AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **61(2)** (2015), Pages 182–191

The smallest 3-uniform bi-hypergraph which is a realization of a given vector

Xiao Zhu* Xiaoxiao Duan†

School of Sciences
Linyi University
Linyi, Shandong, 276005
China

Abstract

For a vector $R = (r_1, r_2, \ldots, r_m)$ of non-negative integers, a mixed hypergraph $\mathcal{H}$ is a realization of R if its chromatic spectrum is R. In this paper, we determine the minimum number of vertices of 3-uniform bi-hypergraphs which are realizations of a special kind of vector R_2. As a result, we partially solve an open problem proposed by Král' in 2004.

1 Introduction

A *mixed hypergraph* on a finite set X is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of X. The members of $\mathcal{C}$ and $\mathcal{D}$ are called $\mathcal{C}$-*edges* and $\mathcal{D}$-*edges*, respectively. A set $B \in \mathcal{C} \cap \mathcal{D}$ is called a *bi-edge*. A *bi-hypergraph* is a mixed hypergraph with $\mathcal{C} = \mathcal{D}$, denoted by $\mathcal{H} = (X, \mathcal{B})$, where $\mathcal{B} = \mathcal{C} = \mathcal{D}$. If $X' \subset X, \mathcal{C}' = \{C \in \mathcal{C} | C \subseteq X'\}$ and $\mathcal{D}' = \{D \in \mathcal{D} | D \subseteq X'\}$, then the hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ is called the *induced sub-hypergraph* of $\mathcal{H}$ on X', denoted by $\mathcal{H}[X']$.

The distinction between $\mathcal{C}$-edges and $\mathcal{D}$-edges becomes substantial when colorings are considered. A *proper k-coloring* of $\mathcal{H}$ is a partition of X into k *color classes* such that each $\mathcal{C}$-edge has two vertices with a *Common* color and each $\mathcal{D}$-edge has two vertices with *Distinct* colors. A *strict k-coloring* is a proper k-coloring with k nonempty color classes, and a mixed hypergraph is *k-colorable* if it has a strict k-coloring. For more information, see [5, 6, 7]. The set of all the values k such that $\mathcal{H}$ has a strict k-coloring is called the *feasible set* of $\mathcal{H}$, denoted by $\mathcal{F}(\mathcal{H})$.

A coloring may also be viewed as a partition (*feasible partition*) of the vertex set, where the color classes (partition classes) are the sets of vertices assigned to

* Corresponding author; qqzhu2009@126.com
† Both authors also at School of Mathematical Sciences, Shandong Normal University, Jinan, 250014, China.

the same color. A mixed hypergraph has a gap at k if its feasible set contains elements larger and smaller than k but omits k. For each k, let r_k denote the number of *feasible partitions* of the vertex set into k nonempty color classes. The vector $R(\mathcal{H}) = (r_1, r_2, \ldots, r_{\overline{\chi}})$ is called the *chromatic spectrum* of $\mathcal{H}$, where $\overline{\chi}$ is the largest possible number of colors in a strict coloring of $\mathcal{H}$. If S is a finite set of positive integers, we say that a mixed hypergraph $\mathcal{H}$ is a *realization* of S if $\mathcal{F}(\mathcal{H}) = S$. A mixed hypergraph $\mathcal{H}$ is a *one-realization* of S if it is a realization of S and all the entries of the chromatic spectrum of $\mathcal{H}$ are either 0 or 1. Moreover, for a vector R of positive integers, a mixed hypergraph $\mathcal{H}$ is called a *realization* of R if $R(\mathcal{H}) = R$.

It is readily seen that if $1 \in \mathcal{F}(\mathcal{H})$, then $\mathcal{H}$ cannot have any $\mathcal{D}$-edges. Let S be a finite set of positive integers with $\min(S) \geq 2$. Jiang et al. [3] proved that a set S of positive integers is a feasible set of a mixed hypergraph if and only if $1 \notin S$ or S is an interval. They also discussed the bound on the number of vertices of a mixed hypergraph with a gap, in particular, the minimum number of vertices of realization of $\{s, t\}$ with $2 \leq s \leq t-2$ is $2t - s$. Moreover, they mentioned that the question of finding the minimum number of vertices in a mixed hypergraph with feasible set S of size at least 3 remains open. Král' [4] proved that there exists a one-realization of S with at most $|S| + 2\max(S) - \min(S)$ vertices, and proposed the following problem: What is the number of vertices of the smallest mixed hypergraph whose spectrum is equal to a given spectrum $(r_1, r_2, \ldots, r_m)$? Bacsó et al. [1] discussed the properties of uniform bi-hypergraphs $\mathcal{H}$ which are one-realizations of S when $|S| = 1$, in this case we also say that $\mathcal{H}$ is *uniquely colorable*. Recently, Bujtás and Tuza [2] gave a necessary and sufficient condition for S being the feasible set of an r-uniform mixed hypergraph, and they raised the following open problem: determine the minimum number of vertices in r-uniform bi-hypergraphs with a given feasible set. Zhao et al. [8] constructed a family of 3-uniform bi-hypergraphs with a given feasible set, and obtained an upper bound on the minimum number of vertices of the one-realizations of a given set. In [9] they improved Král's result and proved that the minimum number of vertices of mixed hypergraphs with a given feasible set S is $2\max(S) - \min(S)$ if $\max(S) - 1 \notin S$ or $2\max(S) - \min(S) - 1$ if $\max(S) - 1 \in S$. Recently, Zhao et al. proved in [10] that the minimum number of vertices of 3-uniform bi-hypergraphs with a given feasible set S is $2\max(S)$ if $\max(S) - 1 \notin S$ or $2\max(S) - 1$ if $\max(S) - 1 \in S$.

We denote by $[n]$ the vertex set $\{1, 2, \ldots, n\}$ for any positive integer n.

In this paper, we determine the minimum number of vertices of 3-uniform bi-hypergraphs which are realizations of a special kind of vector R_2, and we obtain the following result.

Theorem 1.1 *For integers $s \geq 2$, $n_1 > n_2 > \cdots > n_s \geq s$ and $t_1 = 0, t_2, \ldots, t_s \geq 0$, let $R_2 = (r_1, r_2, \ldots, r_{n_1})$ be a non-negative vector with $r_{n_1} = 1, r_{n_i} = 2^{t_i}, i \in \{2, \ldots, s\}$ and $r_j = 0, j \in [n_1] \setminus \{n_1, n_2, \ldots, n_s\}$. If $n_{i-1} - n_i > t_i, i \in \{2, \ldots, s\}$, then*

$$\delta_3(R_2) = \begin{cases} 6, & \text{if } n_1 = 3, n_2 = 2, \\ 2n_1, & \text{if } n_1 > n_2 + 1, \\ 2n_1 - 1, & \text{otherwise,} \end{cases}$$

where $\delta_3(R_2)$ is the minimum number of vertices of a 3-uniform bi-hypergraphs which is a realization of R_2.

This paper is organized as follows. In Section 2, we prove that the number in Theorem 1.1 is a lower bound for $\delta_3(R_2)$. In Section 3, we introduce a basic construction of 3-uniform bi-hypergraphs and discuss the coloring property of 3-uniform bi-hypergraphs. In Section 4, we construct 3-uniform bi-hypergraphs which are realizations of R_2 and meet this lower bound in each case.

2 The lower bound

In this section we shall show that the number $\delta_3(R_2)$ given in Theorem 1.1 is a lower bound on the minimum number of vertices of 3-uniform bi-hypergaphs which are realizations of R_2.

Lemma 2.1

$$\delta_3(R_2) \geq \begin{cases} 6, & \text{if } n_1 = 3, n_2 = 2, \\ 2n_1, & \text{if } n_1 > n_2 + 1, \\ 2n_1 - 1, & \text{otherwise.} \end{cases}$$

Proof. Assume that $\mathcal{H} = (X, \mathcal{B})$ is a 3-uniform bi-hypergraph which is a realization of R_2. Note that $|X| \geq 4$. We divide our proof into the following two cases.

Case 1 $t_2 \geq 1$.

That is to say, $\mathcal{H}$ has a gap at $n_1 - 1$. Suppose $|X| = 2n_1 - 1$. For any strict n_1-coloring $c = \{C_1, C_2, \ldots, C_{n_1}\}$ of $\mathcal{H}$, if there exist two color classes, say C_1 and C_2, such that $|C_1| = |C_2| = 1$, then $c' = \{C_1 \cup C_2, C_3, \ldots, C_{n_1}\}$ is a strict $(n_1 - 1)$-coloring of $\mathcal{H}$, a contradiction. Since $|X| = 2n_1 - 1$, there exists one color class, say C_1, such that $|C_1| = 1$, and $|C_i| = 2$, $i = 2, 3, \ldots, n_1$. Let $C_1 = \{x_1\}$ and $C_i = \{x_i, y_i\}, i = 2, 3, \ldots, n_1$. Then, $c'' = \{\{x_1, x_2, x_3, \ldots, x_{n_1}\}, \{y_2, y_3, \ldots, y_{n_1}\}\}$ is a strict 2-coloring of $\mathcal{H}$, which implies that $n_s = 2$. Note that each element of

$$\{\{\{a_1, a_2, \ldots, a_{n_1}\}, \{b_2, b_3, \ldots, b_{n_1}\}\} | a_1 = x_1, a_i, b_i \in \{x_i, y_i\}, a_i \neq b_i, i \in [n_1] \setminus \{1\}\}$$

is a strict 2-coloring of $\mathcal{H}$. It follows that $r_{n_s} \geq 2^{n_1 - 1} > 2^{t_s}$, a contradiction to that $r_{n_s} = 2^{t_s}$. If $|X| \leq 2n_1 - 2$, then we can get a strict $(n_1 - 1)$-coloring of $\mathcal{H}$ from a strict n_1-coloring of $\mathcal{H}$, also a contradiction.

Case 2 $t_2 = 0$.

That is to say, $r_{n_2} = 2^{t_2} = 1$. If $n_1 > n_2 + 1$, by Case 1, we have $\delta_3(R_2) \geq 2n_1$. If $n_1 = n_2 + 1$, we have two possible cases as follows:

Case 2.1 $S = \{3, 2\}$.

Note that the complete 3-uniform bi-hypergraph K_5^3 is uncolorable, and the bi-hypergraph obtained by deleting any edge from K_5^3 is 2-colorable but not 3-colorable. Furthermore, the bi-hypergraph obtained by deleting any two edges from K_5^3 has two strict 2-colorings but not 3-coloring. We have a similar conclusion for the complete 3-uniform bi-hypergraph K_4^3. Therefore, $\mathcal{H}$ which is a realization of R_2 has at least 6 vertices.

Case 2.2 $S \neq \{3, 2\}$.

That is to say, $n_1 \geq 4, n_s > 2$. Suppose $|X| \leq 2n_1 - 2$. For any strict n_1-coloring $c = \{C_1, C_2, \ldots, C_{n_1}\}$ of $\mathcal{H}$, if there exist three color classes, say C_1, C_2 and C_3, such that $|C_1| = |C_2| = |C_3| = 1$, then $c' = \{C_1 \cup C_2, C_3, \ldots, C_{n_1}\}$ and $c'' = \{C_1, C_2 \cup C_3, C_4, \ldots, C_{n_1}\}$ are two distinct strict n_2-colorings of $\mathcal{H}$, a contradiction to that $r_{n_2} = 1$. Noticing that $|X| \leq 2n_1 - 2$, there exist at least two color classes each of which has one vertex, and each of the other color classes has two vertices. Similar to Case 1, $\mathcal{H}$ has a strict 2-coloring, a contradiction.

The proof is complete. □

3 The basic construction

In this section, we shall construct a family of 3-uniform bi-hypergraphs and discuss their coloring properties. This construction plays an important role in constructing 3-uniform bi-hypergraphs which are realizations of R_2 and meet the bounds in Lemma 2.1.

Construction I. Suppose $n_{i-1} - n_i > t_i$, $i \in \{2, \ldots, s\}$, $n_s \geq s$. Let $l_i = s - i + 1$, and write

$$\alpha_a^0 = (\underbrace{a, a, \ldots, a}_{\sum_{w=1}^{s} 2^{t_w}}, 0) \text{ and}$$

$$\alpha_a^1 = (\underbrace{a, a, \ldots, a}_{\sum_{w=1}^{s} 2^{t_w}}, 1), \ a \in [n_s],$$

$$\beta_{ih}^0 = (\underbrace{n_i + h, \ldots, n_i + h}_{\sum_{w=1}^{i-1} 2^{t_w}}, \underbrace{l_i, \ldots, l_i}_{\sum_{w=i}^{s} 2^{t_w}}, 0) \text{ and}$$

$$\beta_{ih}^1 = (\underbrace{n_i + h, \ldots, n_i + h}_{\sum_{w=1}^{i-1} 2^{t_w}}, \underbrace{n_i, \ldots, n_i}_{2^{t_i}}, \ldots, \underbrace{n_s, \ldots, n_s}_{2^{t_s}}, 1),$$

$$i \in [s] \setminus \{1\}, h \in \{0, t_i + 1, t_i + 2, \ldots, n_{i-1} - n_i - 1\},$$

$$\gamma_{ik}^0 = (\underbrace{n_i + k, \ldots, n_i + k}_{\sum_{w=1}^{i-1} 2^{t_w}}, \underbrace{\underbrace{l_i, \ldots, l_i}_{2^{k-1}}, \underbrace{n_i, \ldots, n_i}_{2^{k-1}}, \ldots, \underbrace{l_i, \ldots, l_i}_{2^{k-1}}, \underbrace{n_i, \ldots, n_i}_{2^{k-1}}}_{2^{t_i}}, \underbrace{l_i, \ldots, l_i}_{\sum_{w=i+1}^{s} 2^{t_w}}, 0)$$

and

$$\begin{aligned}
\gamma^1_{ik} &= (\underbrace{n_i+k,\ldots,n_i+k}_{\sum_{w=1}^{i-1}2^{t_w}},\underbrace{\underbrace{n_i,\ldots,n_i}_{2^{k-1}},\underbrace{l_i,\ldots,l_i}_{2^{k-1}},\ldots,\underbrace{n_i,\ldots,n_i}_{2^{k-1}},\underbrace{l_i,\ldots,l_i}_{2^{k-1}}}_{2^{t_i}},\\
&\quad \underbrace{n_{i+1},\ldots,n_{i+1}}_{2^{t_{i+1}}},\underbrace{n_{i+2},\ldots,n_s}_{\sum_{w=i+2}^{s}2^{t_w}},1), i\in[s]\setminus\{1\}, k\in[t_i],\\
\beta^1_1 &= (n_1,\underbrace{n_2,\ldots,n_2}_{2^{t_2}},\underbrace{n_3,\ldots,n_s}_{\sum_{w=3}^{s}2^{t_w}},1),\ \text{and}\\
X &= \bigcup_{a=1}^{n_s}\{\alpha^0_a,\alpha^1_a\}\cup\bigcup_{i=2}^{s}\{\beta^0_{i0},\beta^1_{i0}\}\cup\bigcup_{i=2}^{s}\bigcup_{h=t_i+1}^{n_{i-1}-n_i-1}\{\beta^0_{ih},\beta^1_{ih}\}\cup\bigcup_{i=2}^{s}\bigcup_{k=1}^{t_i}\{\gamma^0_{ik},\gamma^1_{ik}\}\cup\{\beta^1_1\},\\
\mathcal{B} &= \{\{\theta_1,\theta_2,\theta_3\}|\theta_l\in X, l\in[3], |\{\theta_{1(j)},\theta_{2(j)},\theta_{3(j)}\}|=2, j\in[\sum_{w=1}^{s}2^{t_w}+1]\}\\
&\cup\ \{\{\alpha^0_1,\beta^0_{s0},\alpha^0_{n_s}\}\},
\end{aligned}$$

where $\theta_{l(j)}$ is the j-th entry of the vertex θ_l. Then $\mathcal{H}=(X,\mathcal{B})$ is a 3-uniform bi-hypergraph with $2n_1$ vertices.

Note that, for any $i\in[s], g\in[2^{t_i}], c^g_i=\{X^g_{i1},X^g_{i2},\ldots,X^g_{in_i}\}$ is a strict coloring of $\mathcal{H}$, where X^g_{ij} consists of vertices

$$(x^1_1,x^1_2,\ldots,x^{2^{t_2}}_2,\ldots,x^1_i,\ldots,x^{g-1}_i,j,x^{g+1}_i,\ldots,x^{2^{t_i}}_i,\ldots,x^1_s,\ldots,x^{2^{t_s}}_s,x)\in X.$$

In the following, for a strict coloring c of a 3-uniform bi-hypergraph $\mathcal{H}=(X,\mathcal{B})$, we denote by $c(v)$ the color of the vertex v under c.

Lemma 3.1 *Let $c=\{C_1,C_2,\ldots,C_m\}$ be a strict coloring of $\mathcal{H}$. Then we may reorder the color classes such that the following conditions hold:*

(i) $\alpha^0_a,\alpha^1_a\in C_a, a\in[n_s]$;

(ii) $\gamma^0_{ik},\gamma^1_{ik},\beta^0_{ih}\notin C_a$, *for* $a\in[n_s-1]\setminus\{l_i\}$;

(iii) $\beta^1_{ih},\beta^1_1\notin C_a$ *for* $a\in[n_s-1]$;

(iv) $\beta^0_{s0}\in C_1\cup C_{n_s}$.

Proof. (i) We claim that $c(\alpha^0_a)=c(\alpha^1_a)$ for each $a\in[n_s]$. If not, there exists a $t\in[n_s]$ such that $c(\alpha^0_t)\neq c(\alpha^1_t)$. Without loss of generality, assume that $\alpha^0_1\in C_1$ and $\alpha^1_1\in C_2$. From the edge $\{\alpha^0_{n_s},\alpha^0_1,\alpha^1_1\}$, we have $\alpha^0_{n_s}\in C_1\cup C_2$. Suppose $\alpha^0_{n_s}\in C_1$. The edges $\{\alpha^1_{n_s},\alpha^0_1,\alpha^1_1\}$, $\{\alpha^1_{n_s},\alpha^0_{n_s},\alpha^0_1\}$, $\{\beta^0_{s0},\alpha^0_{n_s},\alpha^1_1\}$, $\{\beta^0_{s0},\alpha^0_{n_s},\alpha^0_1\}$ imply that $\alpha^1_{n_s},\beta^0_{s0}\in C_2$. Therefore, the three vertices of the edge $\{\beta^0_{s0},\alpha^1_{n_s},\alpha^1_1\}$ fall into a common color class, a contradiction. We have the same conclusion for the case of $\alpha^0_{n_s} \in C_2$. Hence our claim is valid.

From the edge $\{\alpha^0_p, \alpha^1_p, \alpha^0_q\}$, we have $c(\alpha^0_p) \neq c(\alpha^0_q)$ for $p, q \in [n_s]$ if $p \neq q$. Hence, we may reorder the color classes such that $\alpha^0_a, \alpha^1_a \in C_a$ for any $a \in [n_s]$, from which it follows that (i) holds.

(ii) For any $a \in [n_s - 1] \setminus \{l_i\}$, the edges $\{\gamma^1_{ik}, \alpha^0_a, \alpha^1_a\}, \{\gamma^0_{ik}, \alpha^0_a, \alpha^1_a\}, \{\beta^0_{ih}, \alpha^0_a, \alpha^1_a\}$ imply that $\gamma^0_{ik}, \gamma^1_{ik}, \beta^0_{ih} \notin C_a$. Hence, (ii) holds.

(iii) For any $a \in [n_s - 1]$, from the edges $\{\beta^1_{ih}, \alpha^0_a, \alpha^1_a\}$ and $\{\beta^1_1, \alpha^0_a, \alpha^1_a\}$, one gets β^1_{ih},$\beta^1_1 \notin C_a$. Hence, (iii) holds.

(iv) The edge $\{\beta^0_{s0}, \alpha^0_{n_s}, \alpha^0_1\}$ implies that $\beta^0_{s0} \in C_1 \cup C_{n_s}$, and so the result follows. □

Lemma 3.2 *Let $c = \{C_1, C_2, \ldots, C_m\}$ be a strict coloring of $\mathcal{H}$ satisfying the conditions (i)–(iv) in Lemma 3.1.*

(i) *Suppose $c(\beta^0_{ph_p}) \neq c(\beta^1_{ph_p})$ for some $p \in [s] \setminus \{1\}$ and $h_p \in \{0, t_p + 1, \ldots, n_{p-1} - n_p - 1\}$. Then for every $i \in [p] \setminus \{1\}$ and $h \in \{0, t_i + 1, \ldots, n_{i-1} - n_i - 1\}$, we have $\beta^0_{ih} \in C_{l_i}$ and $\beta^1_1, \beta^1_{ih} \in C_d$ for some $d \in [m] \setminus [n_s]$.*

(ii) *Suppose $c(\beta^0_{qh_q}) = c(\beta^1_{qh_q})$ for some $q \in [s] \setminus \{1\}$ and $h_q \in \{0, t_q + 1, \ldots, n_{q-1} - n_q - 1\}$. Then for every $i \in [s] \setminus [q - 1]$, $h \in \{0, t_i + 1, \ldots, n_{i-1} - n_i - 1\}$ and $k \in [t_i]$, we have $c(\beta^0_{ih}) = c(\beta^1_{ih})$ i and $c(\gamma^0_{ik}) = c(\gamma^1_{ik})$.*

Proof. (i) From the edge $\{\alpha^0_{l_p}, \beta^0_{ph_p}, \beta^1_{ph_p}\}$, we have $\beta^0_{ph_p} \in C_{l_p}$. For any $i \in [p - 1] \setminus \{1\}$, the edges $\{\beta^1_{ph_p}, \beta^0_{ph_p}, \beta^1_{ih}\}$ and $\{\beta^1_{ph_p}, \beta^0_{ph_p}, \beta^1_1\}$ imply that $c(\beta^1_{ih}) = c(\beta^1_{ph_p}) = c(\beta^1_1)$. Suppose $\beta^1_{ih} \in C_d$ for some $d \in [m] \setminus [n_s]$. Then since $\{\beta^1_{ih}, \beta^0_{ih}, \beta^1_1\}$ and $\{\beta^1_{ih}, \beta^0_{ih}, \alpha^1_{l_i}\}$ are edges, we have $\beta^0_{ih} \in C_{l_i}$. Hence, (i) holds.

(ii) If there exist $p \in \{q, \ldots, s\}$ and $h_p \in \{0, t_p + 1, \ldots, n_{p-1} - n_p - 1\}$ such that $c(\beta^0_{ph_p}) \neq c(\beta^1_{ph_p})$, then by (i) we have $c(\beta^0_{ih}) \neq c(\beta^1_{ih})$ for any $i \in [p] \setminus \{1\}$. It follows that $c(\beta^0_{qh_q}) \neq c(\beta^1_{qh_q})$, a contradiction. Hence, $c(\beta^0_{ih}) = c(\beta^1_{ih})$ for $i \in \{q, \ldots, s\}$. Moreover, for $i \in \{q, \ldots, s\}$, from the edges $\{\gamma^0_{ik}, \beta^0_{ih}, \beta^1_{ih}\}$ and $\{\gamma^1_{ik}, \beta^0_{ih}, \beta^1_{ih}\}$, we have $c(\gamma^0_{ik}) \neq c(\beta^1_{ih})$ and $c(\gamma^1_{ik}) \neq c(\beta^1_{ih})$; and the edge $\{\gamma^0_{ik}, \gamma^1_{ik}, \beta^1_{ih}\}$ implies that $c(\gamma^0_{ik}) = c(\gamma^1_{ik})$. Hence, (ii) holds. □

Lemma 3.3 *Let $c = \{C_1, C_2, \ldots, C_m\}$ be a strict coloring of $\mathcal{H}$ satisfying $c(\beta^0_{p0}) \neq c(\beta^1_{p0})$ for some $p \in [s] \setminus \{1\}$. Then there exists an integer $d \in [m] \setminus [n_s]$ such that*

(i) *$\gamma^0_{pk}, \gamma^1_{pk} \in C_{l_p} \cup C_d$ and $c(\gamma^0_{pk}) \neq c(\gamma^1_{pk})$;*

(ii) *$\gamma^0_{qk} \in C_{l_q}$ and $\gamma^1_{qk} \in C_d$ for any $q \in [p - 1] \setminus \{1\}$.*

Proof. For any $i \in [p] \setminus \{1\}$, by Lemma 3.2, we have $\beta^0_{i0} \in C_{l_i}$ and $\beta^1_{i0} \in C_d$ for some $d \in [m] \setminus [n_s]$. Since $\{\gamma^0_{ik}, \beta^0_{i0}, \beta^1_{i0}\}, \{\gamma^1_{ik}, \beta^0_{i0}, \beta^1_{i0}\}$ are edges, we have $\gamma^0_{ik}, \gamma^1_{ik} \in C_{l_i} \cup C_d$;

and the edges $\{\gamma^0_{ik}, \gamma^1_{ik}, \beta^0_{i0}\}, \{\gamma^0_{ik}, \gamma^1_{ik}, \beta^1_{i0}\}$ imply that $c(\gamma^0_{ik}) \neq c(\gamma^1_{ik})$. Specially, we have $\gamma^0_{pk}, \gamma^1_{pk} \in C_{l_p} \cup C_d$ and $c(\gamma^0_{pk}) \neq c(\gamma^1_{pk})$. Hence,(i) holds.

For any $q \in [p-1] \setminus \{1\}$, from the edge $\{\gamma^0_{pk}, \gamma^1_{pk}, \gamma^1_{qk}\}$, we have $\gamma^1_{qk} \in C_d$. Since $\gamma^0_{qk} \in C_{l_q} \cup C_d$ and $c(\gamma^0_{qk}) \neq c(\gamma^1_{qk})$, we have $\gamma^0_{qk} \in C_{l_q}$.

The proof is complete. □

Lemma 3.4 *Let $c = \{C_1, C_2, \ldots, C_m\}$ is a strict coloring of $\mathcal{H}$ satisfying the conditions (i)-(iv) in Lemma 3.1. Let $b \in [s] \setminus \{1\}$ be the minimum number such that $c(\beta^0_{b0}) = c(\beta^1_{b0})$. Then we may reorder the color classes such that the following conditions hold:*

(i) $\{\beta^0_{ih}, \beta^1_{ih}\} \subseteq C_{n_i+h}$ *for* $i \in [s] \setminus [b-1]$;

(ii) $\{\gamma^0_{ik}, \gamma^1_{ik}\} \subseteq C_{n_i+k}$ *for* $i \in [s] \setminus [b-1]$.

Proof. By Lemma 3.2, we have $c(\beta^0_{ih}) = c(\beta^1_{ih})$ and $c(\gamma^0_{ik}) = c(\gamma^1_{ik})$ for each $i \in [s] \setminus [b-1]$. For $i_1, i_2 \in [s] \setminus [b-1]$ and $i_1 > i_2$, the edges $\{\beta^0_{i_1h_1}, \beta^1_{i_1h_1}, \beta^1_{i_2h_2}\}$, $\{\beta^0_{i_1h_1}, \beta^1_{i_1h_1}, \gamma^1_{i_2k_2}\}$, $\{\gamma^0_{i_1k_1}, \gamma^1_{i_1k_1}, \beta^1_{i_2h_2}\}$ and $\{\gamma^0_{i_1k_1}, \gamma^1_{i_1k_1}, \gamma^1_{i_2k_2}\}$ imply that $c(\beta^1_{i_1h_1}) \neq c(\beta^1_{i_2h_2})$, $c(\beta^1_{i_1h_1}) \neq c(\gamma^1_{i_2k_2}), c(\beta^1_{i_2h_2}) \neq c(\gamma^1_{i_1k_1})$ and $c(\gamma^1_{i_1k_1}) \neq c(\gamma^1_{i_2k_2})$ for any $k_j \in [t_i], h_j \in \{0, t_i+1, t_i+2, \ldots, n_{i-1}-n_i-1\}, j \in \{1,2\}$. Moreover, for $i \in [s] \setminus [b-1]$, the edge $\{\beta^0_{ih_1}, \beta^1_{ih_1}, \beta^1_{ih_2}\}$ implies that $c(\beta^1_{ih_1}) \neq c(\beta^1_{ih_2})$ if $h_1 \neq h_2$; the edge $\{\beta^0_{ih}, \beta^1_{ih}, \gamma^1_{ik}\}$ implies that $c(\beta^1_{ih}) \neq c(\gamma^1_{ik})$; and from the edge $\{\gamma^0_{ik_1}, \gamma^1_{ik_1}, \gamma^1_{ik_2}\}$, we have $c(\gamma^1_{ik_1}) \neq c(\gamma^1_{ik_2})$ if $k_1 \neq k_2$. Hence, we may reorder the color classes such that $\{\beta^0_{ih}, \beta^1_{ih}\} \subseteq C_{n_i+h}, \{\gamma^0_{ik}, \gamma^1_{ik}\} \subseteq C_{n_i+k}$ for $i \in [s] \setminus [b-1]$, which implies that (i) and (ii) holds. □

4 Proof of Theorem 1.1

Next, we shall prove that all the strict colorings of the 3-uniform bi-hypergraph $\mathcal{H}$ are $c^1_1, c^1_2, \ldots, c^{2^{t_2}}_2, c^1_3, \ldots, c^{2^{t_s}}_s$.

Theorem 4.1 *$\mathcal{H}$ is a realization of R_2, where $\mathcal{H}$ satisfying the conditions (i)-(iv) in Lemma 3.1.*

Proof. Suppose $c = \{C_1, C_2, \ldots, C_m\}$ is a strict coloring of $\mathcal{H}$. Then $\mathcal{H}$ satisfying the conditions (i)-(iv) in Lemma 3.1. In particular, $\beta^0_{s0} \in C_1 \cup C_{n_s}$.

Case 1 $\beta^0_{s0} \in C_1$.

In this case, we shall prove that $c \in \{c^g_s | g \in [2^{t_s}]\}$. Note that $\beta^1_{s0} \in C_{n_s}$. For any $i \in [s] \setminus \{1\}$, by Lemma 3.2, we have $\beta^0_{ih} \in C_{l_i}$ and $\beta^1_1, \beta^1_{ih} \in C_{n_s}$. By Lemma 3.3, one gets that

(i) $\gamma^0_{sk}, \gamma^1_{sk} \in C_1 \cup C_{n_s}$ and $c(\gamma^0_{sk}) \neq c(\gamma^1_{sk})$;

(ii) $\gamma^0_{qk} \in C_{l_q}$ and $\gamma^1_{qk} \in C_{n_s}$ for any $q \in [s-1] \setminus \{1\}$.

Then we have $c \in \{c^g_s | g \in [2^{t_s}]\}$.

Case 2 $\beta^0_{s0} \in C_{n_s}$.

Then c satisfies the condition (ii) in Lemma 3.2. In this case, we shall prove that $c \in \{c^g_i | i \in [s-1], g \in [2^{t_i}]\}$. Let $b \in [s] \setminus \{1\}$ be the minimum number such that $c(\beta^0_{b0}) = c(\beta^1_{b0})$. So c satisfies the conditions in Lemma 3.4.

Case 2.1 If $b = 2$, we claim that β^1_1 fall into a new color class $C_l = \emptyset, l \in [m] \setminus [n_s]$. Suppose $C_l \neq \emptyset$. Without loss of generality, there exists a vertex β^1_{p0} such that $\beta^1_{p0} \in C_l$ for $p \in [s] \setminus \{1\}$. Then we have $\beta^0_{p0} \in C_l$. The edge $\{\beta^0_{p0}, \beta^1_{p0}, \beta^1_1\}$ is monochromatic, a contradiction. Hence, our claim is valid. Then we have $\beta^1_1 \in C_{n_1}$ and $c = c^1_1$.

Case 2.2 If $b > 2$, that is to say, for each $p \in [b-1] \setminus \{1\}$, $c(\beta^0_{p0}) \neq c(\beta^1_{p0})$. Similarly to Case 2.1, and so we have $\beta^1_{b-1,0}$ fall into a new color class $C_l = \emptyset$. Hence, we may assume that $\beta^1_{b-1,0} \in C_{n_{b-1}}$ and then $\beta^0_{b-1,0} \in C_{l_{b-1}}$. By lemma 3.2, we have $\beta^0_{ih} \in C_{l_i}$ and $\beta^1_1, \beta^1_{ih} \in C_{n_{b-1}}$ for $i \in [b-1] \setminus \{1\}$; and then by Lemma 3.3, one gets that

(i) $\gamma^0_{b-1,k}, \gamma^1_{b-1,k} \in C_{l_{b-1}} \cup C_{n_{b-1}}$ and $c(\gamma^0_{b-1,k}) \neq c(\gamma^1_{b-1,k})$;

(ii) $\gamma^0_{qk} \in C_{l_q}$ and $\gamma^1_{qk} \in C_{n_{b-1}}$ for any $q \in [b-2] \setminus \{1\}$.

Hence $c \in \{c^g_{b-1} | g \in [2^{t_{b-1}}]\}$.

The proof is complete. □

Note that $\mathcal{H}$ is a desired 3-uniform bi-hypergraph when $n_1 > n_2 + 1$. Then we focus on the case of $n_1 = n_2 + 1$.

Construction II. Suppose $n_{i-1} - n_i > t_i$, $i \in \{2, \ldots, s\}$, $n_s \geq s$. For $s \geq 3$ and $n_1 = n_2 + 1$, let $X' = X \setminus \{\beta^0_{20}\}$ and $\mathcal{H}' = \mathcal{H}[X']$.

Theorem 4.2 *Suppose $s \geq 3$ and $n_1 = n_2 + 1$. Then $\mathcal{H}'$ is a realization of R_2.*

Proof. We have $t_2 = 0$ from the condition $n_1 = n_2 + 1$.

Let $Y = X' \setminus \{\beta^1_1\} \subset X$, then we have that $\mathcal{G} = \mathcal{H}[Y]$ is a induced sub-hypergraph of $\mathcal{H}$ on Y.

By Theorem 4.1, all the strict colorings of $\mathcal{G}$ are as follows:

$$e^g_i = \{Y^g_{i1}, Y^g_{i2}, \ldots, Y^g_{in_i}\}, i \in [s] \setminus \{1\}, g \in [2^{t_i}],$$

where Y^g_{ij} consists of vertices

$$(x^1_2, x^1_2, x^1_3, \ldots, x^{2^{t_3}}_3, \ldots, x^1_i, \ldots, x^{g-1}_i, j, x^{g-1}_i, \ldots, x^{2^{t_i}}_i, \ldots, x^1_s, \ldots, x^{2^{t_s}}_s, x) \in X.$$

Let $c = \{C_1, C_2, \ldots, C_m\}$ be a strict coloring of $\mathcal{H}'$. Then $\mathcal{H}'$ satisfying the conditions (i)-(iv) in Lemma 3.1. There are the following two possible cases.

Case 1 $c|_Y = e^1_2$.

In this case, we shall prove that $c \in \{c_1, c_2\}$. For $i \in [s] \setminus \{1, 2\}$, from the edges $\{\beta^0_{ih}, \beta^1_{ih}, \beta^1_1\}$ and $\{\gamma^0_{ik}, \gamma^1_{ik}, \beta^1_1\}$, we have $\beta^1_1 \notin C_{n_i+k} \cup C_{n_i+h}$. Therefore, we have $c = c^1_2$ if $\beta^1_1 \in C_{n_2}$ and $c = c^1_1$ if $\beta^1_1 \notin C_{n_2}$.

Case 2 $c|_Y = e^g_p, p \in [s] \setminus \{1, 2\}, g \in [2^{t_p}]$.

Note that $\beta^0_{p0} \in C_{l_p}$ and $\beta^1_{p0} \in C_{n_p}$. The edge $\{\beta^1_1, \beta^0_{p0}, \beta^1_{p0}\}$ implies that $\beta^1_1 \in C_{n_p}$. Therefore, $c = c^g_p, g \in [2^{t_p}]$. □

For the case of $s = 2, n_2 > 2$ and $n_1 = n_2 + 1$, Zhao et al. constructed a 3-uniform bi-hypergraph $\mathcal{H}^*$ [10, Construction III] with $2n_1 - 1$ vertices and obtained the following result.

Theorem 4.3 ([10, Theorem 2.6]) *Suppose $s = 2, n_2 > 2$ and $n_1 = n_2 + 1$. Then $\mathcal{H}^*$ is a one-realization of $\{n_1, n_2\}$.*

Note that, when $s = 2, n_2 > 2$ and $n_1 = n_2 + 1$, any one-realization of $\{n_1, n_2\}$ is a realization of R_2. Hence, we get the following result.

Theorem 4.4 *Suppose $s = 2, n_2 > 2$ and $n_1 = n_2 + 1$. Then $\mathcal{H}^*$ is a realization of R_2.*

Combining Theorems 4.1, Theorems 4.2 and Theorem 4.4, the proof of Theorem 1.1 is now complete.

Acknowledgments

The research is supported by NSF of Shandong Province (No. ZR2009AM013) and NSF of China (No. 11226288).

References

[1] G. Bacsó, Zs. Tuza and V. Voloshin, Unique colorings of bi-hypergraphs, *Australas. J. Combin.* 27 (2003), 33–45.

[2] Cs. Bujtás and Zs. Tuza, Uniform mixed hypergraphs: the possible numbers of colors, *Graphs Combin.* 24 (2008), 1–12.

[3] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. West, The chromatic spectrum of mixed hypergraphs, *Graphs Combin.* 18 (2002), 309–318.

[4] D. Král', On feasible sets of mixed hypergraphs, *Electron. J. Combin.* 11 (2004), #R19.

[5] Zs. Tuza and V. Voloshin, Problems and results on colorings of mixed hypergraphs, in *Horizons of Combinatorics*, Bolyai Society Mathematical Studies 17, Springer-Verlag, 2008, pp. 235–255.

[6] V. Voloshin, On the upper chromatic number of a hypergraph, *Australas. J. Combin.* 11 (1995), 25–45.

[7] V. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications*, Amer. Math. Soc., Providence, 2002.

[8] P. Zhao, K. Diao and K. Wang, The chromatic spectrum of 3-uniform bi-hypergraphs, *Discrete Math.* 311 (2011), 2650–2656.

[9] P. Zhao, K. Diao and K. Wang, The smallest one-realization of a given set, *Electron. J. Combin.* 19 (2012), #P19.

[10] P. Zhao, K. Diao, R. Chang and K. Wang, The smallest one-realization of a given set II, *Discrete Math.* 19 (2012), 2946–2951.

(Received 23 Mar 2014; revised 23 Nov 2014)