j'}$ to the front of STACK^j, $\text{STACK}^j \mapsto (\sigma_{j'}, s_1, \ldots, s_{j''})$, or
  - to move s_1 to the end of σ'^j, $\sigma'^j \mapsto \sigma'_1 \cdots \sigma'_{j^*} s_1$.

  If one of σ^j and STACK^j is empty, we are forced to perform the other operation. If both are empty, we conclude the algorithm and output $\sigma' = \sigma'^j$. We say σ has been sorted if $\sigma' = 12 \cdots n$.

It is well-known that a permutation is stack-sortable if and only if it avoids the pattern 231. We can easily extend the notion of stack-sortability to set partitions and left-right arrangements by treating the blocks of $\pi \in \mathcal{P}_n$, alternatively the classes of $\alpha \in \mathcal{A}_n$, as indivisible atoms in the above stack-sorting algorithm. In this case, the output of the algorithm will be a partition π' of $[n]$. For $\pi = B_1 / \cdots / B_k \in \mathcal{P}_n$, we define the *flattening* of π, denoted $\text{FLATTEN}(\pi)$, as the permutation obtained by removing the block dividers and listing elements in increasing order within each block. For example, FLATTEN(135/24/6/7) = 1352467. If there is a way to pass through the above algorithm such that $\text{FLATTEN}(\pi') = 12\cdots n$, then we say π, respectively α, is stack-sortable.

By this description, it is clear that a partition, respectively left-right arrangement, is sortable only if its blocks are contiguous. For example, the left-right arrangement $(145, 2, 3, 678)$ is not sortable, because the block 145 has gaps in it. On the other hand, though it is contiguous, the left-right arrangement $(1, 45, 3, 678, 2)$ is not sortable because in order to get the sequence 12 in σ', we must put the block 678 in front of 3 in the stack. However, $(1, 45, 3, 2, 678)$ is sortable since we obtain the partition $\pi' = 1/2/3/45/678$ after a single run through the above algorithm and $\text{FLATTEN}(\pi') = 12345678$.

Proposition 3.1. *$\alpha \in \mathcal{A}_n$ is sortable if and only if α is contiguous and avoids the pattern 231.*

Proof. Necessity is clear. First, if any class of α is not contiguous, then α cannot be sorted by any means. Second, if α contains 231, then the "3"-class must end up in front of the "2"-class on the stack, which precludes sorting.

For sufficiency, let $\alpha := (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}_n$ avoid 231 and be contiguous. Then α can be encoded as a permutation of $[k]$ in the obvious way. Let $m_1, \ldots, m_k$ be the minima of the classes of α and define $\sigma := \text{REDUCE}(m_1 \cdots m_k)$. Then σ is a 231-avoiding permutation of $[k]$, which is known to be sortable. Once σ is sorted, we obtain an ordering of $[n]$ by substituting the classes $\alpha_1, \ldots, \alpha_k$ for their corresponding element of $[k]$. By contiguity of α, we recover $1 \cdots n$. This completes the proof. □

Let $\text{SORT}(\mathcal{A}_n)$ denote the set of sortable left-right arrangements of $[n]$. The sets $\{\text{SORT}(\mathcal{A}_n)\}_{n\geq 1}$ can be easily enumerated, as we show in the next section. The list of all left-right arrangements of $\{1, 2, 3, 4\}$, along with their contained 231 patterns, is given in Table 3.

3.2 Pattern avoidance for set partitions

The current state of research on pattern avoidance for set partitions is summarized in Chapter 6 of [8], which contains contributions of Mansour and his coauthors as well as others, see e.g., [2, 3, 4, 5, 9, 10, 13]. Pattern avoidance for set partitions is a natural outgrowth of the large industry of pattern avoidance for permutations; see Bóna's book [1] for a survey of this literature. We now briefly discuss pattern avoidance for set partitions within the context of left-right arrangements and projective structure of $\{\mathcal{P}_n\}_{n\geq 1}$. Except for $T_k(n, m)$ in Theorem 3.4, none of the enumerative results here are new; however, our arguments involving a simple recurrence in terms of ordered partition patterns (rather than restricted growth patterns) have not appeared previously. The triangular arrays $\{T_k(n, m)\}_{1\leq m\leq n}$ refine the expression for $\#\mathcal{P}_n(12\cdots k(k-1))$ to $\#\mathcal{P}_n(12\cdots k(k-1); m)$, the number of $12\cdots k(k-1)$-avoiding partitions of $[n]$ with exactly m blocks. By the Wilf-equivalence of patterns

1234	**avoids**	124,3	**avoids**	24,3,1	231
134,2	**avoids**	14,23	**avoids**	14,2,3	**avoids**
14,3,2	**avoids**	123,4	**avoids**	12,4,3	**avoids**
2,34,1	231, 241	13,4,2	342	1,4,23	**avoids**
1,24,3	**avoids**	1,34,2	**avoids**	2,4,13	241
12,34	**avoids**	2,3,14	231	13,24	**avoids**
1,234	**avoids**	1,2,34	**avoids**	1,3,24	**avoids**
3,4,12	341	12,3,4	**avoids**	2,3,1,4	231
13,2,4	**avoids**	1,23,4	**avoids**	1,2,3,4	**avoids**
1,4,3,2	**avoids**	23,4,1	241, 341	2,4,1,3	241
3,4,1,2	341, 342	1,3,4,2	342	1,2,4,3	**avoids**
1,3,2,4	**avoids**	1,4,2,3	**avoids**		

Table 3: Table of 231-avoiding left-right arrangements of $\{1, 2, 3, 4\}$. There are 24 left-right arrangements of $\{1, 2, 3, 4\}$ that avoid 231.

$12 \cdots k(k-1)$ and $12 \cdots k1$, the generating function for $\# \mathcal{P}_n(12 \cdots k(k-1))$ is given in [9]; however, the triangles $\{T_k(n, m)\}_{1 \leq m \leq n}$ appear to be novel. We also give recurrence arguments for other specific partition patterns.

Let $\tau = \tau_1 \cdots \tau_k$ be the restricted growth function of some partition of $[k]$, called a *restricted growth pattern*. We say $\pi \in \mathcal{P}_n$ *aviods* τ if $\rho(\pi)$ avoids τ in the sense described above. For example, the set partition $\pi = 15/2/34/6$ has $\rho(\pi) = 123314$ and contains the patterm $\tau = 1223$ because the subsequence with $(i_1, i_2, i_3, i_4) = 1346$ yields $\textsc{Reduce}(1334) = 1223$. This is the only occurrence of 1223 in π. We write

$$\mathcal{P}_n(\tau) := \{\pi \in \mathcal{P}_n : \pi \nsim \tau\}$$

to denote partitions of $[n]$ that avoid τ.

For partitions, we define a pattern differently than previous authors, although each of our patterns can be rewritten as a restricted growth pattern. For us, a pattern τ^* is an *ordered partition* of $[k]$, denoted $B_1 \text{-} B_2 \text{-} \cdots \text{-} B_m$. In this setting, we say $\pi := B_1/\cdots/B_k$ contains τ^* if there exist indices $i_1 < \cdots < i_m$ and subsets $b_j \subset B_{i_j}$, for each $j = 1, \ldots, m$, so that the reduction of $b_1 \text{-} b_2 \text{-} \cdots \text{-} b_m$ is τ^*. Here, we abuse terminology and speak of the *reduction* of an ordered partition rather than a sequence of integers. In this case, the reduction of $b_1 \text{-} \cdots \text{-} b_m$ is obtained by assigning each element to its *rank* among the elements of $b_1 \cup \cdots \cup b_m$. For example, take $\tau^* = 2\text{-}3\text{-}1$. Then the partition $\pi = 14/25/3$ has a single copy of τ^* by reducing $4\text{-}5\text{-}3$. For an ordered partition pattern τ^*, we adopt the same notation and write $\mathcal{P}_n(\tau^*)$ to denote the subset of partitions of $[n]$ that avoid τ^*. Since the underlying mechanism of both reduction operations is essentially identical, we do not anticipate any confusion.

To begin, we use their correspondence with sortable left-right arrangements to enumerate partitions avoiding $2\text{-}3\text{-}1$.

Theorem 3.2. $\text{Sort}(\mathcal{A}_n)$ *is in bijection with* $\mathcal{P}_n$*(2-3-1) and*

$$\#\text{Sort}(\mathcal{A}_n) = \sum_{k=0}^{n-1} \binom{n-1}{k} Cat_k,$$

where $\{Cat_k\}_{k\geq 1}$ *are the Catalan numbers [14]:A000108.*

Proof. Let $\mathcal{S}_n(231)$ denote the set of 231-avoiding permutations. It is known, e.g., [1], that $\#\,\mathcal{S}_n(231) = \text{Cat}_n$ for each $n \geq 1$. Given $0 \leq k \leq n-1$, we can easily obtain a sortable 231-avoiding left-right arrangement with $k+1$ classes as follows. There is only one arrangement with a single class and so the $k = 0$ case is trivial. Assuming $k \geq 1$, we begin by choosing a subset of size k from $\{2, \ldots, n\}$ and arranging its elements in the order of a 231-avoiding permutation. Let $c_1 \cdots c_k$ be this permutation and let $c_{(1)} < \cdots < c_{(k)}$ be these elements listed in increasing order. For each $i = 1, \ldots, k-1$, define $C_{(i)} := [c_{(i)}, c_{(i+1)})$, $C_{(k)} := [c_{(k)}, n]$ and $C_0 := [1, c_{(1)})$, where we write $[m, M) := \{m, m+1, \ldots, M-1\}$, $m < M$. We then put $\alpha := (C_0, C_1, \ldots, C_k)$, with $C_1, \ldots, C_k$ listed in the order corresponding to $c_1 \cdots c_k$. Clearly, α avoids 231 since $c_1 \cdots c_k$ does, and α is contiguous by construction; therefore, α is sortable. By inverting the above procedure, each $\alpha \in \text{Sort}(\mathcal{A}_n)$ gives rise to a unique 231-avoiding permutation of some subset of $\{2, \ldots, n\}$.

By the correspondence between $\mathcal{P}_n$ and contiguous, inversion-free arrangements, it is clear that $\pi \in \mathcal{P}_n$ avoids 2-3-1 if and only if its corresponding arrangement avoids 231, because an occurrence of 231 in a contiguous, inversion-free arrangement cannot be the result of an inversion. □

By Callan [2], we also have a bijection between sortable left-right arrangements and 321-avoiding flattened partitions. Using a different argument, $\mathcal{P}_n$(2-3-1) has been enumerated previously under the guise of $\mathcal{P}_n(12312)$, since a partition avoids 12312 if and only if it avoids 2-3-1; see [9].

We now discuss the family of $\tau_k := 12 \cdots k(k-1)$ avoiding partitions. It is known (Theorem 6.67 of [8]) that $\#\,\mathcal{P}_n(\tau_k) = \#\,\mathcal{P}_n(12\cdots(k+1))$. Therefore, $\#\,\mathcal{P}_n(\tau_k) := \sum_{j=1}^{k} S(n,j)$ is the number of partitions of $[n]$ with k or fewer blocks, where $S(n,j)$ is the (n,j)-Stirling number of the second kind. In the next theorem, we rederive the number of partitions avoiding τ_k, for $k \geq 2$. We do so by setting up a recurrence relation for $T_k(n,m)$, the number of partitions of $[n]$ that both avoid τ_k and have exactly m blocks. We obtain this by studying avoidance of the ordered partition pattern $\tau_k^* := 1\text{-}2\text{-}\cdots\text{-}(k-2)\text{-}k\text{-}(k-1)$, which is equivalent to avoidance of τ_k.

Lemma 3.3. *Let* $\tau_k := 12 \cdots k(k-1)$ *and* $\tau_k^* := 1\text{-}2\text{-}\cdots\text{-}(k-2)\text{-}k\text{-}(k-1)$*, then* $\mathcal{P}_n(\tau_k) = \mathcal{P}_n(\tau_k^*)$ *for all* $k \geq 3$.

Proof. Suppose $\pi \in \mathcal{P}_n$ contains τ_k. Then there is a subset $A \subseteq [n]$ with $k+1$ elements such that the restriction of π to A reduces to $1/2/\cdots/(k-1)(k+1)/k$, which corresponds to the ordered partition pattern τ_k^*. Conversely, if $\pi \in \mathcal{P}_n$ contains τ_k^*, then there is a subpartition with reduction $1/2/\cdots/(k-1)(k+1)/k$, which corresponds exactly to τ_k. □

Theorem 3.4. *For $k \geq 1$, let τ_k and τ_k^* be as in the preceding lemma. For each $n \geq 1$ and $1 \leq m \leq n$, let*

$$T_k(n,m) := \#\{\pi \in \mathcal{P}_n : \pi \not\sim \tau_k^* \text{ and } \#\pi = m\}$$

be the number of partitions of $[n]$ that avoid τ_k^* and have exactly m blocks. Then $\#\mathcal{P}_n(\tau_k^*) := \sum_{m=1}^n T_k(n,m)$, where

$$T_k(n,m) := ((k-1) \wedge m)T_k(n-1,m) + T_k(n-1,m-1), \tag{1}$$

and we put $T_k(1,1) = 1$ and $T_k(n,m) = 0$ for m outside the range $1 \leq m \leq n$. We write $(k-1) \wedge m$ to denote the minimum of $k-1$ and m.

For fixed $m \geq 1$, let $G_k(x;m) := \sum_{n=1}^\infty T_k(n,m)x^n$ be the generating function for the mth column of $\{T_k(n,m)\}_{1 \leq m \leq n}$. Then

$$G_k(x;m) = \frac{x^m}{(1-kx)^{m-((k-1)\wedge m)} \prod_{j=1}^{(k-1)\wedge m}(1-jx)}. \tag{2}$$

Proof. As in the statement of the theorem, let

$$T_k(n,m) := \#\{\pi \in \mathcal{P}_n : \pi \not\sim \tau_k^* \text{ and } \#\pi = m\}.$$

We use the projective structure of $\mathcal{P}_n$ to set up a recurrence for $T_k(n,m)$ as follows. Suppose $\pi \in \mathcal{P}_n$ avoids τ_k^* and has exactly m blocks. Then, if $m \geq k$, we can add the element $n+1$ to either of the first $k-2$ blocks of π or to the last block of π to obtain a partition of $[n+1]$ that both avoids τ_k^* and has m blocks. We can also obtain a partition of $[n+1]$ avoiding τ_k^* and having m blocks by appending $\{n+1\}$ as a singleton block to any τ_k^*-avoiding partition of $[n]$ with $m-1$ blocks. Since a partition of $[n]$ containing any pattern will always give rise to a partition of $[n+1]$ containing that pattern, we have the recurrence in the case $m \geq k$. When $m < k$, we can add $n+1$ to any block of π and obtain a τ_k^*-avoiding partition with m blocks. The recursion is initialized by putting $T(1,1) = 1$ and $T(n,0) = 0$ for all $n \geq 1$.

The generating function (2) of $T_k(n,m)$ for fixed k and m is obtained from the recursion (1). □

In the appendix, we provide the triangle for $T_k(n,m)$ for $k = 3,4,5$. The sums of the rows of these triangles have also appeared in previous work by Moreria and Reis [11].

Enumeration of the sets avoiding ordered partitions of length 3 (1-2-3, 2-3-1, 3-2-1, 1-3-2, 3-1-2, 2-1-3) now follows as a corollary to the preceding theorem and the connection between left-right arrangements and set partitions.

Corollary 3.5. *The Wilf equivalence classes for length three patterns are as follows:*

- $\#\mathcal{P}_n(\textit{1-2-3}) = 2^{n-1}$.
- $\#\mathcal{P}_n(\textit{2-3-1}) = \#\mathcal{P}_n(\textit{3-2-1}) = \sum \binom{n-1}{k} Cat_k$ *([14]:A007317), where Cat_k is the kth Catalan number.*

- $\#\,\mathcal{P}_n(\mathit{1\text{-}3\text{-}2}) = \#\,\mathcal{P}_n(\mathit{3\text{-}1\text{-}2}) = \#\,\mathcal{P}_n(\mathit{2\text{-}1\text{-}3}) = (3^n + 1)/2.$

Proof. We need only show the cases 1-2-3, 2-3-1, and 1-3-2.

- **1-2-3**: Since we order blocks in increasing order, a partition can avoid 1-2-3 only if it has fewer than three blocks; hence, $\mathcal{P}_n(1\text{-}2\text{-}3) = \#\{\pi \in \mathcal{P}_n : \#\pi \leq 2\} = 2^{n-1}$.

- **2-3-1**: The case 2-3-1 follows from the bijection between sortable arrangements and partitions. We then use the well-known result about 231-avoiding permutations. This is a corollary to Theorem 3.2. We point out that this also follows as a corollary to Callan's enumeration of 321-avoiding flattened partitions, since a flattened partition avoids 321 if and only if the partition avoids 2-3-1.

- **1-3-2**: This is a special case of Theorem 3.4 for $k = 3$.

□

4 Tables of $T_k(n, m)$ for $k = 3, 4, 5$

The triangle for $k = 2$ relates to combinatorial properties of semigroups [7]. For $k \geq 3$, these triangles have not appeared previously, but some of their attributes correspond to other well-known integer sequences.

$T_3(n,m)$	$m=1$	2	3	4	5	6	7	8	9
$n=1$	1								
2	1	1							
3	1	3	1						
4	1	7	5	1					
5	1	15	17	7	1				
6	1	31	49	31	9	1			
7	1	63	129	111	49	11	1		
8	1	127	321	351	209	71	13	1	
9	1	255	769	1023	769	351	97	15	1

Table 4: Number of partitions avoiding 1232 with a specific number of blocks. This table appears to coincide with [14]:A112857, which is cited in connection with [7]. The third column is [14]:A000337, fourth column is the Bjorn-Welker sequence [14]:A055580, fifth column is [14]:A027608, sixth column is [14]:A211386, seventh column is [14]:A211388. In general, the $(m + 1)$st column has generating function $x^m(1 - 2x)^{-m}/(1 - x)$, as we saw in Theorem 3.4.

$T_4(n,m)$	$m=1$	2	3	4	5	6	7	8	9
$n=1$	1								
2	1	1							
3	1	3	1						
4	1	7	6	1					
5	1	15	25	9	1				
6	1	31	90	52	12	1			
7	1	63	301	246	88	15	1		
8	1	127	966	1039	510	133	18	1	
9	1	255	3025	4083	2569	909	187	21	1

Table 5: Number of partitions avoiding 12343 with a specific number of blocks. The third column is the $(n,3)$ Stirling numbers of the second kind [14]:A000392; fourth column appears to coincide with [14]:A163941.

$T_5(n,m)$	$m=1$	2	3	4	5	6	7	8	9
$n=1$	1								
2	1	1							
3	1	3	1						
4	1	7	6	1					
5	1	15	25	10	1				
6	1	31	90	65	14	1			
7	1	63	301	350	121	18	1		
8	1	127	966	1701	834	193	22	1	
9	1	255	3025	7770	5037	1606	281	26	1

Table 6: Number of partitions avoiding 123454 with a specific number of blocks. The fifth column is [14]:A163942.

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