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Pancyclic out-arcs of a vertex in a hypertournament

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Abstract

A k-hypertournament H on n vertices, where $2 \leq k \leq n$, is a pair $H = (V, A_H)$, where V is the vertex set of H and A_H is a set of k-tuples of vertices, called arcs, such that for all subsets $S \subseteq V$ of order k, A_H contains exactly one permutation of S as an arc. Inspired by the successful extension of classical results for tournaments (i.e. 2-hypertournaments) to hypertournaments, by Gutin and Yeo [*J. Graph Theory* 25 (1997), 277–286] and Li et al. [*Discrete Appl. Math.* 161 (2013), 2749–2752], we will prove the following: every strong k-hypertournament on n vertices, where $n \geq k+2 \geq 3$, contains a vertex all of whose out-arcs are pancyclic. This is a generalization of a known result for tournaments, by Yao et al. [*Discrete Appl. Math.* 99 (2000), 245–249]. Furthermore, our result is best possible in the sense that the bound $n \geq k + 2$ is tight.

1 Introduction and Terminology

For all notation not explicitly defined here, we follow [1]. A *directed k-hypergraph H* is a tuple (V, A), where V is the *vertex set* of H and the *arc set* A of H consists of k-tuples of vertices. If the vertex and arc set of H are not specified, we denote them by $V(H)$ and $A(H)$, respectively.

A *digraph D* is a directed 2-hypergraph. Let D be a digraph. Instead of $(x, y) \in A(D)$, we mostly use the notation $xy \in A(D)$ or $x \to y$. If X and Y are two disjoint subsets of $V(D)$, then $X \Rightarrow Y$ conveys that there are no arcs from Y to X and $X \to Y$ implies $xy \in A(D)$ for all $x \in X$ and $y \in Y$. For subdigraphs $D_1, D_2 \subseteq D$ we write $D_1 \Rightarrow D_2$ and $D_1 \to D_2$, to express $V(D_1) \Rightarrow V(D_2)$ and $V(D_1) \to V(D_2)$, respectively.

Let X be a subset of $V(D)$. $D[X] := (X, \{xy \in A(D) \mid x, y \in X\})$ is the *subdigraph of D induced by X*. The *out-neighborhood* of a vertex $x \in X$ in $D[X]$

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is the vertex set $N^+_{D[X]}(x) := \{y \mid xy \in A(D[X])\}$. The *in-neighborhood* $N^-_{D[X]}(x)$ is defined analogously. We write N^+ and N^- instead of N^+_D and N^-_D, respectively. The number of out-neighbors of a vertex x, denoted by $d^+_{D[X]}(x)$, is called *out-degree*. As before, we define the in-degree analogously. $D - X$ denotes the subdigraph $D[V(D) \setminus X]$. If X consists of a single vertex $x \in V(D)$, we write $D - x$ instead of $D - \{x\}$.

For a non-empty vertex set V let $A_V := \{xy \mid xy \in V^2, x \neq y\}$ denote the arc set of a *complete digraph* on the vertex set V.

Let $H = (V, A_H)$ be a directed k-hypergraph on n vertices. An arc $a = (x_1, \ldots, x_k) \in A_H$ is called an *out-arc of* x_1 and an *in-arc of* x_k. The set of all out-arcs of a vertex x is denoted by $\mathrm{Out}_H(x)$. Furthermore, $a^{-1} := (x_k, \ldots, x_1)$ is the *reverse arc* of a and the *converse* directed k-hypergraph $H^{-1} := (V, A(H^{-1}))$ is defined through $A(H^{-1}) := \{a^{-1} \mid a \in A_H\}$.

Let $X \subseteq V$. For $xy \in A_V$, we define $A_H(x,y)\big|_X \subseteq A_H$ as the set of all arcs $a = (x_1, \ldots, x_k) \in A_H$ such that there are indices $1 \leq i_0 < i_1 \leq k$ with $x_{i_0} = x$, $x_{i_1} = y$ and $x_i \in X$ for all $i \in \{1, \ldots, k\} \setminus \{i_0, i_1\}$. Instead of $A_H(x,y)\big|_V$ we write $A_H(x,y)$. Furthermore, $A_H\big|_X$ denotes the set of arcs in A_H that contain only vertices from X. If $\mathrm{Out}_H(x) \subseteq A_H\big|_X$ holds for all $x \in X$, we call X *self-contained*.

An (x_1, x_{l+1})*-path of length* l or l*-path from* x_1 *to* x_{l+1} *in* H is a sequence $P = x_1 a_1 x_2 \ldots a_l x_{l+1}$ such that the vertices $x_1, \ldots, x_{l+1} \in V$ and the arcs $a_1, \ldots, a_l \in A_H$ are pairwise distinct and $a_i \in A_H(x_i, x_{i+1})$ holds for all $1 \leq i \leq l$. An l*-cycle in* H is defined analogously with the exception that $x_1 = x_{l+1}$ holds. If we consider an l-cycle $C = x_1 a_1 x_2 \ldots a_l x_1$ in a directed hypergraph, let x_{l+1} denote x_1, for convenience. An n-cycle ($(n-1)$-path, respectively) in H is called *Hamiltonian*. A vertex (an arc, respectively) of H is called *pancyclic*, if it is contained in an l-cycle for all $l \in \{3, \ldots, n\}$. H is *vertex-pancyclic*, if all of its vertices are pancyclic. For a path $P = x_1 a_1 \ldots a_{l-1} x_l$ in H and two vertices $x_i, x_j \in V(P)$ with $i \leq j$, we define $x_i P x_j$ as the unique (x_i, x_j)-subpath of P. xCy is the corresponding subpath of a cycle C in H.

Since the sequence of vertices of a path (or cycle, respectively) in a digraph D defines the arcs connecting them, in this case, we usually omit the arcs in our notation. If P is an (x,y)-path and Q is a (v,w)-path in a digraph D such that $v \in N^+_D(y)$ and $V(P) \cap V(Q) = \emptyset$ holds, then PQ denotes the path obtained by appending Q to P.

H is called *strong*, if there is an (x,y)-path in H for all $x, y \in V$, $x \neq y$. A *strong component* D' of a digraph D is a maximal strong induced subgraph of D. The strong components $D_1, \ldots, D_r$ of a digraph D can be ordered such that $D_1 \Rightarrow D_2 \Rightarrow \ldots D_r$ holds. The strong components of a digraph D in this order are called the *strong decomposition* of D; D_1 is the *initial*, D_r the *terminal* component of this composition.

For $2 \leq k \leq n$, a k*-hypertournament* $H = (V, A_H)$ *on* n *vertices* is a directed k-hypergraph such that the following statement holds: For every subset $S \subseteq V$ of order k, A_H contains exactly one ordered k-tuple of the vertices contained in S. k-Hypertournaments are therefore a generalization of tournaments (i.e. 2-hypertournaments).

As Volkmann [13] says in one of several surveys on the subject published over the past fifty years, "tournaments are without doubt the best studied class of directed graphs". In recent years, there has also been an increased interest in generalizations of tournaments. The simplest of these generalizations is the class of *semicomplete digraphs*. While in a tournament, every pair of distinct vertices is connected by exactly one arc, in a semicomplete digraph, every such pair is connected by *at least* one arc. Many results for tournaments are easily proven to hold for semicomplete digraphs as well.

Other well-studied generalizations are for example multipartite tournaments [13] and locally-semicomplete digraphs [3] (see [2] for more). A property all of them have in common is that they are all still classes of digraphs. k-Hypertournaments differ from these generalizations in that respect. As a consequence, in general, there is no substructure of a k-hypertournament equivalent to the aforementioned strong decomposition of a digraph. This was shown for example in [5]. This absence of structure constitutes an obstacle, as, during the process of extending known results for tournaments to hypertournaments, one realizes quickly that its existence is integral to many of the proofs.

To circumvent this problem, in 1997, Gutin and Yeo [6] introduced the following auxiliary digraph.

Definition 1.1. Let $H = (V, A_H)$ be a k-hypertournament on $n \geq k \geq 3$ vertices. The *majority digraph* $M(H) = (V, A_{\text{maj}}(H))$ of H is a digraph on the same vertex set V such that for all $xy \in A_V$ the following holds:

$$xy \in A_{\text{maj}}(H) \quad \text{if and only if} \quad |A_H(x,y)| \geq |A_H(y,x)|. \tag{1}$$

Remark 1.2.

- $M(H)$ is obviously a semicomplete digraph.
- Condition (1) is equivalent to:

$$|A_H(x,y)| \geq \frac{1}{2}\binom{n-2}{k-2}. \tag{2}$$

- When the considered hypertournament $H = (V, A_H)$ is evident, we will also use the notation $xy \in A_{\text{maj}}(H)$ to express that $xy \in A_V$ and $|A_H(x,y)| \geq \frac{1}{2}\binom{n-2}{k-2}$ holds, even if we do not consider the majority digraph explicitly.

Using this new substructure, Gutin and Yeo were able to prove generalizations of two classical results for tournaments by Rédei (1.3) and Camion (1.5), respectively.

Theorem 1.3. *[11] Every tournament contains a Hamiltonian path.*

Theorem 1.4. *[6] Every k-hypertournament on $n \geq k+1 \geq 4$ vertices contains a Hamiltonian path.*

Theorem 1.5. *[4] Every strong tournament contains a Hamiltonian cycle.*

Theorem 1.6. *[6] Every strong k-hypertournament on $n \geq k+2 \geq 5$ vertices contains a Hamiltonian cycle.*

Furthermore, in [6], an example for a strong $(n-1)$-hypertournament without a Hamiltonian cycle is given, thus proving that the bound $n \geq k+2$ is best possible. For $k = n \geq 3$, a k-hypertournament obviously contains exactly one arc and hence, no Hamiltonian cycle or path. In addition, the question was raised whether hypertournaments were vertex-pancyclic, a generalization of Moon's theorem for tournaments.

Theorem 1.7. *[9] Every vertex of a strong tournament T is contained in an l-cycle for all $l \in \{3, \ldots, |V(T)|\}$.*

Remark 1.8. Theorem 1.7 obviously holds for strong semicomplete digraphs, since they contain a strong tournament as a subdigraph.

In 2006, Petrovic and Thomassen [10] and Yang [14], in 2009, gave some sufficient conditions for hypertournaments to be vertex-pancyclic. Finally, the general question was answered in the affirmative by Li et al., in 2013.

Theorem 1.9. *[8] Every strong k-hypertournament on $n \geq k+2 \geq 5$ vertices is vertex-pancyclic.*

Inspired by the successful extension of these known results for tournaments to hypertournaments, the goal of this paper is to prove a generalization of the following theorem, by Yao et al.

Theorem 1.10. *[15] A strong tournament contains a vertex u such that all out-arcs of u are pancyclic.*

Theorem 1.10 itself is a generalization of Theorem 1.11, due to Thomassen.

Theorem 1.11. *[12] If T is a strong tournament, then T contains a vertex x such that every arc going out from x is contained in a Hamiltonian cycle.*

The standard method to prove such generalizations usually takes advantage of the fact that many results for tournaments also hold for semicomplete digraphs. Consider for example the proof of Theorem 1.6. If the majority digraph of a hypertournament H is strong, then it is a strong semicomplete digraph and thus, it contains a Hamiltonian cycle C by Remark 1.8. Now it suffices to find pairwise distinct arcs in H that correspond to those in C to find a Hamiltonian cycle in H. By the definition of the majority digraph, this translation is rather elementary in most cases, only a few exceptions remain to consider.

Unfortunately, Theorem 1.10 does not hold for semicomplete digraphs, as illustrated by the following example. Therefore, the proof of its generalization will be somewhat more complex.

Example 1.12. An *opera-ball-digraph* is obtained from a strong tournament by replacing each of its vertices with a complete digraph of order two, called a *couple* or *partners*.

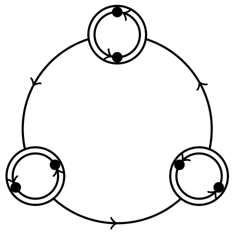

Figure 1: The smallest opera-ball-digraph.

Let D be an opera-ball-digraph. Then by definition, D is a strong semicomplete digraph. Let x be a vertex of D and let y be its partner. Then $xy \in A$ is an out-arc of x. By definition of D, all couples have the same in- and out-neighborhood (except for their respective partners). Furthermore, there are no 2-cycles between couples, since the underlying digraph is a tournament. Therefore, there is no out-neighbor z of y that is also an in-neighbor of x. Thus, the arc xy is not contained in a 3-cycle and is particularly not pancyclic.

Therefore, opera-ball-digraphs, a subclass of semicomplete digraphs, do not contain a vertex whose all out-arcs are pancyclic.

Even if Theorem 1.10 would hold for semicomplete digraphs, a simple majority digraph would still not be the right substructure to consider. The fact that all out-arcs of a vertex are pancyclic in the majority digraph does by no means imply that all out-arcs of said vertex are pancyclic in the hypertournament, as not all arcs of the hypertournament are represented in the majority digraph. Therefore, we introduce a new kind of majority digraph tailored to our needs in the following proofs.

Definition 1.13. Let $H = (V, A_H)$ be a k-hypertournament on $n \geq k \geq 3$ vertices and let $X \subseteq V$. A semicomplete digraph $D = (V, A_D)$ is then called an *X-out-arc-majority digraph of H*, if there is a function $R_D : A_V \to \mathcal{P}(A_H)$ such that the following conditions are met:

(a) For all $xy \in A_V$ we have:

(i)	$R_D(xy) \subseteq A_H(x, y)$.		
(ii)	$R_D(xy) \neq \emptyset$	implies	$xy \in A_D$.
	$xy \in A_D \setminus A_{\text{maj}}(H)$	implies	$R_D(xy) \neq \emptyset$.
(iii)	$R_D(xy) = R_D(yx) = \emptyset$	implies	$\{xy, yx\} \cap A_{\text{maj}}(H) \subseteq A_D$.
	$\{xy, yx\} \subseteq A_D$	implies	$R_D(xy) = R_D(yx) = \emptyset$ or $R_D(xy) \neq \emptyset \neq R_D(yx)$.

(b) For all $xy \in A_X$ we have $R_D(xy) \subseteq \text{Out}_H(x)$.

(c) For all $a \in A_H$ there is exactly one $xy \in A_D$ with $a \in R_D(xy)$.

We call R_D a *representative function of D* and denote the set of all such functions by REP_D. Condition (c) allows us to define a *quasi-inverse function $R_D^{\downarrow}$ of R_D*:

$$R_D^{\downarrow} : A_H \to A_D,\ a \mapsto xy :\Leftrightarrow a \in R_D(xy).$$

By $\mathrm{OAMD}_X(H)$, we denote the set of all X-out-arc-majority digraphs of H. A V-out-arc-majority digraph of H is also simply called an *out-arc-majority digraph of* H and the set of all such digraphs is denoted by $\mathrm{OAMD}(H)$.

The motivation for these rather technical definitions will become more apparent through the proof of the following easy theorem, which illustrates how to obtain an out-arc-majority digraph of an arbitrary k-hypertournament.

Theorem 1.14. *Let $H = (V, A_H)$ be a k-hypertournament on $n \geq k \geq 3$ vertices. Then* $\mathrm{OAMD}(H) \neq \emptyset$.

Proof. We construct a semicomplete digraph $D = (V, A_D)$ and a representative function R_D of D.

0. We start with $D := (V, \emptyset)$ and $R_D : A_V \to \mathcal{P}(A_H),\ xy \mapsto \emptyset$.

1. Now for every arc $a = (x_1, \ldots, x_k) \in A_H$, we choose an $i \in \{2, \ldots, k\}$. The out-arc a of x_1 in the hypertournament H shall be represented by the out-arc x_1x_i of x_1 in the digraph D. Thus, we add x_1x_i to A_D and add a to $R_D(x_1x_i)$ (the set of arcs of the hypertournament represented by x_1x_i).

   After step 1, conditions 1.13 (a)(i), (a)(ii), (b) and (c) are met.

2. For all vertices $x, y \in V$, $x \neq y$, that are not yet adjacent in D, we add $\{xy, yx\} \cap A_{\mathrm{maj}}(H)$ to A_D to guarantee that D is semicomplete.

   After step 2, condition 1.13 (a)(iii) is met. The conditions (a)(i), (a)(ii), (b) and (c) remain unaffected.

□

Ideally, we will find a strong out-arc-majority digraph $D = (V, A_D)$ of H, i.e. for every vertex $x \in V$, an out-arc $a \in A_H$ of x is represented by an out arc $xy \in A_D$ of x. In this case, all we need to do is to find a vertex in D, whose all out-arcs are pancyclic in D and can easily translate the cycles involved to corresponding cycles in H via the representative function. But such an out-arc-majority digraph need not exist. All we can guarantee is a strong X-out-arc-majority digraph D for a suitable vertex set $X \subseteq V$. The task is to find such a suitable vertex set that, at the same time, contains a vertex, whose all out-arcs are pancyclic in D, to allow for the translation mentioned above. To make things even more complicated, remember that in general, semicomplete digraphs such as out-arc-majority digraphs need not contain such a vertex. Thus, we will rather have to find a collection of X-out-arc-majority digraphs and a vertex $x \in X$ such that every out-arc of x is pancyclic in at least one of these digraphs.

To this end, in the following section, we give several technical lemmata for later use in the proof of our main result:

Theorem 1.15. *Let H be a strong k-hypertournament on $n \geq k + 2 \geq 5$ vertices. Then H contains a vertex, whose all out-arcs are pancyclic.*

Remark 1.16. The example of a strong $(n-1)$-hypertournament on n vertices without a Hamiltonian cycle given in [6] implies that the bound $n \geq k + 2$ is best possible.

2 Preliminaries

First, we gather some known results. We begin with two lemmata originally formulated for tournaments by Yeo, but they hold for semicomplete digraphs as well.

Lemma 2.1. *[16] Let $D = (V, A_D)$ be a non-strong semicomplete digraph, let $D_1, \ldots, D_r$ be the strong decomposition of D, $1 \le i < j \le r$, $x \in V(D_i)$, $y \in V(D_j)$ and $l \in \{1, \ldots, |\bigcup_{i \le s \le j} V(D_s)| - 1\}$. Then there is an (x, y)-path of length l in D.*

Lemma 2.2. *[16] Let $D = (V, A_D)$ be a strong digraph and let $x \in V$ such that $D - x$ is semicomplete and $d^+_D(x) + d^-_D(x) \ge |V|$. Then there is an l-cycle containing x in D for all $l \in \{2, \ldots, |V|\}$.*

Furthermore, we will use the following version of Hall's marriage theorem and the subsequent obvious corollary.

Theorem 2.3. *[7] Let S be a set, let J be a finite index set and let $(T_i)_{i \in J}$ be a family of subsets of S. Then there is an injective function $r : J \to S$ with $r(i) \in T_i$ for all $i \in J$ if and only if $|I| \le |\bigcup_{i \in I} T_i|$ holds for all $I \subseteq J$.*

Corollary 2.4. *Let $H = (V, A_H)$ be a k-hypertournament, where $k \ge 3$, let $X \subseteq V$, $D = (V, A_D) \in \mathrm{OAMD}_X(H)$ and let C be a cycle in D. If $|I| \le |\bigcup_{vw \in I} A_H(v, w)|$ for all $I \subseteq A(C)$, then every arc in $\bigcup_{vw \in A(C)} A_H(v, w)$ is contained in a cycle C_H in H on the same vertex set as C, particularly of the same length.*

Proof. Let $C = x_1 \ldots x_l x_1$. Theorem 2.3 guarantees the existence of an injective function $r : A(C) \to A_H$ with $r(vw) \in A_H(v, w)$ for all $vw \in A(C)$. Thus, $C_H := x_1 r(x_1x_2) x_2 \ldots x_l r(x_l x_1) x_1$ is a cycle in H. If $a \in A_H(v, w)$ for some $vw \in A(C)$ is not contained in C_H, simply exchange $r(vw)$ for a in C_H. □

Lemma 2.5. *[5] Let $H = (V, A_H)$ be a strong 3-hypertournament on $n \ge 5$ vertices, let $D = (V, A_D)$ be a strong semicomplete digraph on the vertex set of H, $B_D \subseteq A_D$ with $A_D \setminus B_D \subseteq A_{\mathrm{maj}}(H)$ and $r : B_D \to A_H$ an injective function, such that $r(xy) \in A_H(x, y)$ holds for all $xy \in B_D$. Then for every cycle C in D, there is a cycle C_H in H on the same vertex set. Furthermore, if C contains an arc $xy \in B_D$, then C_H can be chosen, such that $r(xy)$ is contained in C_H.*

The following lemma is easy to verify.

Lemma 2.6. *Let $k \ge 4$ and $n \ge k + 2$.*

- *If $(n, k) \notin \{(6, 4), (7, 4), (7, 5)\}$, then $\binom{n-2}{k-2} \ge 2n - 1$ holds.*
- *If $(n, k) \ne (6, 4)$, then $\binom{n-2}{k-2} \ge 2n - 4$ holds.*

To allow us to exchange undesirable arcs of an out-arc-majority digraph for more suitable ones, we give the following definition.

Definition 2.7. Let $H = (V, A_H)$ be a k-hypertournament on $n \geq k \geq 3$ vertices, let $X \subseteq V$, $D = (V, A_D) \in \mathrm{OAMD}_X(H)$, $R_D \in \mathrm{REP}_D$, $xy \in A_V$ and $a \in \mathrm{Out}_H(x) \cap A_H(x, y) \setminus R_D(xy)$. We define $D(R_D, x, a, y) := (V, A_{D(R_D,x,a,y)})$ through:

(i) $(A_D \cup \{xy\}) \setminus \{yx, R^{\downarrow}_D(a), R^{\downarrow}_D(a)^{-1}\} \quad \subseteq \quad A_{D(R_D,x,a,y)}$.

(ii) $A_{D(R_D,x,a,y)} \quad \subseteq \quad A_D \cup \{xy, R^{\downarrow}_D(a)^{-1}\}$.

(iii) $yx \in A_{D(R_D,x,a,y)}$ if and only if $R_D(yx) \neq \emptyset$.

(iv) $R^{\downarrow}_D(a) \in A_{D(R_D,x,a,y)}$ if and only if $R_D(R^{\downarrow}_D(a)) \neq \{a\}$ or $R_D(R^{\downarrow}_D(a)^{-1}) = \emptyset$ and $R^{\downarrow}_D(a) \in A_{\mathrm{maj}}(H)$.

(v) $R^{\downarrow}_D(a)^{-1} \in A_{D(R_D,x,a,y)}$ if and only if $R_D(R^{\downarrow}_D(a)^{-1}) \neq \emptyset$ or $R_D(R^{\downarrow}_D(a)) = \{a\}$ and $R^{\downarrow}_D(a)^{-1} \in A_{\mathrm{maj}}(H)$.

The representative function $R_{D(R_D,x,a,y)} : A_V \to \mathcal{P}(A_H)$ is defined trough:

$$vw \mapsto \begin{cases} R_D(vw), & \text{if } vw \in A_V \setminus \{xy, R^{\downarrow}_D(a)\}. \\ R_D(vw) \cup \{a\}, & \text{if } vw = xy. \\ R_D(vw) \setminus \{a\}, & \text{if } vw = R^{\downarrow}_D(a). \end{cases}$$

It is easy to check that $D(R_D, x, a, y) \in \mathrm{OAMD}_X(H)$ and $R_{D(R_D,x,a,y)} \in \mathrm{REP}_D$ hold, given the assumptions of Definition 2.7. Essentially, we change the representative of the arc $a \in A_H(x, y)$. It is now represented by xy in $D(R_D, x, a, y)$ and no longer by $R^{\downarrow}_D(a)$. All we then have to do, is to consider the reverse arcs of xy and $R^{\downarrow}_D(a)$ to guarantee that the resulting digraph is indeed in $\mathrm{OAMD}_X(H)$. Thus, D and $D(R_D, x, a, y)$ differ in at most four arcs.

We will put this new definition to work immediately in the following lemma.

Lemma 2.8. *Let $H = (V, A_H)$ be a k-hypertournament on $n \geq k \geq 3$ vertices, $D = (V, A_D) \in \mathrm{OAMD(H)}$ and let X be the vertex set of the terminal component of the strong decomposition of D. If there is a vertex $x \in X$ with an out-arc $a \in A_H$ that contains a vertex $y \in V \setminus X$, then there exists a $D' = (V, A_{D'}) \in \mathrm{OAMD(H)}$ such that $|X| < |X'|$ holds for the vertex set X' of the terminal component of the strong decomposition of D'.*

Proof. Let $R_D \in \mathrm{REP}_D$. Since $D[X]$ is a strong semicomplete digraph, by Remark 1.8, there is either a Hamiltonian cycle $x_1x_2 \dots x_lx_1$ in $D[X]$ or $D[X]$ consists of the single vertex x. In the former case let $l := |X|$. Without loss of generality, we may assume that $x = x_l$ and we have $yx_1 \in A_D$ and $xy \notin A_D$, since $x_1, x \in X$ and y is contained in a component preceding X. In the case $X = \{x\}$, a contains a vertex $x_1 \in V \setminus (X \cup \{y\})$, since $k \geq 3$. Without loss of generality, we may assume that $yx_1 \in A_D$. Otherwise, we rename x_1 and y. We define $l = 2$ and $x_l = x_2 := x$. As in the first case, we then have $xy \notin A_D$. Particularly, $a \notin R_D(xy)$ holds in both cases and $D' = (V, A_{D'}) := D(R_D, x, a, y)$ is well-defined.

By Definition 2.7 (i) and (ii), D and D' differ in at most the arcs xy, yx, $R^{\downarrow}_D(a)$ and $R^{\downarrow}_D(a)^{-1}$, which are all incident with $x = x_l$. Hence, $yx_1x_2 \dots x_{l-1}$

is a path in D', since $yx_i \in A_D$ for all $1 \leq i \leq l$ and yx_1 not incident with x. Analogously, $x_{l-1}x_l \notin \{xy, yx, R_D^{\downarrow}(a), R_D^{\downarrow}(a)^{-1}\}$, implies $x_{l-1}x_l \in A_{D'}$. If $x_{l-1}x_l \in \{xy, yx, R_D^{\downarrow}(a), R_D^{\downarrow}(a)^{-1}\}$, then $x_{l-1}x_l = R_D^{\downarrow}(a)^{-1}$, since $R_D^{\downarrow}(a)$ is an out-arc of $x = x_l$ and $y \neq x_{l-1}$. Consequently, we then have $a \in R_D(R_D^{\downarrow}(a)) = R_D(x_l x_{l-1})$ and therefore, $x_l x_{l-1} \in A_D$, by Definition 1.13 (ii). From $\{x_{l-1}x_l, x_l x_{l-1}\} \subseteq A_D$ and $a \in R_D(x_l x_{l-1})$, we get $R_D(R_D^{\downarrow}(a)^{-1}) = R_D(x_{l-1}x_l) \neq \emptyset$, by Definition 1.13 (iii). Thus, as in the first case, we have $x_{l-1}x_l = R_D^{\downarrow}(a)^{-1} \in A_{D'}$, by Definition 2.7 (v). Altogether $C := yx_1x_2 \dots x_l y$ is an $(l+1)$-cycle in D'.

Suppose that $V(C)$ is not a subset of X' (the vertex set of the terminal component of the strong decomposition of D'). By the definition of the strong decomposition, there is a vertex $z \in X' \setminus V(C)$ such that $x_l z \in A_{D'}$ holds. Since $R_D^{\downarrow}(a)$ is an out-arc of x, $z \neq y$ and $A_{D'} \setminus A_D \subseteq \{xy, R_D^{\downarrow}(a)^{-1}\}$, we have $x_l z \in A_D$. Thus, $x_l z$ is an arc from a vertex x_l from the terminal component of the strong decomposition of D to a vertex z from a component preceding it, a contradiction. Therefore, $|X'| \geq |V(C)| = l + 1 > l = |X|$ holds. □

Lemma 2.9. *Let $H = (V, A_H)$ be a strong k-hypertournament on $n \geq k \geq 3$ vertices and $D \in \mathrm{OAMD}(H)$ such that the cardinality of the vertex set X of the terminal component of the strong decomposition of D is maximum. Then $|X| \geq k+1$ holds or H contains a vertex without an out-arc.*

Proof. Suppose that every vertex of H has an out-arc. If X contains less than $k+1$ vertices, then at most one arc of H contains solely vertices from X. The existence of such an arc obviously implies $|X| = k \geq 3$. Combined with the fact that every vertex has an out-arc, it follows that there is a vertex $x \in X$ with an out-arc $a \in A_H$ that contains a vertex $y \in V \setminus X$, a contradiction to the maximality of X, by Lemma 2.8. □

With the next two lemmata we lay some groundwork for the cases $(n, k) \in \{(6,4), (7,4), (7,5)\}$, which we will have to consider separately from all other cases.

Lemma 2.10. *Let $H = (V, A_H)$ be a strong k-hypertournament on $n = k+2$ vertices. Then there exists a strong $D \in \mathrm{OAMD}(H)$ or H contains a vertex without an out-arc.*

Proof. Suppose that every vertex of H has an out-arc. Let $D \in \mathrm{OAMD}(H)$ such that the cardinality of the vertex set X of the terminal component of the strong decomposition of D is maximum. By Lemma 2.9, X contains at least $k+1$ vertices. Suppose that $|X| = k+1 = n-1$. Let $y \in V \setminus X$. Since H is strong, there is an arc $a \in A_H(x', y)$ for some vertex $x' \in V \setminus \{y\} = X$. Obviously, a is an out-arc of some vertex $x \in X$. By Lemma 2.8, there exists a $D' \in$ OAMD(H) such that $n - 1 = |X| < |X'|$ holds for the vertex set X' of the terminal component of the strong decomposition of D'. Therefore, D' is strong. □

Lemma 2.11. *Let $H = (V, A_H)$ be a strong 4-hypertournament on 7 vertices. Then there exists a strong $D \in \mathrm{OAMD}(H)$ or H contains a vertex whose all out-arcs are pancyclic.*

Proof. Suppose that every vertex of H has an out-arc. Let $D \in \mathrm{OAMD}(H)$ such that the cardinality of the vertex set X of the terminal component of the strong decomposition of D is maximum. By Lemma 2.9, X contains at least 5 vertices. If $|X| = 6 = n - 1$, we find a strong $D' \in \mathrm{OAMD(H)}$ as in the proof of Lemma 2.10.

Suppose that $X = \{x_3, \dots, x_7\}$, i.e. $|X| = 5$. By Lemma 2.8, out-arcs of vertices from X contain solely vertices from X. Since $\binom{5}{4} = 5 = |X|$, $x_i \in X$ has exactly one out-arc $a_i \in A_H$ for all $i \in \{3, \dots, 7\}$ and $A_H\big|_X = \{a_3, \dots, a_7\}$ holds. We consider such an out-arc $a_i \in A_H\big|_X$. Without loss of generality, we may assume that $i = 3$. a_3 contains x_3 and three more vertices from X. Without loss of generality, we may assume that these vertices are x_4, x_5 and x_6, where their order is irrelevant. Since every vertex from X is contained in exactly four arcs from $A_H\big|_X$, x_3 is contained in at least two of the arcs from $\{a_4, a_5, a_6\}$. Without loss of generality, we may assume that these arcs are a_5 and a_6. Conversely, every arc from $A_H\big|_X$ contains exactly four vertices from X, and thus, a_4 contains at least one vertex from $\{x_5, x_6\}$. Without loss of generality, we may assume a_4 contains x_6. Altogether, $C_{a_3,3} := x_3a_3x_4a_4x_6a_6x_3$ is a 3-cycle in H that contains a_3.

Since x_7 is not contained in a_3, it is contained in a_4, a_5 and a_6. Furthermore, a_7 contains either x_5 or x_6. Without loss of generality, we may assume that it contains x_6. Then $C_{a_3,4} := x_3a_3x_4a_4x_7a_7x_6a_6x_3$ is a 4-cycle in H that contains a_3.

If a_7 does not contain the vertex x_5, then it is contained in a_6 and we obtain a 5-cycle $C_{a_3,5} := x_3a_3x_4a_4x_7a_7x_6a_6x_5a_5x_3$ in H that contains a_3. If a_7 contains x_5 (in addition to x_6), then we consider a_4, which, again, contains at least one of these two vertices. Without loss of generality, we may assume that a_4 contains x_6, whereby we obtain the 5-cycle $C_{a_3,5} := x_3a_3x_4a_4x_6a_6x_7a_7x_5a_5x_3$ in H that contains a_3. Since $i = 3$ was chosen arbitrarily, $a_i \in A_H\big|_X$ is contained in an l-cycle in H that consists solely of arcs from $A_H\big|_X$ for all $l \in \{3, 4, 5\}$ and all $i \in \{3, \dots, 7\}$.

Let $\{x_1, x_2\} := V \setminus X$. Since H is strong, there is an arc $a \in A_H$ from X to $\{x_1, x_2\}$. Without loss of generality, we may assume that $a \in A_H(x_7, x_1)$. By Lemma 2.8, all arcs that contain a vertex from $\{x_1, x_2\}$ are also an out-arc of a vertex from $\{x_1, x_2\}$. Thus, $a \in A_H$ is an out-arc of x_2. Conversely, there is an out-arc $b \in A_H$ of x_1 that contains x_2. a and b are obviously pairwise distinct from all arcs in $A_H\big|_X$. Let $i_0 \in \{3, \dots, 7\}$ such that $a_{i_0} \in A_H\big|_X$ does not contain x_7 and let $C = y_1b_1y_2b_2 \dots y_5b_5y_1$ be a 5-cycle in H that consists solely of arcs from $A_H\big|_X$. Without loss of generality, we may assume that $y_5 = x_7$. $b_5 \neq a_{i_0}$ follows by the choice of a_{i_0}. Furthermore, for all $j \in \{1, 2\}$, let a_j be an arc that contains x_j, y_1 and two more vertices from X. By Lemma 2.8, a_j is an out-arc of x_j that contains y_1. By definition, it is also distinct from all arcs in $\{a, b\} \cup A_H\big|_X$ for all $j \in \{1, 2\}$. Consequently, $C_{a_{i_0},6} := y_1b_1y_2b_2 \dots y_5ax_1a_1y_1$ is a 6-cycle and $C_{a_{i_0},7} := y_1b_1y_2b_2 \dots y_5ax_1bx_2a_2y_1$ is a 7-cycle in H that contains a_{i_0}. Altogether, a_{i_0}, the sole out-arc of x_{i_0}, is pancyclic in H. □

We will use the following lemma in the case that there is no strong X-out-arc-majority digraph containing a vertex whose all out-arcs are vertex pancyclic. As mentioned in the introduction, we will then consider different X-out-arc-majority digraphs for each out-arc of a suitable vertex.

Lemma 2.12. *Let $H = (V, A_H)$ be a k-hypertournament on $n \geq k+2 \geq 5$ vertices. Let $X \subseteq V$ be self-contained, let $D = (V, A_D) \in \mathrm{OAMD}_X(H)$ and let $R_D \in \mathrm{REP}_D$. For every $xy \in A_D \cap A_X$ with $R_D(xy) \neq \emptyset$, there exist $D_{xy} = (V, A_{D_{xy}}) \in \mathrm{OAMD}_X(H)$ and $R_{D_{xy}} \in \mathrm{REP}_{D_{xy}}$ with the following properties:*

(i) $A_D \cup A_X \supseteq A_{D_{xy}} \supseteq A_D \setminus (\{zy \mid zy \in A_D, z \in X \setminus \{x\}\} \cup \{yx\})$.

(ii) $R_{D_{xy}}(xy) = R_D(xy)$.

(iii) $d^+_{D_{xy}}(x) = d^+_D(x)$.

(iv) $d^+_{D_{xy}}(y) \geq 1$.

(v) $d^+_{D_{xy}}(y) \geq d^+_D(y)$ *or* $|A_{D_{xy}} \cap A_X| < |A_D \cap A_X|$.

(vi) $yx \notin A_{D_{xy}}$.

Proof. We will prove the following by inverse induction on m:
for all $m \in \{0, \ldots, |R_D(yx)|\}$, there are $D_m \in \mathrm{OAMD}_X(H)$ and $R_{D_m} \in \mathrm{REP}_{D_m}$ such that

$$A_D \cup A_X \supseteq A_{D_m} \supseteq A_D \setminus (\{zy \in A_D \mid z \in X \setminus \{x\}\} \cup \{yx\}),$$

$$R_{D_m}(xy) = R_D(xy), \;\; R_{D_m}(yx) \subseteq R_D(yx), \;\; |R_{D_m}(yx)| = m,$$

$$d^+_{D_m}(x) = d^+_D(x), \;\; d^+_{D_m}(y) \geq 1 \;\text{ and}$$

$$(d^+_{D_m}(y) - |A_{D_m} \cap A_X|) - (d^+_D(y) - |A_D \cap A_X|) \geq 0$$

hold. The base case is trivial ($D_{|R_D(yx)|} := D$).

Let the statement be true for an $m \in \{1, \ldots, |R_D(yx)|\}$. By induction hypothesis, we have $|R_{D_m}(yx)| = m \geq 1$. Thus, we may choose an arc $a = (x_1, \ldots, x_k) \in R_{D_m}(yx)$. Definition 1.13 (b) implies $x_1 = y$, since $R_{D_m}(yx) \subseteq \mathrm{Out}_H(y)$. Furthermore, $x = x_i$ holds for some $i \in \{2, \ldots, k\}$, by Definition 1.13 (a)(i). Therefore, a is contained in $\mathrm{Out}_H(y) \cap A_H(y, x_j) \setminus R_{D_m}(yx_j)$ for some $j \in \{2, \ldots, k\} \setminus \{i\}$, since $a \in R_{D_m}(yx_i)$ and $k \geq 3$. Hence, $D_{m-1} := D(R_{D_m}, y, a, x_j) \in \mathrm{OAMD}_X(H)$ and $R_{D_{m-1}} := R_{D(R_{D_m}, y, a, x_j)} \in \mathrm{REP}_{D_{m-1}}$ are well-defined. By Definition 2.7 (i) and (ii), $A_{D_m} \cup \{yx_j, xy\} \supseteq A_{D_{m-1}} \supseteq (A_{D_m} \cup \{yx_j\}) \setminus \{x_j y, yx, xy\}$ holds, which implies $d^+_{D_{m-1}}(y) \geq 1$. Furthermore, we have $xy \in A_{D_{m-1}}$, by Definition 2.7 (iv), since $R_{D_m}(xy) \neq \emptyset$. Thus, $d^+_{D_{m-1}}(x) = d^+_{D_m}(x) = d^+_D(x)$. Altogether, we have

$$\begin{aligned} A_D \cup A_X &\supseteq A_{D_m} \cup A_X \\ &\supseteq A_{D_{m-1}} \\ &\supseteq A_{D_m} \setminus (\{zy \mid zy \in A_{D_m}, z \in X \setminus \{x\}\} \cup \{yx\}) \\ &\supseteq A_D \setminus (\{zy \mid zy \in A_D, z \in X \setminus \{x\}\} \cup \{yx\}), \end{aligned}$$

by induction hypothesis. In addition, we have $R_{D_{m-1}}(xy) = R_{D_m}(xy) = R_D(xy)$, $R_{D_{m-1}}(yx) = R_{D_m}(yx) \setminus \{a\} \subseteq R_D(yx)$ and $|R_{D_{m-1}}(yx)| = |R_{D_m}(yx) \setminus \{a\}| = m - 1$, by Definition 2.7 of $R_{D_{m-1}} = R_{D(R_{D_m},y,a,x_j)}$ and induction hypothesis. Since X is self-contained, $x_1 = y \in X$ implies $x_j \in X$.

Suppose that $d^+_{D_{m-1}}(y) < d^+_{D_m}(y)$. By Definition 2.7 (i), $A_{D_{m-1}}$ is a superset of $(A_{D_m} \cup \{yx_j\}) \setminus \{x_jy, yx, xy\}$, which implies $d^+_{D_{m-1}}(y) = d^+_{D_m}(y) - 1$, $yx_j \in A_{D_m}$ and $yx \notin A_{D_{m-1}}$, since $xy \in A_{D_{m-1}}$. Furthermore, we have $R_{D_m}(xy) = R_D(xy) \neq \emptyset$ and $|R_{D_m}(yx)| = m \geq 1$ and thus, $xy, yx \in A_{D_m}$, by Definition 1.13 (a)(ii). $A_{D_{m-1}} \subseteq A_D$ and $yx \in A_{D_m} \setminus A_{D_{m-1}}$ follow, since $A_{D_{m-1}} \subseteq A_{D_m} \cup \{yx_j, xy\}$, by Definition 2.7 (ii). Consequently,

$$|A_{D_{m-1}} \cap A_X| \;\leq\; |(A_{D_m} \cap A_X) \setminus \{yx\}| \;=\; |A_{D_m} \cap A_X| - 1$$

and thus,

$$\begin{aligned} & (d^+_{D_{m-1}}(y) - |A_{D_{m-1}} \cap A_X|) - (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) \\ \geq\; & (d^+_{D_m}(y) - 1 - (|A_{D_m} \cap A_X| - 1)) - (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) \\ =\; & 0 \end{aligned}$$

hold. Suppose now that $|A_{D_{m-1}} \cap A_X| > |A_{D_m} \cap A_X|$. We have $R_{D_m}(xy) = R_D(xy) \neq \emptyset$ and thus, $xy \in A_{D_m}$, by Definition 1.13 (a)(ii). Therefore, by Definition 2.7 (i), $A_{D_{m-1}} \subseteq A_{D_m} \cup \{yx_j, xy\}$, implies $A_{D_{m-1}} = A_{D_m} \cup \{yx_j\}$ and $yx_j \in A_{D_{m-1}} \setminus A_{D_m}$. Hence,

$$\begin{aligned} & (d^+_{D_{m-1}}(y) - |A_{D_{m-1}} \cap A_X|) - (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) \\ =\; & (d^+_{D_m}(y) + 1 - (|A_{D_m} \cap A_X| + 1)) - (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) \\ =\; & 0 \end{aligned}$$

holds. Finally, if $d^+_{D_{m-1}}(y) \geq d^+_{D_m}(y)$ and $|A_{D_{m-1}} \cap A_X| \leq |A_{D_m} \cap A_X|$, then

$$(d^+_{D_{m-1}}(y) - |A_{D_{m-1}} \cap A_X|) - (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) \;\geq\; 0$$

is a direct consequence.

Altogether, we have

$$\begin{aligned} & (d^+_{D_{m-1}}(y) - |A_{D_{m-1}} \cap A_X|) - (d^+_{D}(y) - |A_{D} \cap A_X|) \\ =\; & (d^+_{D_{m-1}}(y) - |A_{D_{m-1}} \cap A_X|) - (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) \\ +\; & (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) - (d^+_{D}(y) - |A_{D} \cap A_X|) \\ \geq\; & (d^+_{D_m}(y) - |A_{D_m} \cap A_X|) - (d^+_{D}(y) - |A_{D} \cap A_X|) \\ \geq\; & 0, \end{aligned}$$

by induction hypothesis.

Therefore, our statement holds by inverse induction. We consider $D_{xy} := D_0$. D_{xy} obviously has the properties (i), (ii), (iii) and (iv). Property (v) is implied by $(d^+_{D_0}(y) - |A_{D_0} \cap A_X|) - (d^+_D(y) - |A_D \cap A_X|) \geq 0$. Since $R_{D_{xy}}(xy) = R_{D_0}(xy) = R_D(xy) \neq \emptyset$ and $|R_{D_{xy}}(yx)| = |R_{D_0}(yx)| = 0$, we have $xy \in A_{D_{xy}}$, by Definition 1.13 (a)(ii), and $\{xy, yx\} \nsubseteq A_{D_{xy}}$, by Definition 1.13 (a)(iii). Thus (vi) holds as well. □

Before we give some more technical lemmata, we will need some additional notation for certain classes of X-out-arc-majority digraphs.

Definition 2.13. Let $H = (V, A_H)$ be a strong k-hypertournament on $n \geq k \geq 3$ vertices, $X \subseteq V$ and let $D = (V, A_D) \in \mathrm{OAMD}_X(H)$. We then define the following classes of X-out-arc-majority digraphs of H:

$$\mathrm{EXC}_X(D) := \{D' = (V, (A_D \setminus A_X) \cup B) \in \mathrm{OAMD}_X(H)\} \mid B \subseteq A_X\}$$

$$\mathrm{EXC}_X^S(D) := \{D' \in \mathrm{EXC}_X(D) \mid D' \text{ is strong.}\}$$

$$\mathrm{MIN}_X^S(D) := \{D' \in \mathrm{EXC}_X^S(D) \mid \min_{D'' \in \mathrm{EXC}_X^S(D)} \{|A_{D''} \cap A_X|\} = |A_{D'} \cap A_X|\}$$

While the previous lemmata dealt with properties of certain X-out-arc-majority digraphs, the following ones will deal with vertices that are candidates for having pancyclic out-arcs, particularly those with small out-degree.

Lemma 2.14. *Let $D = (V, A_D)$ be a strong semicomplete digraph and let $v_1, \ldots, v_{|V|}$ be an enumeration of its vertices such that $d_D^+(v_1) \leq \ldots \leq d_D^+(v_{|V|})$. Furthermore, let $x \in \{v_1, v_2\}$ and $y \in V \setminus \{x\}$. If $D - y$ is not strong and $D_1, \ldots, D_r$ is the strong decomposition of $D - y$, then at least one of the following conditions holds:*

- *$x \in V(D_r)$.*
- *$xy \notin A_D$, $V(D_{r-1}) = \{x\}$, $D_r - v_1$ is a complete digraph and $zy \in A_D$ for all $z \in V(D_r - v_1)$. Particularly, $d_D^+(x) = d_D^+(z)$ follows for all $z \in V(D_r - v_1)$.*
- *$x = v_2$ and $V(D_r) = \{v_1\}$. In particular, $\delta^+(D) = 1$.*

Proof. By the definition of the strong decomposition, all vertices $z \in V(D_r)$ only have out-arcs to vertices in $V(D_r) \cup \{y\}$. Thus, $d_D^+(z) \leq |V(D_r)|$ for all $z \in V(D_r)$. Suppose that $x \in V(D_i)$ for some $i \in \{1, \ldots, r-1\}$, which implies $xz \in A_D$ for all $z \in V(D_r)$ and thus, $d_D^+(x) \geq |V(D_r)| \geq d_D^+(z)$ for all $z \in V(D_r)$. If $V(D_r) = \{v_1\}$, then we have $x = v_2$ and $d_D^+(v_1) \leq 1$. Therefore, $\delta^+(D) = 1$, since D is strong. If $V(D_r) \neq \{v_1\}$, then $d_D^+(x) = |V(D_r)| = d_D^+(z)$ holds for all $z \in V(D_r - v_1)$, by choice of x. As a direct consequence, $D_r - v_1$ is a complete digraph, $zy \in A_D$ for all $z \in V(D_r - v_1)$ and x has no out-arcs that do not end in D_r. Therefore, we have $xy \notin A_D$, $i = r - 1$, since otherwise $x\bar{z} \in A_D$ for all $\bar{z} \in V(D_{r-1})$, and $V(D_{r-1}) = \{x\}$, since otherwise the strong connectivity of D_{r-1} would require an arc $x\bar{z} \in A_D$ for some $\bar{z} \in V(D_{r-1})$. □

Lemma 2.15. *Let $H = (V, A_H)$ be a strong k-hypertournament on $n \geq k + 2 \geq 5$ vertices. Let $X \subseteq V$ be self-contained, let $D = (V, A_D) \in \mathrm{MIN}_X^S(D)$ and $R_D \in \mathrm{REP}_D$. Furthermore, let $v_1, \ldots, v_n$ be an enumeration of V such that $d_D^+(v_1) \leq \ldots \leq d_D^+(v_n)$. If $x \in \{v_1, v_2\}$ such that $D - x$ is strong and $xy \in A_D \cap A_X$ such that $R_D(xy) \neq \emptyset$ and $d_D^+(x) \leq d_D^+(y)$, then either $\delta^+(D) = 1$ holds or there exist $D_{xy} = (V, A_{D_{xy}}) \in \mathrm{EXC}_X^S(D)$ such that xy is contained in $A_{D_{xy}}$ and pancyclic in D_{xy}, and $R_{D_{xy}} \in \mathrm{REP}_{D_{xy}}$ such that $R_{D_{xy}}(xy) = R_D(xy)$.*

Proof. Suppose that $D - y$ is not strong. If x is not contained in the terminal component of the strong decomposition of $D - y$, then $xy \in A_D$ implies $\delta^+(D) = 1$, by Lemma 2.14. Suppose now that x is contained in the terminal component. Since D is strong, there is a vertex x_1 in the initial component of the strong decomposition of $D - y$ such that $yx_1 \in A_D$. By Lemma 2.1, there is a Hamiltonian path $x_1 \dots x_{n-1}$ in $D - y$ with $x_{n-1} = x$. Thus, $C := x_1 \dots x_{n-1} y x_1$ is a Hamiltonian cycle in D with $x_{n-1} = x$.

Suppose now that $D - y$ is strong. If $D - \{x, y\}$ is not strong, then there is a vertex x_1 in the initial component of the strong decomposition of $D - \{x, y\}$ and a vertex x_{n-2} in the terminal component such that $yx_1, x_{n-2}x \in A_D$, since $D - x$ and $D - y$ are strong. By Lemma 2.1, there is a Hamiltonian path $x_1 \dots x_{n-2}$ in $D - \{x, y\}$. Thus, $C := x_1 \dots x_{n-2} xyx_1$ is a Hamiltonian cycle in D.

Let us now consider the X-out-arc-majority digraph D_{xy} of H from Lemma 2.12. Then we have $R_{D_{xy}}(xy) = R_D(xy) \neq \emptyset$ and therefore, $xy \in A_{D_{xy}}$ by Definition 1.13 (a)(ii). Suppose that there is a Hamiltonian cycle $C = x_1 \dots x_{n-2} xyx_1$ in D. Then C is also a Hamiltonian cycle in D_{xy}, since $A_{D_{xy}} \supseteq A_D \setminus (\{zy \mid zy \in A_D, z \in X \setminus \{x\}\} \cup \{yx\})$.

Suppose now that there is no such Hamiltonian cycle C in D. We then already know that $D - y$ and $D - \{x, y\}$ are strong. D_{xy} is strong as well, since $D_{xy} - y \supseteq D - y$, $d^+_{D_{xy}}(y) \geq 1$ and $xy \in A_{D_{xy}}$, by Lemma 2.12 (i), (iv) and (vi), respectively. Lemma 2.12 (i) implies $D_{xy} \in \mathrm{EXC}^S_X(D)$. Let $\bar{C} = x_1 \dots x_{n-2} x_1$ be a Hamiltonian cycle in $D - \{x, y\}$. Since $D \in \mathrm{MIN}^S_X(D)$ and $D_{xy} \in \mathrm{EXC}^S_X(D)$, we have $|A_{D_{xy}} \cap A_X| \geq |A_D \cap A_X|$, by Definition 2.13. Therefore, Lemma 2.12 (v) and (iii) imply $d^+_{D_{xy}}(y) \geq d^+_D(y) \geq d^+_D(x) = d^+_{D_{xy}}(x)$. Furthermore, $yx \notin A_{D_{xy}}$ holds, by Lemma 2.12 (vi) and thus, $d^+_{D_{xy}-x}(y) > d^+_{D_{xy}-y}(x)$. Consequently, there is an index $i \in \{1, \dots, n-2\}$ such that $yx_i \in A_{D_{xy}}$ but $xx_{i-1} \notin A_{D_{xy}}$ (where $x_0 := x_{n-2}$). Without loss of generality, we may assume $i = 1$. Hence, $x_{n-2}x \in A_{D_{xy}}$, since D_{xy} is semicomplete, and therefore, $C := x_1 \dots x_{n-2} xyx_1$ is a Hamiltonian cycle in D_{xy}.

Thus, in both cases, D_{xy} is strong and contains a Hamiltonian cycle $C = x_1 \dots x_{n-2} xyx_1$. We now define the digraph

$$D'_{xy} := ((V \setminus \{x, y\}) \cup \{v_{x,y}\}, A_{D'_{xy}}),$$

where $v_{x,y} \notin V$, through

$$A_{D'_{xy}} := (A_{D_{xy}} \cap A_{V \setminus \{x,y\}}) \cup \{v_{x,y}z \mid yz \in A_{D_{xy}}\} \cup \{zv_{x,y} \mid zx \in A_{D_{xy}}\}.$$

D'_{xy} contains the Hamiltonian cycle $x_1 \dots x_{n-2} v_{x,y} x_1$ and is therefore strong. Furthermore, $D'_{xy} - v_{x,y} \subseteq D_{xy}$ is semicomplete and

$$\begin{aligned} d^+_{D'_{xy}}(v_{x,y}) + d^-_{D'_{xy}}(v_{x,y}) &= d^+_{D_{xy}-x}(y) + d^-_{D_{xy}-y}(x) = d^+_{D_{xy}}(y) + d^-_{D_{xy}}(x) \\ &\geq d^+_{D_{xy}}(y) + (n - 1 - d^+_{D_{xy}}(x)) \geq n - 1 = |D'_{xy}| \end{aligned}$$

holds, since D_{xy} is semicomplete, $yx \notin A_{D_{xy}}$ and $d^+_{D_{xy}}(y) \geq d^+_{D_{xy}}(x)$. By Lemma 2.2, there is an l-cycle $C'_l = x'_1 \dots x'_l x'_1$ in D'_{xy} that contains $v_{x,y}$ for all $2 \leq l \leq |D'_{xy}| =$

$n-1$. Without loss of generality, we may assume that $x'_l = v_{x,y}$. Consequently, the arc $xy \in D_{xy}$ is contained in an $(l+1)$-cycle $C_{l+1} := x'_1 \dots x'_{l-1}xyx'_1$ for all $2 \le l \le |D'_{xy}| = n-1$ and thus, is pancyclic in D_{xy}. □

Lemma 2.16. *Let $H = (V, A_H)$ be a strong k-hypertournament on $n \ge k+2 \ge 5$ vertices. Let $X \subseteq V$ be self-contained, let $D = (V, A_D) \in \mathrm{MIN}^S_X(D)$ and $R_D \in \mathrm{REP}_D$. Furthermore, let $v_1, \dots, v_n$ be an enumeration of V such that $d^+_D(v_1) \le \dots \le d^+_D(v_n)$. Then there is a vertex $x \in \{v_1, v_2\}$ such that the following statement holds: For all arcs $xy \in A_D \cap A_X$ with $R_D(xy) \ne \emptyset$, there exist $D_{xy} = (V, A_{D_{xy}}) \in \mathrm{EXC}^S_X(D)$ such that $xy \in A_{D_{xy}}$ is pancyclic in D_{xy}, and $R_{D_{xy}} \in \mathrm{REP}_{D_{xy}}$ such that $R_{D_{xy}}(xy) = R_D(xy)$.*

Proof. If $\delta^+(D) = 1$, then v_1 has exactly one out-arc $v_1y \in A_D$. This arc is pancyclic in D, since D, a strong semicomplete digraph, is vertex-pancyclic. Suppose now that $d^+_D(v_1) = \delta^+(D) \ge 2$.

Suppose that $D - v_1$ is strong. $d^+_D(v_1) \le d^+_D(y)$ is trivially true for all arcs $v_1y \in A_D \cap A_X$ with $R_D(v_1y) \ne \emptyset$ and thus, we obtain a strong X-out-arc-majority digraph D_{v_1y} via Lemma 2.15.

Suppose now that $D - v_1$ is not strong. Let $D_1, \dots, D_r$ be the strong decomposition of $D - v_1$. Suppose that $v_2 \notin V(D_r)$. Then we have $v_2v_1 \notin A_D$ by Lemma 2.14 and thus, $v_1v_2 \in A_D$. Furthermore, the completeness of D_r and the existence of $zv_1 \in A_D$ for all $z \in V(D_r)$, guaranteed by Lemma 2.14, imply the existence of a (z, v_1)-path P^l_{z,v_1} of length l in $D[V(D_r) \cup \{v_1\}]$ for all $z \in V(D_r)$ and for all l in $\{1, \dots, |V(D_r)|\}$. In addition, there is an $x_1 \in V(D_1)$ such that $v_1x_1 \in A_D$, since D is strong. By Lemma 2.1, there is an (x_1, v_2)-path $P^l_{x_1v_2}$ of length l in $D[\bigcup_{1\le s\le r-1} V(D_s)]$ for all $l \in \{1, \dots, |\bigcup_{1\le s\le r-1} V(D_s)| - 1\}$. Let $v_2z \in A_D$ and $l \in \{3, \dots, n\}$ be arbitrarily chosen. If $l \le |V(D_r)| + 2$, then v_2z is contained in the l-cycle $v_2P^{l-2}_{z,v_1}v_2$ in D. If $l > |V(D_r)| + 2$, then v_2z is contained in the l-cycle $P^{|V(D_r)|+2-l}_{x_1,v_2}P^{|V(D_r)|}_{z,v_1}x_1$ in D. Therefore, all out-arcs of v_2 are pancyclic in D. Thus, we may assume that $v_2 \in V(D_r)$.

Suppose that $D - v_2$ is not strong. Let $\tilde{D}_1, \dots, \tilde{D}_t$ be the strong decomposition of $D - v_2$. It follows analogously that all out-arcs of v_1 are pancyclic in D or that $v_1 \in V(\tilde{D}_t)$. Thus, we may assume that $v_1 \in V(\tilde{D}_t)$. By the definition of the strong decomposition, we have $zv \notin A_D$ for all $zv \in (V \setminus (V(D_1) \cup \{v_1\})) \times V(D_1)$ and all $zv \in (V \setminus (V(\tilde{D}_1) \cup \{v_2\})) \times V(\tilde{D}_1)$. If there are vertices $z \in V(\tilde{D}_1) \setminus V(D_1)$ and $v \in V(\tilde{D}_s) \cap V(D_1)$ for an $s \in \{2, \dots, t\}$, then $zv \in (V \setminus (V(D_1) \cup \{v_1\})) \times V(D_1)$ and $zv \in A_D$ hold by the definition of the strong composition, a contradiction. If $V(\tilde{D}_1) \setminus V(D_1) \ne \emptyset \ne V(\tilde{D}_1) \cap V(D_1)$, we reach the same contradiction by consideration of an arc $zv \in (V(\tilde{D}_1) \setminus V(D_1)) \times (V(\tilde{D}_1) \cap V(D_1)) \subseteq (V \setminus (V(D_1) \cup \{v_1\})) \times V(D_1)$, which exists, since $\tilde{D}_1$ is strong. Therefore, $V(\tilde{D}_1) \subseteq V(D_1)$ holds. But then $v \in V(\tilde{D}_1) \subseteq V(D_1)$ already implies that $v_2v \in (V \setminus (V(D_1) \cup \{v_1\})) \times V(D_1)$ and thus, $v_2v \notin A_D$. Consequently, D is not strong, a contradiction.

Hence, $D - v_2$ is strong. For all arcs $v_2y \in (A_D \cap A_X) \setminus \{v_2v_1\}$ with $R_D(v_2y) \ne \emptyset$ we have $d^+_D(v_2) \le d^+_D(y)$ and thus, we obtain a suitable strong X-out-arc-majority digraph D_{v_2y} via Lemma 2.15. Since D is strong, there is a vertex $x_1 \in V(D_1)$ such that $v_1x_1 \in A_D$. By Lemma 2.1 there exists an (x_1, v_2)-path $P^l_{x_1,v_2}$ of length l in $D - v_1$ for all $l \in \{1, \ldots, n-2\}$. If $v_2v_1 \in A_D$, then v_2v_1 is contained in the l-cycle $v_2v_1P^{l-2}_{x_1,v_2}$ in D for all $l \in \{3, \ldots, n\}$. Thus, all out-arcs of v_2 are pancyclic. □

The final lemma of this section will provide a self-contained vertex set $X \subseteq V$ and an appropriate X-out-arc-majority digraph that will enable us to apply the previous lemmata.

Lemma 2.17. *Let $H = (V, A_H)$ be a strong k-hypertournament on $n \geq k + 2 \geq 5$ vertices such that every vertex of H has at least one out-arc. Then there exist a self-contained vertex set $X \subseteq V$ and a strong digraph $D \in \mathrm{OAMD}_X(H)$ such that the following statement holds: For all digraphs $D' \in \mathrm{EXC}_X(D)$ and enumerations $v_1, \ldots, v_n$ of V such that $d^+_{D'}(v_1) \leq \ldots \leq d^+_{D'}(v_n)$ we have $\{v_1, v_2\} \subseteq X$.*

Proof. If there is a strong $D \in \mathrm{OAMD}(H)$, then we are finished. Thus, we may assume that there is no such digraph. Let $D'' = (V, A_{D''}) \in \mathrm{OAMD}(H)$ such that the cardinality of the vertex set X of the terminal component of the strong decomposition of D'' is maximum and let $R_{D''} \in \mathrm{REP}_{D''}$. By Lemma 2.8, X is self-contained. We construct a suitable X-out-arc-majority digraph $D = (V, A_D)$ of H and an $R_D \in \mathrm{REP_D}$ as follows:

0. We start with $D := (V, A_{D''} \cap A_X)$ and

   $$R_D : A_V \to \mathcal{P}(A_H),\ xy \mapsto \begin{cases} R_{D''}(xy), & \text{if } xy \in A_X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

   After this step, D restricted to A_X is identical to $R_{D''} \in \mathrm{REP}_{D''}$. Thus, the conditions 1.13 (a) and (b) hold for all $xy \in A_X$ as well as $R_D(xy) \subseteq A_H(x,y)\big|_X$, since otherwise, there would exist an arc $a = (x_1, \ldots, x_k) \in A_H(x, y)$ such that $x_1 = x \in X$ and an index $i_0 \in \{2, \ldots, k\}$ such that $x_{i_0} \in V \setminus X$, in contradiction to Lemma 2.8. Furthermore, for all $a \in A_H$ there is at most one $xy \in A_D$ with $a \in R_D(xy)$.

1. For all $xy \in (V \setminus X) \times X$ we have $A_H(x,y)\big|_X \subseteq \mathrm{Out}_H(x)$, since otherwise, there would exist an arc $a = (x_1, \ldots, x_k) \in A_H(x,y)\big|_X$ such that $x_1 \in X$ and an index $i_0 \in \{2, \ldots, k\}$ with $x_{i_0} = x \in V \setminus X$, in contradiction to Lemma 2.8. Particularly, $A_H(x_1, y_1)\big|_X \cap A_H(x_2, y_2)\big|_X = \emptyset$ holds for all $x_1, x_2 \in V \setminus X$, $x_1 \neq x_2$ and $y_1, y_2 \in X$. Furthermore, we have

   $$|A_H(x,y)\big|_X| = \binom{|X|-1}{k-2} \geq \binom{|X|-1}{(|X|-1)-2} = \binom{|X|-1}{2}$$
   $$= \frac{|X|^2 - 3|X| + 2}{2} \geq |X| - 1$$

   for all $xy \in (V \setminus X) \times X$ and $k \geq 4$, since $k \leq |X| - 1$ by choice of D'' and Lemma 2.9. For $k = 3$, $|A_H(x,y)\big|_X| = |X| - 1$ follows directly.

   Let $x \in V \setminus X$ and $I \subseteq \{x\} \times X$. If $|I| > |\bigcup_{xy \in I} A_H(x,y)\big|_X|$, then $|I| = |X|$ and $|\bigcup_{xy \in I} A_H(x,y)\big|_X| = |X| - 1 = |A_H(x,y)\big|_X|$ holds for all $xy \in I$ and thus, $A_H(xy_1)\big|_X = A_H(xy_2)\big|_X$ for all $xy_1, xy_2 \in I$. Therefore, every arc in $\bigcup_{xy\in I} A_H(x,y)\big|_X$ contains at least $|I| = |X| \geq k+1$ vertices, a contradiction. Hence, $|I| \leq |\bigcup_{xy\in I} A_H(x,y)\big|_X|$ and we obtain an injective function $r_x : \{x\} \times X \to A_H$ such that $r_x(xy) \in A_H(x,y)\big|_X$ via Hall's marriage theorem (2.3). Since $A_H(x_1,y_1)\big|_X \cap A_H(x_2y_2)\big|_X = \emptyset$ for all $x_1, x_2 \in V \setminus X$, $x_1 \neq x_2$ and $y_1, y_2 \in X$,

$$r : (V \setminus X) \times X \to A_H,\ xy \mapsto r_x(y)$$

is an injective function such that $r(xy) \in A_H(x,y)\big|_X$ for all $xy \in (V \setminus X) \times X$. For all $xy \in (V \setminus X) \times X$, we add xy to A_D and define $R_D(xy) := \{r(xy)\}$.

For an arc xy added to A_D in step 1, we have $R_D(xy) = \{r(xy)\}$ and thus, $\emptyset \neq R_D(xy) \subseteq A_H(x,y)\big|_X \subseteq A_H(x,y)$, $xy \notin A_X$ and $yx \notin A_D$. Therefore, the conditions 1.13 (a) and (b) are met. Furthermore, $R_D(x_1y_1) \cap R_D(x_2y_2) = \emptyset$ holds for all $x_1y_1, x_2y_2 \in ((V \setminus X) \times X) \cap A_D$, $x_1y_1 \neq x_2y_2$, since r is injective. Thus, for all arcs $a \in A_H$, there is at most one $xy \in A_D$ such that $a \in R_D(xy)$, since arcs in $R_D(xy)$ contain only vertices from X, if $xy \in A_X \cap A_D$, and contain exactly one vertex from $V \setminus X$, if $xy \in ((V \setminus X) \times X) \cap A_D$.

2. For $xy \in A_{V\setminus X}$, we add the arc xy to A_D, if $|A_H(x,y)\big|_X| \geq |A_H(y,x)\big|_X|$ holds and define $R_D(xy) := A_H(x,y)\big|_X$.

   After step 2, D is semicomplete, since either xy or yx were added to A_D for all $xy \in A_{V\setminus X}$, $D[X]$ was already semicomplete as an induced subgraph of D'' and $(V \setminus X) \times X \subseteq A_D$ holds, by step 1. Furthermore, D is not strong and the terminal component of the strong decomposition of D is identical to the one of D''.

   For an arc xy added to A_D in step 2, we have $R_D(xy) = A_H(x,y)\big|_X \subseteq \mathrm{Out}_H(x)$, since otherwise, there would be an arc $a = (x_1, \ldots, x_k) \in A_H(x,y)\big|_X$ such that $x_1 \in X$ and an index $i_0 \in \{2, \ldots, k\}$ such that $x_{i_0} = x \in V \setminus X$, in contradiction to Lemma 2.8. We know $R_D(xy) \subseteq A_H(x,y)$ and in addition, $R_D(xy) \neq \emptyset$ holds if and only if $xy \in A_D$, which implies the conditions 1.13 (a)(i) and (ii), respectively. Since $\{xy, yx\} \subseteq A_D$ if and only if $A_H(x,y)\big|_X = A_H(yx)\big|_X$ and thus, $R_D(xy) \neq \emptyset \neq R_D(yx)$, condition 1.13 (a)(iii) is met as well. Condition 1.13 (b) remains unaffected, since only arcs $xy \notin A_X$ were added to A_D.

   $xy \in A_{V\setminus X} \cap A_D$ is the sole arc in D such that arcs in $R_D(xy)$ contain x, $y \in V \setminus X$ in this order. Combined with the fact that the arcs in $R_D(xy)$ contain at most one vertex from $V \setminus X$ for all $xy \in (A_X \cup (V \setminus X) \times X) \cap A_D$, we see that for all $a \in A_H$, there exists at most one arc $xy \in A_D$ such that $a \in R_D(xy)$.

3. Since H is strong, there is a shortest path $P = y_1a_1y_2 \ldots y_l$ in H from a vertex y_1 in the terminal component of the strong decomposition of D to a vertex y_l in the initial component. For $i \in \{1, \ldots, l-1\}$, we add the arc y_iy_{i+1} to A_D if and only if $y_iy_{i+1} \notin A_D$. In this case, we define $R_D(y_iy_{i+1}) := \{a_i\}$ and remove a_i from $R_D(xy)$ for all arcs $xy \in A_D \setminus \{y_iy_{i+1}\}$ with $a_i \in R_D(xy)$.

After step 3, D is strong. Let $i \in \{1, \ldots, l-1\}$. Suppose that before step 3, there was an arc $xy \in A_D \setminus \{y_i y_{i+1}\}$ with $a_i \in R_D(xy)$. If $xy \in A_X \cup ((V \setminus X) \times X)$, then $R_D(xy) \subseteq A_H(x,y)\big|_X \cap \mathrm{Out}_H(x)$ holds, by step 0 or 1 and thus, $y_{i+1} \in X$, in contradiction to the choice of P as a shortest path. Therefore, $xy \in A_{V\setminus X}$ was added to A_D in step 2. Hence, $|R_D(xy)| = |A_H(x,y)\big|_X| \geq |A_H(y,x)\big|_X|$ held before step 3, i.e.

$$|R_D(xy)| \geq \frac{1}{2}\binom{|X|}{k-2} \geq \frac{1}{2}\binom{k+1}{k-2} \geq \frac{1}{2}\binom{4}{1} = 2,$$

by Lemma 2.9. Suppose that $R_D(xy)$ contains a_j for an index $j \in \{1, \ldots, l-1\} \setminus \{i\}$ as well. Without loss of generality, we may assume that $i < j$. Then there is a vertex $\tilde{y} \in \{y_{i+1}, y_j, y_{j+1}\} \setminus \{x, y\}$, since $R_D(xy) \subseteq \mathrm{Out}_H(x)$ and thus, $x \notin \{y_{i+1}, y_{j+1}\}$. Furthermore, $\tilde{y} \in X$ holds, since all arcs in $R_D(xy)$ contain only vertices from X, except for x and y, by construction. This constitutes a contradiction to the choice of P as a shortest path. Consequently, only a_i is removed from $R_D(xy)$ in step 3 and thus, $|R_D(xy)| \geq 1$ holds. Particularly, $R_D(xy) \neq \emptyset$ before step 3 implies $R_D(xy) \neq \emptyset$ after step 3 for all $xy \in A_D$.

In addition, we have $\emptyset \neq R_D(y_i y_{i+1}) = \{a_i\} \subseteq A_H(x,y)$ and $y_i y_{i+1} \notin A_X$, since P is a shortest path. If the arc $y_{i+1} y_i$ is contained in A_D, then it must have been added in step 1 or 2 and thus, $R_D(y_{i+1} y_i) \neq \emptyset$. Therefore, the conditions 1.13 (a) and (b) are met. By removing a_i from all $R_D(xy)$ it was contained in before step 3, we still have: For all $a \in A_H$ exists at most one $xy \in A_D$ with $a \in R_D(xy)$.

4. For all arcs $a = (x_1, \ldots, x_k) \in A_H$ such that there is no $xy \in A_D$ with $a \in R_D(xy)$, we choose an index $i \in \{2, \ldots, k\}$, add $x_1 x_i$ to A_D and define $R_D(x_1 x_i) := R_D(x_1 x_i) \cup \{a\}$.

   After step 4, condition 1.13 (c) is obviously met. Let $a = (x_1, \ldots, x_k) \in A_H$ be an arc considered in step 4 and let $x_1 x_i$ be the corresponding arc added to A_D for an $i \in \{2, \ldots, k\}$. Then $\emptyset \neq \{a\} \subseteq R_D(x_1 x_i) \subseteq A_H(x_1, x_i)$ implies the conditions 1.13 (a)(i) and (ii) for $x_1 x_i$. Suppose that $x_1 \in X$. Then there is exactly one arc $xy \in A_{D''}$ with $a \in R_{D''}(xy) \subseteq A_H(x,y) \cap \mathrm{Out}_H(x)$, since $A_{D''} \in \mathrm{OAMD}(H)$ and $R_{D''} \in \mathrm{REP}_{D''}$. It follows that $x = x_1 \in X$ and thus, $y \in X$, by Lemma 2.8. After step 0 in the construction of D, we then have $xy \in A_D$ and $a \in R_D(xy)$, a contradiction to the choice of a. Therefore, $x_1 \notin X$ and thus, $x_1 x_i, x_i x_1 \notin A_X$. Consequently, condition 1.13 (b) for D remains unaffected by step 4 and $x_i x_1 \in A_D$ implies that $x_i x_1$ was added to A_D in step 1 to 4 and thus, $R_D(x_i x_1) \neq \emptyset$. Therefore, condition 1.13 (a)(iii) is met by D as well. Altogether, we have $D \in \mathrm{OAMD}_X(H)$ is strong and $R_D \in \mathrm{REP}_D$.

Let $D' \in \mathrm{EXC}_X(D)$ and let $v_1, \ldots, v_n$ be an enumeration of V such that $d^+_{D'}(v_1) \leq \ldots \leq d^+_{D'}(v_n)$. Then, by step 1 of the construction of D and Definition 2.13, $xy \in A_D \setminus A_X \subseteq A_{D'}$ holds for all $xy \in (V \setminus X) \times X$. Consequently, we have $d^+_{D'}(v) \geq |X|$ for all $v \in V \setminus X$. By the construction of D, the arc $y_1y_2 \in A_D$ added in step 3 is the only one from X to $V \setminus X$ in D and thus, the only such arc in D', since $A_{D'} \subseteq A_D \cup A_X$. Therefore, $d^+_{D'}(v) \leq |X| - 1 < |X| \leq d^+_{D'}(w)$ holds for all $v \in X \setminus \{y_1\}$ and all $w \in V \setminus X$. The choice of D'' combined with Lemma 2.9 implies $|X| \geq k+1 \geq 4$ and thus, $\{v_1, v_2, v_3\} \subseteq X$. □

3 Main results

In this section, we will combine the gathered lemmata to prove our main result.

Lemma 3.1. *Let $H = (V, A_H)$ be a strong k-hypertournament on $n \geq k+2 \geq 5$ vertices. Then there exist a vertex set $X \subseteq V$ and a vertex $x \in X$ such that the following statement holds: For every out-arc $a \in A_H$ of x there is a strong $D_a = (V, A_{D_a}) \in \mathrm{OAMD}_X(H)$ and an $R_{D_a} \in \mathrm{REP}_{D_a}$ such that $R^{\downarrow}_{D_a}(a)$ is contained in A_{D_a} and pancyclic in D_a. If there is a strong $D \in \mathrm{OAMD}(H)$, then we can choose $X = V$.*

Proof. Suppose that every vertex of H has an out-arc. Let D' be a strong out-arc-majority digraph of H, if one exists. In this case, let $X = V$. Otherwise, let $X \subseteq V$ be self-contained and let D' be a strong X-out-arc-majority digraph of H, whose existence Lemma 2.17 guarantees. Then $D' \in \mathrm{EXC}^S_X(D')$ and thus, $\mathrm{MIN}^S_X(D') \neq \emptyset$. Let $D \in \mathrm{MIN}^S_X(D') \subseteq \mathrm{EXC}^S_X(D')$, $R_D \in \mathrm{REP}_D$ and let $v_1, \ldots, v_n$ be an enumeration of V such that $d^+_D(v_1) \leq \ldots \leq d^+_D(v_n)$. Then $\{v_1, v_2\} \subseteq X$ holds by Lemma 2.17 as well as $D \in \mathrm{MIN}^S_X(D)$, since $\mathrm{EXC}^S_X(D) = \mathrm{EXC}^S_X(D')$. By Lemma 2.16, there is a vertex $x \in \{v_1, v_2\} \subseteq X$ such that the following holds: For all arcs $xy \in A_D \cap A_X$ with $R_D(xy) \neq \emptyset$, there exist $D_{xy} = (V, A_{D_{xy}}) \in \mathrm{EXC}^S_X(D)$ such that $xy \in A_{D_{xy}}$ is pancyclic in D_{xy}, and $R_{D_{xy}} \in \mathrm{REP}_{D_{xy}}$ such that $R_{D_{xy}}(xy) = R_D(xy)$. Let $a = (x_1, \ldots, x_k) \in A_H$ be an out-arc of x. By definition 1.13 (c), there is exactly one $vw \in A_D$ with $a \in R_D(vw)$. Definition 1.13 (a)(i) implies $a \in A_H(v, w)$, i.e. there exist indices $1 \leq i < j \leq k$ such that $v = x_i$ and $w = x_j$. By assumption, $x_1 = x \in X$ implies $x_i, x_j \in X$, i.e. $x_ix_j \in A_D \cap A_X$ and $x_i = x$, by Definition 1.13 (b). Thus, by choice of x, the fact that $a \in R_D(xx_j)$ holds, implies the existence of a $D_a := D_{xx_j} = (V, A_{D_{xx_j}}) \in \mathrm{EXC}^S_X(D)$ such that $xx_j \in A_{D_a}$ is pancyclic in D_a and $R_{D_a} := R_{D_{xx_j}} \in \mathrm{REP}_{D_a}$ such that $R_{D_a}(xx_j) = R_D(xx_j)$. Thus, we have $a \in R_{D_a}(xx_j)$ and therefore, $R^{\downarrow}_{D_a}(a) = xx_j$ holds, by Definition 1.13 (c). □

Lemma 3.2. *Let $H = (V, A_H)$ be a strong k-hypertournament on $n \geq k+2 \geq 5$ vertices and let $(n, k) \notin \{(6, 4), (7, 4), (7, 5)\}$. Then H contains a vertex, whose all out arcs are pancyclic.*

Proof. Let $X \subseteq V$ and $x \in X$ be chosen as in Lemma 3.1. Let $a \in A_H$ be an out-arc of x. Then there exists a strong $D_a = (V, A_{D_a}) \in \mathrm{OAMD}_X(H)$ and an $R_{D_a} \in \mathrm{REP}_{D_a}$ such that $R^{\downarrow}_{D_a}(a) \in A_{D_a}$ is pancyclic in D_a, by the previous Lemma. Let $l \in \{3, \ldots, n\}$, let $C = x_1 \ldots x_l x_1$ be an l-cycle in D_a with $x_lx_1 = R^{\downarrow}_{D_a}(a)$ and let $B_{D_A} := \{vw \mid vw \in A_{D_a}, R_{D_a}(vw) \neq \emptyset\}$. Then $A_{D_a} \setminus B_{D_a} = \{vw \mid vw \in A_{D_a}, R_{D_a}(vw) = \emptyset\} \subseteq A_{\mathrm{maj}}(H)$ holds, by Definition 1.13 (a)(ii). For all $vw \in B_{D_a}$, we choose an arc $r(vw) \in R_{D_a}(vw)$, particularly $r(x_l x_1) := a$. By Definition 1.13 (c) and (a)(i), $r : B_{D_a} \to A_H$ is an injective function and $r(vw) \in A_H(v, w)$ holds for all $vw \in B_{D_a}$.

Case 1. $k = 3$. Lemma 2.5 implies the existence of an l-cycle C_H in H on the same vertex set as C, which contains $a = r(x_l x_1)$.

Case 2. $k \geq 4$. It follows that for all $i \in \{1, \ldots, l-1\}$ with $x_i x_{i+1} \notin B_{D_a}$, $x_i x_{i+1} \in A_{\text{maj}}(H)$ holds, by Definition 1.13 (a)(ii) and thus,

$$|A_H(x_i, x_{i+1})| \geq \lceil \frac{1}{2}\binom{n-2}{k-2} \rceil \geq \lceil \frac{1}{2}(2n-1) \rceil = n,$$

by Lemma 2.6. Hence, there is an injective function

$$r' : \{x_i x_{i+1} \mid 1 \leq i \leq l\} \to A_H$$

such that $r'(vw) = r(vw)$ for all $vw \in B_{D_a} \cap \{x_i x_{i+1} \mid 1 \leq i \leq l\}$ and $r'(vw) \in A_H(v, w)$ for all $vw \in \{x_i x_{i+1} \mid 1 \leq i \leq l\}$. Consequently, $C_H := x_1 r(x_1 x_2) x_2 r(x_2 x_3) \ldots x_l r(x_l x_1) x_1$ is an l-cycle in H, which contains $a = r(x_l x_1)$. Since $a \in \text{Out}_H(x)$ and $l \in \{3, \ldots, n\}$ were chosen arbitrarily, all out-arcs of x are pancyclic in H. □

Lemma 3.3. *Let $H = (V, A_H)$ be a strong k-hypertournament on n vertices and $(n, k) \in \{(7, 4), (7, 5)\}$. Then H contains a vertex, whose all out-arcs are pancyclic.*

Proof. Without loss of generality, we may assume that there is a $D \in \text{OAMD}(H)$, since otherwise, Lemma 2.10 or Lemma 2.11 would give the result. By Lemma 3.1, there exists a vertex $x \in V$ such that the following holds for every out-arc $a \in A_H$ of x: There is a strong $D_a = (V, A_{D_a}) \in \text{OAMD}(H)$ and an R_{D_a} such that $R^{\downarrow}_{D_a}(a) \in A_{D_a}$ is pancyclic in D_a. Let $a \in \text{Out}_H(x)$, $l \in \{3, \ldots, n\}$ and let $C = x_1 \ldots x_l x_1$ be an l-cycle in D_a with $x_l x_1 = R^{\downarrow}_{D_a}(a)$. Furthermore, let $I \subseteq \{x_1 x_2, \ldots, x_{l-1} x_l, x_l x_1\}$,

$$I_1 := \{vw \mid vw \in I, R_{D_a}(vw) \neq \emptyset\} \text{ and } I_2 := I \setminus I_1.$$

By Definition 1.13 (a)(i) and (b), we have $R_{D_a}(vw) \subseteq A_H(v, w) \cap \text{Out}_H(v)$ for all $vw \in A_{D_a}$, which implies $|\bigcup_{vw \in I} A_H(v, w)| \geq |I_1|$. Furthermore, by Definition 1.13 (a)(ii), $vw \in A_{\text{maj}}(H)$ holds for all $vw \in I_2$ and thus,

$$|A_H(v, w)| \geq \lceil \frac{1}{2}\binom{n-2}{k-2} \rceil \geq \lceil \frac{1}{2}(2n-4) \rceil = 5,$$

by Definition 1.1 and Lemma 2.6.

(∗) If there are two non-incident arcs $v_1 w_1, v_2 w_2 \in I_2$, then

$$|A_H(v_1, w_1) \cap A_H(v_2, w_2)| \leq \binom{n-4}{k-4} \leq \binom{3}{1} = 3$$

and thus,

$$|\bigcup_{vw \in I} A_H(v, w)| \geq |A_H(v_1, w_1) \cup A_H(v_2, w_2)| \geq 7.$$

(†) If there is an $i \in \{1, \dots, l-1\}$ such that $x_i x_{i+1} \in I_2$ and $x_{i+1}x_{i+2} \in I_1$, then there exists at least one arc $a_{i+1} \in A_H(x_{i+1}, x_{i+2}) \cap \mathrm{Out}_H(x_{i+1})$, since $\emptyset \neq R_{D_a}(x_{i+1}x_{i+2}) \subseteq A_H(x_{i+1}, x_{i+2}) \cap \mathrm{Out}_H(x_{i+1})$. Thus, a_{i+1} cannot be contained in $A_H(x_i, x_{i+1})$ and therefore, we have

$$|\bigcup_{vw\in I} A_H(v,w)| \;\geq\; |A_H(x_i, x_{i+1}) \cup A_H(x_{i+1}, x_{i+2})| \;\geq\; 6.$$

By Corollary 2.4, we may assume that $|I| > |\bigcup_{vw\in I} A_H(v,w)|$. Because of $|\bigcup_{vw\in I} A_H(v,w)| \geq |I_1|$, there exists a $vw \in I_2$. $|A_H(v,w)| \geq 5$ implies $|I| \geq 6$.

Case 1. $|I| = 6$. If $l = 7$, then the arcs in I form an $(l-1)$-path $y_1 \dots y_l$. If $y_{l-1}y_l \in I_1$, let $i_0 := \max\{i \mid 1 \leq i \leq l-2, y_i y_{i+1} \in I_2\}$. By definition, we have $y_{i_0+1}y_{i_0+2} \in I_1$ and thus, $|\bigcup_{vw\in I} A_H(v,w)| \geq 6 = |I|$ by (†), a contradiction. For $l = 6$, we reach the same contradiction by consideration of the $(l-1)$-path $y_1 \dots y_l := x_2 \dots x_6 x_1$, since $x_6x_1 \in I_1$. Thus we may assume that $l = 7$ and $y_6y_7 \in I_2$.

If $I_1 = I \setminus \{y_6y_7\}$, then $\emptyset \neq R_{D_a}(y_iy_{i+1}) \subseteq A_H(y_i, y_{i+1}) \cap \mathrm{Out}_H(y_i)$ for all $i \in \{1, \dots, 5\}$ combined with $|A_H(y_6, y_7)| \geq 5$ and $|\bigcup_{vw\in I} A_H(v,w)| < |I| = 6$ imply $\{a_i\} = R_{D_a}(y_iy_{i+1}) \subseteq \mathrm{Out}_H(y_i)$ for all $i \in \{1, \dots, 5\}$ and $\bigcup_{vw\in I} A_H(v,w) = A_H(y_6, y_7) = \{a_1, \dots, a_5\}$. Hence, we have

$$|A_H(y_6, y_5) \setminus \{a_1, \dots, a_5\}| \geq \binom{n-2}{k-2} - |A_H(y_5, y_6) \cup \{a_1, \dots, a_5\}| = 5$$

If $R_{D_a}(y_7y_1) \neq \emptyset$, we choose $a_7 \in R_{D_a}(y_7y_1)$. Particularly, if $y_7y_1 = x_7x_1$, we choose $a_7 = a$. Then $a_7 \notin \{a_1, \dots, a_5\}$ holds, since $R_{D_a}(y_7y_1) \subseteq \mathrm{Out}_H(y_7)$. If $R_{D_a}(y_7y_1) = \emptyset$, then $y_7y_1 \in A_{\mathrm{maj}}(H)$, by Definition 1.13 (a)(ii) and thus, $|A_H(y_7, y_1)| \geq 5$. Since we have $a_1 \in \mathrm{Out}_H(y_1)$ and therefore, $a_1 \notin A_H(y_7, y_1)$, we can choose an $a_7 \in A_H(y_7, y_1) \setminus \{a_1, \dots, a_5\}$. Finally, we choose an $a_6 \in A_H(y_6, y_5) \setminus \{a_1, \dots, a_5, a_7\}$. In addition, we have $a_4 \in A_H(y_4, y_6)$ and $a_5 \in A_H(y_5, y_7)$, because of $a_i \in \mathrm{Out}_H(y_i) \cap A_H(y_6, y_7)$ for all $i \in \{4, 5\}$. Hence, $C_H := y_1a_1y_2a_2y_3a_3y_4a_4y_6a_6y_5a_5y_7a_7y_1$ is an l-cycle in H that contains a.

If $I_1 \neq I \setminus \{y_6y_7\}$, then $(*)$ implies

$$I_1 = \{y_1y_2, y_2y_3, y_3y_4, y_4y_5\} \text{ and } I_2 = \{y_5y_6, y_6y_7\}.$$

Since $|A_H(y_5, y_6)| \geq 5$, $|A_H(y_6, y_7)| \geq 5$ and $|\bigcup_{vw\in I} A_H(v,w)| < |I| = 6$, we then have $A_H(y_5, y_6) = A_H(y_6, y_7) = \bigcup_{vw\in I} A_H(v,w)$. Since $\emptyset \neq R_{D_a}(y_iy_{i+1}) \subseteq A_H(y_i, y_{i+1}) \cap \mathrm{Out}_H(y_i)$ holds for all $i \in \{1, \dots, 4\}$, we can choose arcs $a_i \in R_{D_a}(y_iy_{i+1}) \subseteq A_H(y_i, y_{i+1}) \cap \mathrm{Out}_H(y_i)$ for all $i \in \{1, \dots, 4\}$ such that $\{a_1, \dots, a_4, b\} = \bigcup_{vw\in I} A_H(v,w)$. Particularly, in the case $y_iy_{i+1} = x_7x_1$, we choose $a_i = a$. Then we have

$$|A_H(y_5, y_4) \setminus \{a_1, \dots, a_4, b\}| \geq \binom{n-2}{k-2} - |A_H(y_4, y_5) \cup \{a_1, \dots, a_4, b\}| = 5.$$

If $R_{D_a}(y_7y_1) = \{b\}$, then we obtain an l-cycle C_H in H that contains a as in the case above, by consideration of the path $\tilde{y}_1 \dots \tilde{y}_7 = y_7y_1 \dots y_6$ with $R_{D_a}(\tilde{y}_1\tilde{y}_2) = \{b\}$ and

$R_{D_a}(\tilde{y}_i\tilde{y}_{i+1}) = \{a_{i-1}\}$ for all $i \in \{2,\dots,5\}$ and $A_H(\tilde{y}_6,\tilde{y}_7) = \{a_1,\dots,a_4,b\}$. Suppose that $R_{D_a}(y_7y_1) \neq \{b\}$. If $R_{D_a}(y_7y_1) \neq \emptyset$, we choose $a_7 \in R_{D_a}(y_7y_1) \setminus \{b\}$. In the case $y_7y_1 = x_7x_1$ and $a \neq b$, we choose $a_7 = a$, in particular. Then $a_7 \notin \{a_1,\dots,a_4\}$, because of $R_{D_a}(y_7y_1) \subseteq \mathrm{Out}_H(y_7)$. If $R_{D_a}(y_7y_1) = \emptyset$, then $y_7y_1 \in A_{\mathrm{maj}}(H)$, by Definition 1.13 (a)(ii) and thus, $|A_H(y_7,y_1)| \geq 5$. Since $a_1 \in \mathrm{Out}_H(y_1)$ holds and therefore, $a_1 \notin A_H(y_7,y_1)$, we can choose an $a_7 \in A_H(y_7,y_1)\setminus\{a_1,\dots,a_4,b\}$. Finally, we choose an $a_5 \in A_H(y_5,y_4) \setminus \{a_1,\dots,a_4,b,a_7\}$. Since $a_i \in \mathrm{Out}_H(y_i) \cap A_H(y_5,y_6)$ for all $i \in \{3,4\}$, we have $a_3 \in A_H(y_3,y_5)$ and $a_4 \in A_H(y_4,y_6)$. Hence, $C_H := y_1a_1y_2a_2y_3a_3y_5a_5y_4a_4y_6by_7a_7y_1$ is an l-cycle in H that contains a.

Case 2. $|I| = 7$. Obviously, we have $l = 7$. Since there is a $vw \in I_2$ and $x_7x_1 \in I_1$ holds, there exists an index $i \in \{1,\dots,6\}$ such that $x_ix_{i+1} \in I_2$ and $x_{i+1}x_{i+2} \in I_1$. Without loss of generality, we may assume that $i = 6$. If $I_1 = I \setminus \{x_6x_7\}$, then $\emptyset \neq R_{D_a}(x_ix_{i+1}) \subseteq A_H(x_i,x_{i+1}) \cap \mathrm{Out}_H(x_i)$ for all $i \in \{1,\dots,5,7\}$, $|A_H(x_6,x_7)| \geq 5$ and $|\bigcup_{vw\in I} A_H(v,w)| < |I| = 7$ imply $\{a_i\} = R_{D_a}(x_ix_{i+1}) \subseteq \mathrm{Out}_H(x_i)$ for all $i \in \{1,\dots,5,7\}$, $A_H(x_6,x_7) = \{a_1,\dots,a_5\}$ and $\bigcup_{vw\in I} A_H(v,w) = \{a_1,\dots,a_5,a_7\}$. Thus, we have

$$|A_H(x_6,x_5) \setminus \{a_1,\dots,a_5,a_7\}| \geq \binom{n-2}{k-2} - |A_H(x_5,x_6) \cup \{a_1,\dots,a_5,a_7\}| = 4.$$

Hence, we may choose an $a_6 \in A_H(x_6,x_5) \setminus \{a_1,\dots,a_5,a_7\}$. Furthermore, $a_4 \in A_H(x_4,x_6)$ and $a_5 \in A_H(x_5,x_7)$ hold, because of $a_i \in \mathrm{Out}_H(x_i) \cap A_H(x_6,x_7)$ for all $i \in \{4,5\}$. Hence, a is contained in the l-cycle $C_H := x_1a_1x_2a_2x_3a_3x_4a_4x_6a_6x_5a_5x_7a_7x_1$ in H.

If $I_1 \neq I \setminus \{x_6x_7\}$, then $(*)$ implies $I_1 = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_7x_1\}$ and $I_2 = \{x_5x_6, x_6x_7\}$. Since $\emptyset \neq R_{D_a}(x_ix_{i+1}) \subseteq A_H(x_i,x_{i+1}) \cap \mathrm{Out}_H(x_i)$ for all $i \in \{1,\dots,4,7\}$, we can choose arcs $a_i \in R_{D_a}(x_ix_{i+1}) \subseteq A_H(x_i,x_{i+1}) \cap \mathrm{Out}_H(x_i)$ for all $i \in \{1,\dots,4,7\}$. Since $|A_H(x_6,x_7)| \geq 5$ and $a_7 \in \mathrm{Out}_H(x_7)$, we have $\{a_1,\dots,a_4,b\} = A_H(x_6,x_7)$ and $\bigcup_{vw\in I} A_H(v,w) = \{a_1,\dots,a_4,b,a_7\}$, for an arc $b \neq a_7$, since otherwise, $|\bigcup_{vw\in I} A_H(v,w)| \geq |I|$ would hold. Hence, we have

$$\begin{aligned}|A_H(x_5,x_4) \setminus \{a_1\dots,a_4,b,a_7\}| &\geq \binom{n-2}{k-2} - |A_H(x_4,x_5) \cup \{a_1\dots,a_4,b,a_7\}| \\ &\geq 4\end{aligned}$$

and analogously, $|A_H(x_4,x_3) \setminus \{a_1\dots,a_4,b,a_7\}| \geq 4$. Thus, we may choose an arc $a_5 \in A_H(x_5,x_4) \setminus \{a_1,\dots,a_4,b,a_7\}$ and an arc $\tilde{a}_4 \in A_H(x_4,x_3) \setminus \{a_1,\dots,a_5,b,a_7\}$. If $a_3 \in A_H(x_5,x_6)$, then $a_3 \in A_H(x_3,x_5)$ as well as $a_4 \in A_H(x_4,x_6)$ holds, since $a_i \in \mathrm{Out}_H(x_i) \cap A_H(x_{i+2},x_{i+3})$ for all $i \in \{3,4\}$. Thus,

$$C_H := x_1a_1x_2a_2x_3a_3x_5a_5x_4a_4x_6bx_7a_7x_1$$

is an l-cycle in H that contains a. If $a_3 \notin A_H(x_5,x_6)$, then we have $a_2 \in A_H(x_5,x_6)$, since otherwise, $|\{a_1,\dots,a_4,a_7\} \cup A_H(x_5,x_6)| \geq 7$ would hold. Because of $a_i \in \mathrm{Out}_H(x_i) \cap A_H(x_{i+3},x_{i+4})$ for all $i \in \{2,3\}$, we therefore have $a_2 \in A_H(x_2,x_5)$ and $a_3 \in A_H(x_3,x_6)$. Thus, the l-cycle $C_H := x_1a_1x_2a_2x_5a_5x_4\tilde{a}_4x_3a_3x_6bx_7a_7x_1$ in H contains a. $\square$

Lemma 3.4. *Let $H = (V, A_H)$ be a strong 4-hypertournament on 6 vertices. Then H contains a vertex, whose all out-arcs are pancyclic.*

We omit our proof of Lemma 3.4, since it is structurally similar to the proof of Lemma 3.3 and consists mainly of a case by case analysis, which is about as long as all previous proofs combined.

We merge Lemmas 3.2, 3.3 and 3.4 to the following theorem, which constitutes a generalization of Theorem 1.10 for hypertournaments.

Theorem 1.15. *Let H be a strong k-hypertournament on $n \geq k + 2 \geq 5$ vertices. Then H contains a vertex, whose all out-arcs are pancyclic.*

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