AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **62(2)** (2015), Pages 155–171

Binomial determinants and positivity of Chern-Schwartz-MacPherson classes

Leonardo Constantin Mihalcea*

Department of Mathematics
460 McBryde, Virginia Tech University
Blacksburg VA 24060
U.S.A.
lmihalce@math.vt.edu

Abstract

We give a combinatorial interpretation of a certain positivity conjecture of Chern-Schwartz-MacPherson classes, as stated by P. Aluffi and the author in a previous paper. It translates into a positivity property for a sum of $p \times p$ determinants consisting of binomial coefficients, generalizing the classical Theorem of Lindström-Gessel-Viennot et al. which computes these determinants in terms of non-intersecting lattice paths. We prove this conjecture for $p = 2, 3$.

1 Introduction

Let X be a projective nonsingular variety of dimension n over the complex numbers. One of the most important classes in the cohomology of X is $c(TX)$ — the total Chern class of its tangent bundle TX. For example, the classical Gauss-Bonnet Theorem states that $c_n(TX) \cap [X]$ equals the topological Euler characteristic of X. There have been sustained efforts to find analogues of the class $c(TX)$ for *singular* varieties. A class with particularly good properties was conjectured by Deligne and Grothendieck; they stated that there is a unique functor $c_* : \mathcal{F}(X) \to H_*(X)$ from the group of constructible functions of X to its homology which is compatible with proper push-forwards and for which $c_*(1_X) = c(TX) \cap [X]$ if X is nonsingular. This conjecture was proved by MacPherson [8], and it turned out that the homology class $c_*(1_X)$ coincided with a class defined earlier by M. H. Schwartz [9, 10]; hence its name Chern-Schwartz-MacPherson (CSM), and the notation $c_{SM}(X) = c_*(1_X)$. We refer the reader to the excellent survey [1] for details about this and other similar constructions.

* During revision stage, this work was partially supported by NSA Young Investigator Award H-98230-13-1-0208.

The current note studies a positivity conjecture made by P. Aluffi and the author [2] in the case when X is the Grassmannian manifold $\mathrm{Gr}(p,n)$ which parametrizes the linear subspaces of dimension p in $\mathbb{C}^n$. In this case, the (integral) homology $H_*(\mathrm{Gr}(p,n))$ has a basis given by Schubert classes $[\Omega_\lambda]$ where λ is a *partition* included in the $p\times(n-p)$ rectangle; this means that λ is a sequence of non-negative integers $(\lambda_1 \geq \cdots \geq \lambda_p)$ such that $\lambda_1 \leq n-p$. The Schubert varieties Ω_λ are closures of Schubert cells Ω_λ^o, which are the orbits of a (fixed) Borel subgroup of the general linear group $\mathrm{GL}_n(\mathbb{C})$. The characteristic function $1_{\Omega_\lambda^o}$ of the Schubert cell determines a homology class $c_*(1_{\Omega_\lambda^o}) \in H_*(\mathrm{Gr}(p,n))$, and therefore an identity

$$c_*(1_{\Omega_\lambda^o}) = \sum_\mu c(\lambda,\mu)[\Omega_\mu].$$

From the definition of the CSM classes (cf. [2]) it follows that $c(\lambda,\mu)=0$ unless $\mu\subset\lambda$ which means that $\mu_i \leq \lambda_i$ for $1\leq i\leq p$. This and the inclusion-exclusion property satisfied by the CSM classes implies that the CSM class of the Schubert *variety* $c_{SM}(\Omega_\lambda)$ is equal to the sum

$$c_{SM}(\Omega_\lambda) = \sum_{\mu\subset\lambda} c_{SM}(1_{\Omega_\mu^o}).$$

Based on substantial computer checking we conjectured in [2, Conj. 1] that $c(\lambda,\mu)\geq 0$. This conjecture was proved in loc. cit. for $p=2$ and for any p but $\mu=(i)$. The proof involved combinatorics of lattice paths (as we shall see below), and some subtle analysis of a generating function for $c(\lambda,\mu)$. B. Jones [5] used small resolutions of Schubert varieties to find different formulas for $c(\lambda,\mu)$, then proved the positivity conjecture when Ω_μ has codimension 1 in Ω_λ (i.e. $(\lambda_1+\cdots+\lambda_p)-(\mu_1+\cdots+\mu_p)=1$). In his thesis, Stryker [11] analyzed carefully the formulas from [2] to extend the positivity to codimensions ≤ 4, and found some particular configurations of the skew-shape λ/μ where positivity holds in general. Finally, after the current paper was finished, J. Huh [4] proved the full positivity conjecture by realizing each homogeneous component of $c_*(1_{\Omega_\lambda^o})$ as the class of an irreducible subvariety in the Grassmannian. The subvariety corresponds to a degeneracy locus on a particular desingularization of a Schubert variety in the Grassmannian, which has finitely many Borel orbits. As noted by Huh, this geometric description does not give a *combinatorial* formula for $c(\lambda,\mu)$, and the question of finding a manifestly positive formula for the coefficients $c(\lambda,\mu)$ remains open. The aim of this paper is to provide such a formula in the case $p\leq 3$. We proved the following statement:

Theorem 1.1 *Let λ,μ be two partitions with p parts, such that $\mu\subset\lambda$. Then:*

1. $c(\lambda,\mu)=\sum_{s\in S(\lambda)}\det M(s)$ *where the sum is over a set $S(\lambda)$ depending only on λ and $M(s)$ is a $p\times p$ matrix. Further,* $\det M(s)$ *is equal to the sum of* signed p-tuples of non-intersecting lattice paths *from some initial points $A_1(s),\dots,A_p(s)$ to some end points $B_1,\dots,B_p$, which depend only on μ.*

2. *Let $p = 2$. Then each determinant* $\det M(s)$ *is positive.*

3. *Let $p = 3$. In this case the set $S(\lambda)$ can be written as the disjoint union $S(\lambda) = \bigcup_{f \in F} S(\lambda; f)$ which induces a decomposition*

$$c(\lambda, \mu) = \sum_{f \in F} c(\lambda, \nu; f).$$

   Then $c(\lambda, \mu; f) \geq 0$ and there exists $f_0 \in F$ such that $c(\lambda, \mu; f_0) > 0$.

In particular, $c(\lambda, \mu) > 0$ for $p \leq 3$.

Parts 1 and 2 of the theorem are re-statements of results from [2], but they are considered now in a uniform format, using lattice paths. The bulk of the paper consists of the proof of the third part. In this case, it is possible that $\det M(s) < 0$; we will illustrate this in an example below. There is a precise definition of the triples of lattice paths which count $c(\lambda, \mu)$, given in Corollary 3.7 below.

Besides the geometric interest, the combinatorics of the sum $\sum_{s \in S(\lambda)} \det M(s)$ is quite interesting. Recall that to any $2p$ lattice points $A_1, \dots, A_p$ and $B_1, \dots, B_p$ in $\mathbb{Z}^2$ one can associate a $p \times p$ matrix of binomial coefficients $M = (m_{ij})$, where m_{ij} is equal to the number of lattice paths from A_i to B_j, with each segment oriented either North-South, or West-East (see Figure 1 below). It is a classical result about binomial determinants (see e.g. [7, 3] or see [6] and references therein) that if the points A_i respectively B_j are arranged, in order, from North-East to South-West, then the determinant of M is non-negative, and counts p-tuples of non-intersecting lattice paths $\pi = (\pi_1, \dots, \pi_p)$, where π_i is a path from A_i to B_i. The theorem appeared and was rediscovered in many places, and we will refer it as the LGV Theorem. A slightly more general version of this theorem implies immediately the calculation $\det M(s)$. The unexpected result here is the positivity of the *sumi* $\sum_{s \in S(\lambda)} \det M(s)$. i In this context, the positivity of $c(\lambda, \mu)$ can be interpreted as a *family* version of the LGV theorem, where a single determinant associated with $2p$ lattice points is replaced by a family of such determinants. Even though a particular determinant might be negative, their sum remains positive. After searching the literature and discussing with experts in this area, this seems to be the first example of such a phenomenon.

2 Preliminaries

2.1 A family of lattice paths

In this note a **path** π will be a lattice path in $\mathbb{Z}^2$ with the horizontal steps to the right and the vertical steps going down. For $A, B \in \mathbb{Z}^2$ the notation $\pi : A \to B$ means that π starts at A and ends at B. See Figure 1 below.

By a **partition** λ we mean a non-increasing sequence of nonnegative integers

$$\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p \geqslant 0).$$

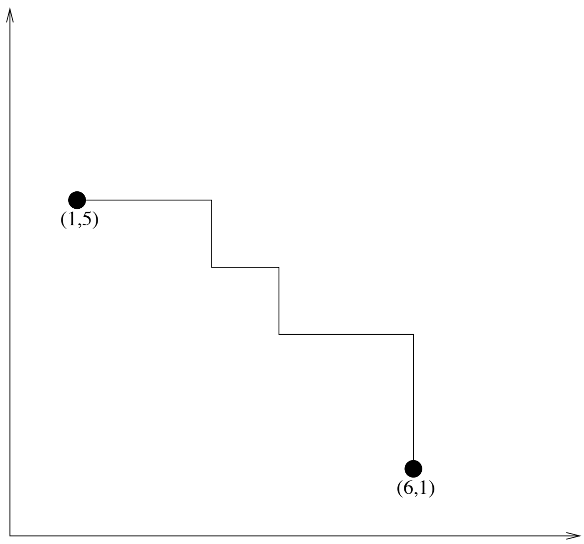

Figure 1: A path from $(1,5)$ to $(6,1)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_p)$ be two such partitions such that $\mu_i \leq \lambda_i$. To this data we associate a family of $2p$ points $A_1(s), \ldots, A_p(s)$ and $B_1, \ldots B_p$ in $\mathbb{Z}^2$. The points $A_i(s)$ will depend on a sequence of parameters s in a set $S(\lambda)$ which we define in the next paragraph. The points B_j, for $1 \leqslant j \leqslant p$ are defined simply by:

$$B_j := (\mu_j + p - j + 1, \mu_j + p - j + 1).$$

The set $S(\lambda)$ consists of sequences $s = (a_{i,j})$ of nonnegative integers indexed as the elements of a square matrix of order $p-1$, situated on or below the main diagonal:

$$\begin{array}{cccc} a_{1,1} & & & \\ a_{2,1} & a_{2,2} & & \\ \vdots & \vdots & \ddots & \\ a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,p-1} \end{array} \tag{2.1}$$

Definition 2.1 *Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition. We say that the integers $a_{i,j}$ are in* **triangular order with respect to λ** *if:*

1. $0 \leqslant a_{i,j} \leqslant \lambda_{j+1}$.
2. *The partial sums from the $j-th$ column, from row $j+1$ to $i+j$, for all $1 \leqslant i \leqslant p-1-j$, are less than $a_{j,j}$, i.e.*

$$a_{j+1,j} + \cdots + a_{j+i,j} \leqslant a_{j,j}. \tag{2.2}$$

To simplify the notation in the upcoming formulae, we let R_j, for $2 \leqslant j \leqslant p-1$, respectively C_j, for $1 \leqslant j \leqslant p-2$ denote the partial sum on the $j-$th row, respectively $j-$th column of 2.1, excluding $a_{j,j}$:

$$R_j := a_{j,1} + \cdots + a_{j,j-1}. \tag{2.3}$$

$$C_j := a_{j+1,j} + \cdots + a_{p-1,j}. \tag{2.4}$$

Set also $R_1 = C_{p-1} = 0$. Denote the set of all triangular sequences with respect to λ by $S(\lambda)$ and an element of it by $s = (a_{ij})$. We define the lattice points $A_j(s)$ by

$$A_j(s) := (p - j + 1 + a_{j,j} - R_j, \lambda_j + p - j + 1 - R_j)$$

for $1 \leqslant j \leqslant p - 1$. If $j = p$, let the $x-$coordinate of $A_p(s)$ be

$$x_{A_p(s)} := 1 + (C_1 - a_{1,1}) + (C_2 - a_{2,2}) + \cdots + (C_{p-1} - a_{p-1,p-1})$$

and the $y-$coordinate to be

$$y_{A_p(s)} := \lambda_p + x_{A_p(s)}.$$

For $s \in S(\lambda)$ define the matrix $M(s) = (m_{ij}(s))$ by

$$m_{ij}(s) = \#\mathcal{P}(A_i(s) \to B_j),$$

where the right hand side denotes the number of paths from $A_i(s)$ to B_j. Explicitly, the matrix $M(s)$ is given by:

$$\begin{pmatrix} \binom{\lambda_1 - a_{1,1}}{\mu_1 + R_1 - a_{1,1}} & \binom{\lambda_1 - a_{1,1}}{\mu_2 - 1 + R_1 - a_{1,1}} & \cdots & \binom{\lambda_1 - a_{1,1}}{\mu_p - (p-1) + R_1 - a_{1,1}} \\ \binom{\lambda_2 - a_{2,2}}{\mu_1 + 1 + R_2 - a_{2,2}} & \binom{\lambda_2 - a_{2,2}}{\mu_2 + R_2 - a_{2,2}} & \cdots & \binom{\lambda_2 - a_{2,2}}{\mu_p - (p-2) + R_2 - a_{2,2}} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{\lambda_{p-1} - a_{p-1,p-1}}{\mu_1 + p - 2 + R_{p-1} - a_{p-1,p-1}} & \binom{\lambda_{p-1} - a_{p-1,p-1}}{\mu_2 + p - 3 + R_{p-1} - a_{p-1,p-1}} & \cdots & \binom{\lambda_{p-1} - a_{p-1,p-1}}{\mu_p - 1 + R_{p-1} - a_{p-1,p-1}} \\ \binom{\lambda_p}{\mu_1 + p - 1 + \sum_{s=1}^{p-1}(a_{s,s} - C_s)} & \binom{\lambda_p}{\mu_2 + p - 2 + \sum_{s=1}^{p-1}(a_{s,s} - C_s)} & \cdots & \binom{\lambda_p}{\mu_p + \sum_{s=1}^{p-1}(a_{s,s} - C_s)} \end{pmatrix} \tag{2.5}$$

i.e. the binomial coefficient on the row r and column c, for $1 \leqslant r \leqslant p - 1$ is equal to

$$\binom{\lambda_r - a_{r,r}}{\mu_c + r - c + R_r - a_{r,r}}.$$

For example, in the case $p = 2$, the triangular sequence $S(\lambda)$ consists of all (a_{11}) such that $0 \leqslant a_{11} \leqslant \lambda_2$ and $M(a_{11})$ is given by:

$$\begin{pmatrix} \binom{\lambda_1 - a_{1,1}}{\mu_1 - a_{1,1}} & \binom{\lambda_1 - a_{1,1}}{\mu_2 - 1 - a_{1,1}} \\ \binom{\lambda_2}{\mu_1 + 1 + a_{1,1}} & \binom{\lambda_2}{\mu_2 + a_{1,1}} \end{pmatrix}. \tag{2.6}$$

Similarly, in the case $p = 3$, the triangular sequences consist of triples (a_{21}, a_{22}, a_{11})[1] such that

$$0 \leqslant a_{21} \leqslant a_{11}; \qquad 0 \leqslant a_{22} \leqslant \lambda_3; \qquad 0 \leqslant a_{11} \leqslant \lambda_2,$$

[1]We used the ordering (a_{21}, a_{22}, a_{11}) rather than (a_{11}, a_{12}, a_{22}) to be consistent with the notation $(a_{21}, a_{22}, a_{11}) = (i, j, k)$ used later in the paper.

and $M(s)$ is:

$$\begin{pmatrix} \binom{\lambda_1-a_{1,1}}{\mu_1-a_{1,1}} & \binom{\lambda_1-a_{1,1}}{\mu_2-1-a_{1,1}} & \binom{\lambda_1-a_{1,1}}{\mu_3-2-a_{1,1}} \\ \binom{\lambda_2-a_{2,2}}{\mu_1+1+a_{2,1}-a_{2,2}} & \binom{\lambda_2-a_{2,2}}{\mu_2+a_{2,1}-a_{2,2}} & \binom{\lambda_2-a_{2,2}}{\mu_3-1+a_{2,1}-a_{2,2}} \\ \binom{\lambda_3}{\mu_1+2+(a_{1,1}-a_{2,1})+a_{2,2}} & \binom{\lambda_3}{\mu_2+1+(a_{1,1}-a_{2,1})+a_{2,2}} & \binom{\lambda_3}{\mu_3+(a_{1,1}-a_{2,1})+a_{2,2}} \end{pmatrix}. \tag{2.7}$$

The matrix $M(s)$ appeared in our previous work [2], and the following was proved in [2, Thm. 3.8]:

Proposition 2.2 *Given the expansion $c_*(1_{\Omega_\lambda^\circ}) = \sum_{\mu\subset\lambda} c(\lambda,\mu)[\Omega_\mu]$ as in [2], the following identity holds:*

$$c(\lambda,\mu) = \sum_{s\in S(\lambda)} \det M(s). \tag{2.8}$$

Remark 2.1 *In the cases $p = 2$ respectively $p = 3$, explicit positive combinatorial formulae for $c(\lambda,\mu)$, in terms of lattice paths, are given respectively in Corollaries 2.4 and 3.7 below.*

Remark 2.2 *For arbitrary p and if $\mu = (r)$ (the row partition with r boxes), it was proved in [2] that $c(\lambda,(r))$ from equation (2.8) is the coefficient of t^r in*

$$\prod_{i\geqslant 1}(1+it)^{\lambda_i-\lambda_{i+1}}.$$

The proof used a different approach than the one considered here, and involved residues and generating functions. A positive formula in the case when $|\mu| = |\lambda| - 1$, where $|\lambda| = \lambda_1 + \cdots + \lambda_p$, was obtained by B. Jones in [5]. It is an interesting combinatorial question to find whether these formulas have natural interpretations using lattice paths.

2.2 Proof of the first two parts of Theorem 1.1

We recall next the (unweighted) version of the classical LGV Theorem which allows any configuration of the initial points and end points. In what follows $\mathcal{P}(E \to F)$ denotes the set of lattice paths from the point E to the point F.

Theorem 2.3 (Theorem 1 in [3]; see also [7]) *Let E_i, F_j be $2p$ lattice points, with $1 \leqslant i,j \leqslant p$. Then the determinant $\det(\#\mathcal{P}(E_i \to F_j)_{1\leqslant i,j\leqslant p}$ is equal to $\sum_{\pi_w} \varepsilon(w)$, where w is a permutation in $Sym(p)$, $\varepsilon(w)$ is its signature and the sum is over all $p-$tuples of paths*

$$\pi_w = (\pi_1^w, \ldots, \pi_p^w)$$

with $\pi_i^w : E_{w(i)} \to F_i$, such that no two paths π_i^w and π_j^w intersect.

Applying this theorem to $\det M(s)$ yields immediately the first part of the Main Theorem 1.1. For the second part, note that in the case $p = 2$ a triangular sequence (a_{ij}) has just one element, denote it i, satisfying $0 \leqslant i \leqslant \lambda_2$. Then one can easily see that for any i, the lattice point $A_1(i)$ is strictly North and strictly East of $A_2(i)$ (i.e. $x_{A_1(i)} > x_{A_2(i)}$ and $y_{A_1(i)} > y_{A_2(i)}$) and similarly B_1 is strictly North and strictly East of B_2. Therefore there cannot be non-intersecting lattice paths which contribute negatively to $\det M(s)$. Moreover, because of the strict inequalities one can find at least one pair of non-intersecting lattice paths. To state the precise formula, let $\Pi(i)$ denote the set of all *non-intersecting* pairs of paths (π_1, π_2), with $\pi_r : A_r(i) \to B_r$. We get the following identity, stated also in [2, Thm. 4.5]:

Corollary 2.4 (Positivity for $p = 2$) *Let $\lambda = (\lambda_1, \lambda_2)$ be a partition. Then for any partition μ, the coefficient $c(\lambda, \mu)$ is equal to*

$$\sum_{i=0}^{\lambda_2} \#\Pi(i)\,.$$

2.3 An example for $p = 3$.

The remainder of the paper deals with the proof of the third part of Theorem 1.1. We begin first by illustrating the result by an example, for $\lambda = (3, 3, 3)$ and $\mu = (2, 2, 1)$. To avoid carrying subscripts, we identify the triangular sequence (a_{21}, a_{22}, a_{11}) to (i, j, k), so that

$$0 \leqslant k \leqslant \lambda_2 = 3;\ 0 \leqslant i \leqslant k;\ 0 \leqslant j \leqslant \lambda_3 = 3 \quad .$$

The lattice points A_ℓ, B_ℓ, for $A_\ell = A_\ell(i, j, k)$, $1 \leqslant \ell \leqslant 3$ are given by $A_1 = (k + 3, \lambda_1 + 3) = (k + 3, 6)$, $A_2 = (2 - i + j, \lambda_2 + 2 - i) = (2 - i + j, 5 - i)$, $A_3 := (1 - k + i - j, \lambda_3 + 1 - k + i - j) = (1 - k + i - j, 4 - k + i - j)$ and

$$B_1 = (5, 5), B_2 = (4, 4), B_3 = (2, 2) \quad .$$

By Theorem 2.3 above, each of the determinants of matrices $M(i, j, k)$ counts *signed* triples of non-intersecting lattice paths. The content of the Theorem 1.1 is that all triples of non-intersecting lattice paths which are counted negatively are cancelled by the positive ones. In fact, we are proving more: if the sum $j + k$ is fixed, say $j + k = f$ then

$$c(\lambda, \mu; f) := \sum_{(i,j,k) \in S(\lambda), j+k=f} M(i, j, k) \tag{2.9}$$

is non-negative and there exists an f such that this sum is positive. As an example, let $j + k = 2$ (so $f = 2$). The configurations arising from this situation are those from Figure 2 below. Then $c((3, 3, 3), (2, 2, 1); 2)$ is the sum of 6 determinants, and it can be written as:

$$c((3, 3, 3), (2, 2, 1); 2) = 0 + 3 + 6 + 3 + 3 - 3 = 12 \quad .$$

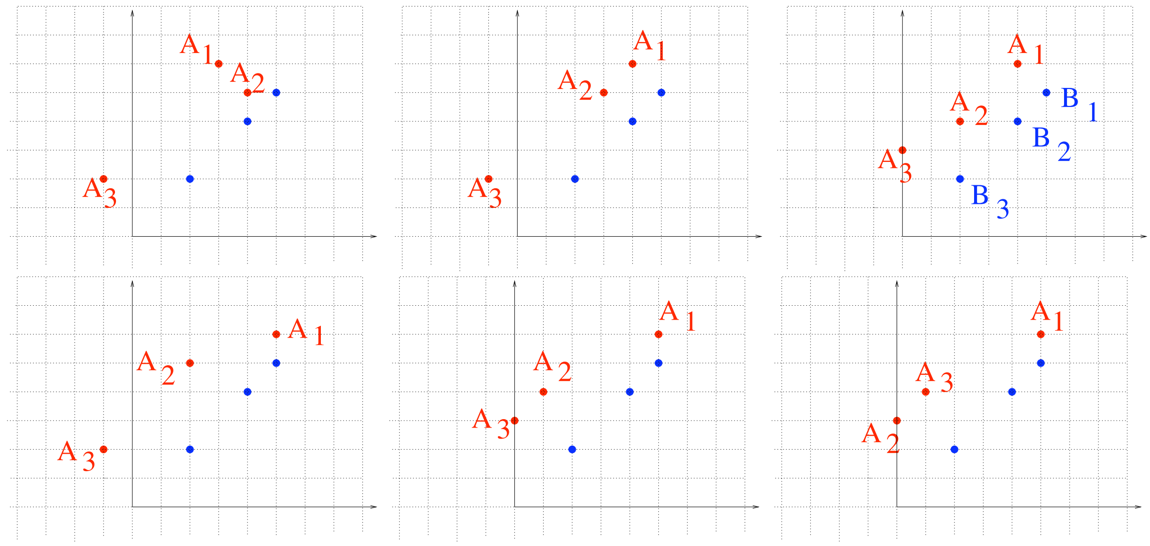

Figure 2: Configurations of A's and B's for $\lambda = (3,3,3), \mu = (2,2,1)$ and $j + k = 2$; figures correspond, left-right, top-down to $(0,2,0)$, $(0,1,1)$, $(1,1,1)$, $(0,0,2)$, $(1,0,2)$, $(2,0,2)$. The blue dots represent the points B_ℓ, which do not vary with (i,j,k).

Note, for example, that $\det M(0,2,0) = 1 - 1 = 0$ since there is one triple of non-intersecting paths, $(A_1, A_2, A_3) \to (B_1, B_2, B_3)$ counted with $+1$, and one triple $(A_2, A_1, A_3) \to (B_1, B_2, B_3)$ counted negatively. This is different from the case $(i,j,k) = (2,0,2)$ when all the triples are counted negatively (in fact, this is just an ordinary LGV determinant, with the second and the third row swapped).

2.4 Idea of proof

To show that the coefficient $c((\lambda_1, \lambda_2, \lambda_3), (\mu_1, \mu_2, \mu_3); f)$ is nonnegative it is enough to prove that there is an injective map from the non-intersecting triples which count negatively to those counting positively. We show how this map is constructed for the negative paths computing $\det M(0,2,0)$ and $\det M(2,0,2)$ in Figure 2 above. To shorten notation, we denote by $\mathcal{P}((A_1, A_2, A_3) \longrightarrow (B_1, B_2, B_3))$ the set of *non-intersecting* triples of lattice paths $\pi = (\pi_1, \pi_2, \pi_3)$ where $\pi_\ell : A_\ell \to B_\ell$.

The map will distinguish between two cases: one when w is the transposition (12) and one when $w = (23)$; $(i,j,k) = (0,2,0)$ corresponds to the first case, while $(2,0,2)$ to the second. We will show among other things, in §3.4, that these are the only configurations resulting in (non-intersecting) triples counted negatively.

If $(i,j,k) = (0,2,0)$, from a triple of paths

$$(\pi_1, \pi_2, \pi_3) \in \mathcal{P}((A_2, A_1, A_3) \longrightarrow (B_1, B_2, B_3)),$$

we construct a triple

$$(\pi_1^*, \pi_2^*, \pi_3^*) \in \mathcal{P}((A_1^*, A_2^*, A_3^*) \longrightarrow (B_1, B_2, B_3))$$

where (A_1^*, A_2^*, A_3^*) is the triple corresponding to $(i^*, j^*, k^*) = (0, 1, 1)$; the path π_3 remains unchanged, so $\pi_3^* = \pi_3$. As for π_1^* respectively π_2^*, they are constructed using certain 'surgery' on π_1 and π_2 respectively. This process, described below, is shown in Figure 3. First, one translates the source A_2 of π_1 horizontally to left, say x units, until it hits π_2. Let A_2^* be this intersection point and define π_2^* to be the portion of π_2 starting at A_2^*. Similarly, given the x units from the previous step, one translates the portion of π_2 from A_1 to A_2^* horizontally to the right x units, and form the new path π_1^*. Note that in this case, the triple (i^*, j^*, k^*) corresponding to (A_1^*, A_2^*, A_3^*) is

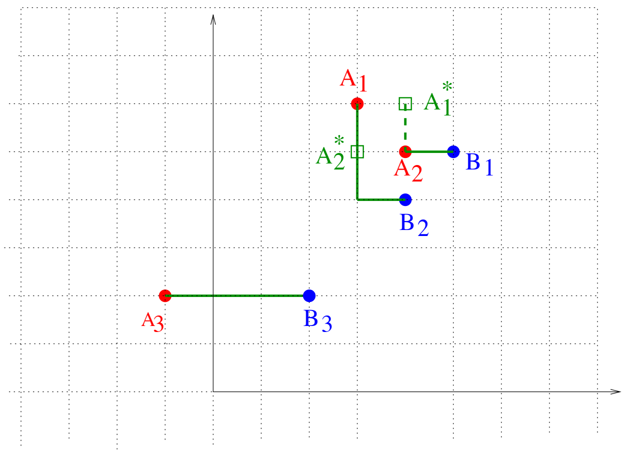

Figure 3: Construction of paths π_1^* and π_2^* corresponding to the inversion (12)

obtained from the initial (i, j, k) by making

$$i^* := i, \ j^* := j - x, \ k^* := k + x, \tag{2.10}$$

and such a transformation leaves $S(\lambda)$ and the sum $j + k$ invariant, provided that x is small enough. The condition on x will be satisfied for each triple of paths which contributes negatively to its corresponding determinant.

A similar procedure, using now a *diagonal translation* with slope 1, can be used to construct a positive triple out of one corresponding to the inversion (23). This is illustrated in Figure 4. In this case, the newly obtained triple (A_1^*, A_2^*, A_3^*) is via the transformation

$$i^* := i - x, \ j^* := j, \ k^* := k \quad , \tag{2.11}$$

which again preserves $S(\lambda)$ and the sum $j + k$ for small x. It is not hard to verify that in this example, the map defined from all the non-intersecting triples of paths which count negatively has four key properties:

1. it sends a triple of non-intersecting paths into another triple of non-intersecting paths;
2. it is injective;

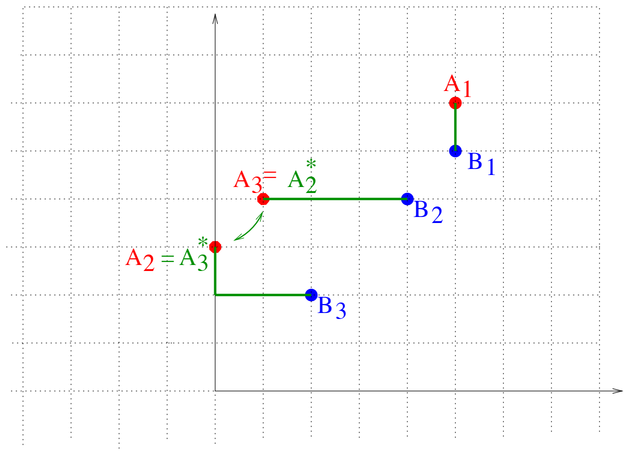

Figure 4: Construction of paths π_2^* and π_3^* corresponding to the inversion (23)

3. it sends the initial points corresponding to a triangular sequence (i, j, k) into points determined by a sequence (i^*, j^*, k^*) from the same set $S(\lambda)$. Moreover, $j + k = j^* + k^*$.

4. it sends each triple which counts "negatively" to a triple counting "positively".

We will prove all these properties in general.

3 Proof of part 3 of Theorem 1.1

3.1 Possible configurations for the points $A_\ell(s)$ and B_ℓ.

We use the notation from §2.1. We fix two partitions $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_p)$. For now p is arbitrary, but we will soon restrict to the case $p = 3$. Since the parts of the partition μ are decreasing it follows that for $\ell_1 > \ell_2$, the point B_{ℓ_1} is strictly North-East of B_{ℓ_2} (see figure below). The points $A_\ell = A_\ell(s)$, for a fixed triangular sequence s, are not arranged as nicely. However, the following holds:

Lemma 3.1 *(a) Let s be a triangular sequence. Then $A_1(s)$ is strictly North-East of $A_p(s)$, i.e. $x_{A_1(s)} > x_{A_p(s)}$ and $y_{A_1(s)} > y_{A_p(s)}$.*
(b) $A_1(s)$ is strictly North of $A_\ell(s)$, for all $2 \leqslant \ell \leqslant p-1$.

Proof: This is a straightforward computation. □

3.2 Two distance functions between paths

Let π and π' be respectively two paths between A and B, and A' and B'. We define two distances between π and π'. One is a horizontal distance, denoted $D_h(\pi, \pi')$ and the other is a diagonal distance, denoted $D_d(\pi, \pi')$. These distances will be the

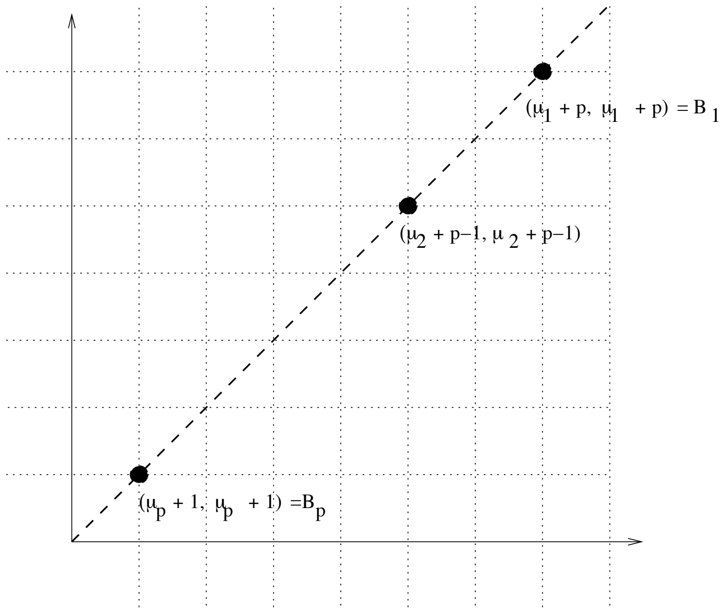

Figure 5: The end points B_ℓ.

quantities x used to define the transformations described in equations (2.10) and (2.11).

We define D_h first. This distance will only be defined provided that $y_A > y_{A'}$, i.e. that A' is strictly to the South of A. The reader can refer to Figure 3, with $(A, B) = (A_1, B_2)$ and $(A', B') = (A_2, B_1)$. Assume that a horizontal line L' passing through A' intersects π at a point C. Then $D_h(\pi, \pi') = l$ where l is the length of the segment $A'C$. If L' doesn't intersect π then we set $D_h(\pi, \pi') = \infty$. In Figure 3, $D_h = 1$.

To define D_d, we take diagonal lines L and L' of slope 1 starting respectively from A and A' in both directions (in Figure 4, $(A, B) = (A_3, B_2), (A', B') = (A_2, B_3)$, and the lines L and L' happen to coincide). If at least one of L or L' intersects respectively π' and π (say at C' or C), then $D_d(\pi, \pi')$ is the (diagonal) length of the segment AC' or $A'C$ (necessarily just one) formed in this way. If neither of L and L' intersects the associated paths, define $D_d(\pi, \pi') = \infty$. The following is immediate:

Lemma 3.2 *Assume that $\pi_r : A_r \to B_r$, $1 \leqslant r \leqslant 2$ are two paths, and that B_1 and B_2 are on the main diagonal. Then $D_d(\pi, \pi') < \infty$.*

3.3 Two swaps

Let $\pi_1 : A_1 \to B_2$ and $\pi_2 : A_2 \to B_1$ be two paths as in Figure 3. As in §2.4, we will define two 'swaps' between π_1 and π_2; one horizontal and one diagonal provided that the corresponding distance between the paths is not infinity. The result will be a new pair of paths, (π_1^*, π_2^*) and new sets of points A_r^*, $r = 1, 2$. The end points B_r remain fixed.

In fact, we only define the *horizontal swap* as in Figure 3 above and we let the reader to fill in the details for the diagonal swap, using the Figure 4. Let $A_r = (x_{A_r}, y_{A_r})$, $r = 1, 2$.

Assume that $D_h(\pi_1, \pi_2) = l$ and that this distance is realized by the segment $A_2A_2^*$, with $A_2^* \in \pi_1$. Then let π_2^* to be the partial path obtained from π_1 by chopping off the part from A_1 to A_2^*. To define π_1^* we translate horizontally the path from A_1 to A_2^* and attach it to π_2 such that A_2^* becomes A_2. Then A_1^* will be the new starting point of π_1^*. In terms of coordinates, if (x_r, y_r) and (x_r^*, y_r^*) are the coordinates respectively of A_r and A_r^*, then

$$(x_1^*, y_1^*) = (x_1 \pm l, y_1) \text{ and } (x_2^*, y_2^*) = (x_2 \mp l, y_2)$$

where $\pm$ is decided by the orientation of the segment $A_2A_2^*$: plus if $x_{A_2^*} < x_{A_2}$, minus otherwise. Similarly, if $D_d(\pi_1, \pi_2) = l'$ then $(x_r^*, y_r^*) = (x_r \pm l', y_r \pm l')$.

3.4 Positivity for $p = 3$

We are now ready to prove the positivity statement from Theorem 1.1(3). To shorten notation, as before, let $(a_{21}, a_{22}, a_{11}) = (i, j, k)$. The constraints for the triangular sequence with respect to λ translate to:

$$0 \leqslant k \leqslant \lambda_2; \qquad 0 \leqslant i \leqslant k; \qquad 0 \leqslant j \leqslant \lambda_3. \tag{3.1}$$

Fix such a sequence (i, j, k); recall that

$$A_1 = (k+3, \lambda_1+3); \quad A_2 = (2-i+j, \lambda_2+2-i); \quad A_3 := (1-k+i-j, \lambda_3+1-k+i-j).$$

Fix also a partition $\mu = (\mu_1 \geqslant \mu_2 \geqslant \mu_3)$ which determines B_1, B_2, B_3:

$$B_r := (\mu_r + 3 - r + 1, \mu_r + 3 - r + 1), \quad 1 \leqslant r \leqslant 3.$$

Invoking again the classical LGV Theorem, if A_1, A_2, A_3 are arranged, in order, (weakly) from NE to SW, the corresponding determinant will be nonnegative. Since A_1 is always strictly NE of A_3, and strictly North of A_2 (by Lemma 3.1) it follows that there are only two possibilities which may yield a negative determinant:

Case 1. *A_2 is strictly SE of A_1 and there is a triple of non-intersecting paths $\pi = (\pi_1, \pi_2, \pi_3)$ such that $\pi_1 : A_1 \to B_2$, $\pi_2 : A_2 \to B_1$ and $\pi_3 : A_3 \to B_3$* (see e.g. the configuration corresponding to $(i, j, k) = (0, 2, 0)$ in Figure 2). In this case, let $l := D_h(\pi_1, \pi_2)$ be the horizontal distance between π_1 and π_2. Clearly $l < \infty$. Then perform the horizontal swap to (π_1, π_2) to define (π_1^*, π_2^*), with $\pi_r^* : A_r^* \to B_r$. The new starting points A_1^* and A_2^* are given by the sequence (i^*, j^*, k^*) where

$$i^* = i; \qquad j^* = j - l; \qquad k^* = k + l. \tag{3.2}$$

Note that π_3 is not affected; denote by $\pi^* = (\pi_1^*, \pi_2^*, \pi_3)$ the new obtained triple. We have to show that (i^*, j^*, k^*) satisfy the constraints conditions in 3.1 - see also

Property (3) in §2.4. Indeed, the relative position of the paths π_1 and π_2 implies that $x_{A_2} - l \geqslant x_{A_1}$, i.e. that

$$2 - i + j - l \geqslant k + 3.$$

In particular

$$j - l \geqslant k + i + 1 > 0 \text{ and } k + l \leqslant -i + j - 1 < \lambda_2.$$

Another important feature of the swaps performed is that the paths in the new triple π^* are non-intersecting. (This proves Property (1) from §2.4.) This follows immediately from their construction: A_3 is SW of A_1 and π_1^* is obtained by moving the 'head' of π_1, containing A_1, horizontally to the East.

Case 2. *A_2 is* **not** *weakly North-East of A_3 and there is a triple of non-intersecting paths $\pi = (\pi_1, \pi_2, \pi_3)$ such that $\pi_1 : A_1 \to B_1$, $\pi_2 : A_2 \to B_3$ and $\pi_3 : A_3 \to B_2$.* There are three situations, according to A_2 being NW, SE or SW of A_3 (see subcases 2.1, 2.2 and 2.3 below). In all three situations one performs a diagonal swap to (π_2, π_3). We obtain a new triple $\pi^* = (\pi_1, \pi_2^*, \pi_3^*)$, with $\pi_r^* : A_r^* \to B_r$ and the new sequence (i^*, j^*, k^*) defining A_r^* $(r = 2, 3)$ is given by:

$$i^* = i - l; \qquad j^* = j; \qquad k^* = k. \tag{3.3}$$

As in Case 1, we have to show that (i^*, j^*, k^*) satisfies the constraints (3.1), i.e. that $i \geqslant l$.

Subcase 2.1. A_2 is NW of A_3, i.e. $x_{A_2} < x_{A_3}$ and $y_{A_2} \geqslant y_{A_3}$. In this case

$$l \leqslant x_{A_3} - x_{A_2} = (1 - k + i - j) - (2 - i + j) = 2i - k - j - 1$$

which shows that

$$i - l \geqslant k - i + j + 1 > 0.$$

Subcase 2.2. A_2 is SE of A_3, i.e. $x_{A_2} \geqslant x_{A_3}$ and $y_{A_2} < y_{A_3}$. In this case

$$l \leqslant y_{A_3} - y_{A_2} = (1 + \lambda_3 - k + i - j) - (2 + \lambda_2 - i) = (\lambda_3 - \lambda_2) + 2i - k - j - 1$$

which shows that

$$i - l \geqslant (\lambda_2 - \lambda_3) + k - i + j + 1 > 0.$$

Subcase 2.3. A_2 is SW of A_3, i.e. $x_{A_2} \leqslant x_{A_3}$ and $y_{A_2} \leqslant y_{A_3}$, but $A_2 \neq A_3$. In this case

$$l \leqslant \max\{x_{A_3} - x_{A_2}, y_{A_3} - y_{A_2}\}$$

and the computation reduces to one from Subcases 2.1 or 2.2 above.

We also need to show that the diagonal swap produces a non-intersecting triple of paths. This is done separately for each of the subcases above, and it should be clear from the construction.

To finally show positivity, let $c(\lambda, \mu; f)$ be the partial sum from equation (2.9) defining the coefficients of CSM classes in the case $p = 3$, obtained by fixing $k+j = f$.

Theorem 3.3 *Let $0 \leqslant f \leqslant \lambda_2 + \lambda_3$. Then the partial sum $c(\lambda, \mu; f)$ is nonnegative and $c(\lambda, \mu; 0) > 0$ if $\mu \subset \lambda$ (i.e. $\mu_r \leqslant \lambda_r$, for $1 \leqslant r \leqslant 3$).*

Proof: The second part of the theorem is immediate: if $f = 0$, then $(i, j, k) = (0, 0, 0)$, and the points A_1, A_2, A_3 are arranged *strictly* from NW to SE, thus satisfying the hypothesis of the LGV Theorem. Since $\mu \subset \lambda$, each B_ℓ is SE of A_ℓ, so there is at least one non-intersecting triple of paths $\pi : (A_1, A_2, A_3) \to (B_1, B_2, B_3)$.

To prove the first part we need to show that there is an injective map from the non-intersecting triples of paths $\pi : (A_1, A_2, A_3) \to (B_1, B_2, B_3)$ where the initial points A_1, A_2, A_3 are *not* arranged in the NE-SW configuration (and may count negatively) to those triples where the initial points *are* arranged in NE-SW (and count positively). Let $\mathcal{S}_1$ respectively $\mathcal{S}_2$ the sets of all such triples. We define the map $\Psi : \mathcal{S}_1 \to \mathcal{S}_2$ as follows: by definition the points A_1, A_2, A_3 are not arranged in the NE-SW configuration, therefore they are exactly in one of the configurations described in Case 1 and Case 2 above. In each of these cases, $\Psi(\pi) = \pi^*$, where π^* is the swap applied to $\pi : (A_1, A_2, A_3) \to (B_1, B_2, B_3)$ (and described explicitly in the Cases 1 and 2). By our proofs in Cases 1 and 2, the map Ψ satisfies properties (1),(3),(4) mentioned in §2.4. It remains to show that Ψ is injective. For that, it is enough to show that there cannot be triples $\pi = (\pi_1, \pi_2, \pi_3)$ and $\pi^{**} = (\pi_1^{**}, \pi_2^{**}, \pi_3^{**})$ such that:

1. π and π^{**} are triples of non-intersecting paths.
2. π creates a (12) inversion, as in Case 1 above, i.e. $\pi_1 : A_1(i, j, k) \to B_2$, $\pi_2 : A_2(i, j, k) \to B_1$, $\pi_3 : A_3(i, j, k) \to B_3$.
3. π^{**} creates a (23) inversion, as in Case 2 above, i.e. $\pi_1^{**} : A_1(i^{**}, j^{**}, k^{**}) \to B_1$, $\pi_2^{**} : A_2(i^{**}, j^{**}, k^{**}) \to B_3$, $\pi_3^{**} : A_3(i^{**}, j^{**}, k^{**}) \to B_2$.
4. The new triples obtained by applying a horizontal swap to (π_1, π_2) in π and a diagonal swap to (π_2^{**}, π_3^{**}) in π^{**} are equal.

We assume there are such triples, and recall that i, j, k is the sequence corresponding to π; to shorten notation, let A_1, A_2, A_3 be the starting points of the paths determined by π and let A_1^*, A_2^* be the initial points of the paths $\pi_1^* : A_1^* \to B_1$ and $\pi_2^* : A_2^* \to B_2$ obtained from the horizontal swap of (π_1, π_2). Let also $l_1 = D_h(\pi_1, \pi_2)$, $l_2 = D_d(\pi_2^*, \pi_3)$ and let (i^*, j^*, k^*) be the sequence determining A_1^*, A_2^* and $A_3^* = A_3$. Refer to Figure 3 for the configuration of A_1, A_2, A_3. The next lemma shows the relations between i, i^*, i^{**} and so on, needed later.

Lemma 3.4 *(a) $i^* = i$ and $i^{**} = i + l_2$.*
(b) $k^* = k + l_1$ and $k^{**} = k^*$.
(c) $j^* = j - l_1$ and $j^{**} = j^*$.

Proof: This follows from the equations 3.2 and 3.3 which record the transformations of i, j, k after a horizontal or diagonal swap. □

Since π creates an (12) inversion, it must be that A_2 is strictly S and strictly E of A_1, i.e.

$$x_{A_2} > x_{A_1} \text{ and } y_{A_2} < y_{A_1} \tag{3.4}$$

(use Lemma 3.1 and the fact that if $x_{A_2} \leqslant x_{A_1}$ then π_1 and π_2 must intersect). Moreover, by the definition of l_1,

$$x_{A_2} - x_{A_1} \geqslant l_1 \Leftrightarrow (2 - i + j) - (k + 3) \geqslant l_1 \Leftrightarrow j - i - l_1 \geqslant k + 1.$$

In particular,

$$j - i \geqslant l_1 + 1. \tag{3.5}$$

Lemma 3.5 *A_3 is strictly S and strictly W of A_2^*, i.e. $x_{A_2^*} > x_{A_3}$ and $y_{A_2^*} > y_{A_3}$.*

PROOF: We have

$$y_{A_2^*} - y_{A_3} = (\lambda_2 - \lambda_3) + (k - i) + (j - i) + 1 \geqslant 2 > 0,$$

where '$\geqslant$' follows from $\lambda_2 \geqslant \lambda_3$, $k \geqslant i$ and equation (3.5). As for $x_{A_2^*} > x_{A_3}$, this happens since $x_{A_2^*} \geqslant x_{A_1} > x_{A_3}$; the first inequality holds because performing a horizontal swap to paths π_1, π_2 creating a (12) inversion implies $A_2^* \in \pi_1$, therefore A_2^* is weakly East of A_1; for the second inequality use Lemma 3.1. □

This lemma, together with the definition of l_2, implies that

$$l_2 \geqslant \min\{x_{A_2^*} - x_{A_3}, y_{A_2^*} - y_{A_3}\}.$$

The triple (i^{**}, j^{**}, k^{**}) must satisfy the constraints (3.1), so in particular

$$i^{**} \leqslant k^{**} \Leftrightarrow i + l_2 \leq k + l_1.$$

Then the theorem follows from the following lemma, which contradicts the existence of such a triple, and therefore of π^{**}.

Lemma 3.6 $i + \min\{x_{A_2^*} - x_{A_3}, y_{A_2^*} - y_{A_3}\} > k + l_1$.

PROOF: We first show that $i + y_{A_2^*} - y_{A_3} > k + l_1$. This is equivalent to

$$i + (\lambda_2 - \lambda_3) + k - i + 1 + j - i > k + l_1 \Leftrightarrow (\lambda_2 - \lambda_3) + 1 + j - i - l_1 > 0$$

and the last expression is true by equation (3.5) above. Similarly, taking into account that $x_{A_2^*} = 2 - i + j - l_1$, we have

$$i + x_{A_2^*} - x_{A_3} > k + l_1 \Leftrightarrow 1 + j - l_1 + j - i - l_1 > 0$$

which is true again by the equation (3.5) above. This finishes the proof of the lemma and of the theorem. □

The proof of the theorem suggests a positive formula to compute $c(\lambda, \mu)$: for a fixed triple (i, j, k), the triples of paths $\pi = (\pi_1, \pi_2, \pi_3)$ which contribute to $c(\lambda, \mu)$ must satisfy the following:

(P1) π is non-intersecting and it corresponds to identity permutation, i.e. $\pi_r : A_r \to B_r$, for $1 \leqslant r \leqslant 3$.

(P2) Let l_h be the horizontal distance between π_1 and π_2 and let l_d be the diagonal distance between π_2 and π_3. Recall that $l_d < \infty$ (Lemma 3.2). Then either $l_h = \infty$, or, if $l_h < \infty$, neither of triples

$$(i, j + l_h, k - l_h) \text{ or } (i + l_d, j, k)$$

is triangular, i.e. neither of them satisfies the conditions from (3.1). For the first triple, this can happen, for example, if $j + l_h > \lambda_3$ or if $k < l_h$.

We call the triples $\pi = (\pi_1, \pi_2, \pi_3)$ satisfying (P1) and (P2) **balanced**. The properties (P1) and (P2) mean that one cannot perform a horizontal transformation to π_1 and π_2, or a diagonal transformation to π_2 and π_3, and obtain a triple of paths with initial points A_1, A_2, A_3 coming from a triangular sequence in $S(\lambda)$. Informally, π_2 is 'far enough' from either π_1 and π_3, so one cannot do either transformation. In summary:

Corollary 3.7 *$c(\lambda, \mu)$ is equal to*

$$\sum_{s \in S(\lambda)} \#\mathcal{P}_{bal}\big((A_1(s), A_2(s), A_3(s)) \to (B_1, B_2, B_3)\big)$$

where $\mathcal{P}_{bal}$ indicates that only the balanced triples from $\mathcal{P}\big((A_1(s), A_2(s), A_3(s)) \to (B_1, B_2, B_3)\big)$ are considered.

Acknowledgments

I am grateful to Paolo Aluffi for some inspiring conversations, to Christian Krattenthaler and Piotr Pragacz for valuable suggestions and encouragement, and to the referees for careful reading and valuable suggestions. Most of this work was done while enjoying the hospitality and support of Max-Planck Institute für Mathematik, Bonn.

References

[1] P. Aluffi, Characteristic classes of singular varieties, *Topics in cohomological studies of algebraic varieties*, *Trends Math.*, Birkhäuser, Basel (2005), 1–32.

[2] P. Aluffi and L. C. Mihalcea, Chern classes of Schubert cells and varieties, *J. Algebraic Geom.* **18** (2009), no. 1, 63–100.

[3] I. Gessel and G. Viennot, Determinants, paths and plane partitions, (1989). (Preprint available at: `http://people.brandeis.edu/~gessel/homepage/papers/index.html`.)

[4] J. Huh, Positivity of Chern classes of Schubert cells and varieties, *J. Algebraic Geom.* (to appear). Available at: arXiv: 1302.5852.

[5] B. Jones, Singular Chern Classes of Schubert Varieties via Small Resolution, *Int. Math. Res. Not.* **8** (2010), 1371–1416.

[6] C. Krattenthaler, Advanced determinant calculus: a complement, *Linear Algebra Appl.* **411** (2005), 68–166.

[7] B. Lindström, On the vector representations of induced matroids, *Bull. London Math. Soc.* **5** (1973), 85–90.

[8] R. MacPherson, Chern Classes for singular algebraic varieties, *Ann. Math.* (2) **100** (1974), 423–432.

[9] M.-H. Schwartz, Classes caractéristiques définis par une stratification d'une variété analytique complexe I, *C.R. Acad. Sci. Paris* **260** (1965), 3262–3264.

[10] M.-H. Schwartz, Classes caractéristiques définis par une stratification d'une variété analytique complexe II, *C.R. Acad. Sci. Paris* **260** (1965), 3535–3537.

[11] J. Stryker, *Chern-Schwartz-MacPherson classes of graph hypersurfaces and Schubert varieties*, Ph.D. Thesis, Florida State University, (2011).

(Received 17 Aug 2013; revised 6 Apr 2015)