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# The measurements of closeness to graceful graphs

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**Abstract**

The beta-number, $\beta(G)$, of a graph $G$ is defined to be either the smallest positive integer $n$ for which there exists an injective function $f : V(G) \to \{0, 1, \ldots, n\}$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $\{c, c+1, \ldots, c+|E(G)|-1\}$ for some positive integer $c$, or $+\infty$ if there exists no such integer $n$. If $c = 1$, the resulting beta-number is called the strong beta-number of $G$ and denoted by $\beta_s(G)$. In this paper, some necessary conditions for a graph to have finite (strong) beta-number are presented, which lead us to sufficient conditions for a graph to have infinite (strong) beta-number. By means of these, the formulas for the (strong) beta-number of certain graphs are determined. Moreover, nontrivial trees and forests are shown to have finite strong beta-number. Finally, three open problems are proposed.

## 1 Introduction

The descriptive terminology and notation used in this paper will generally follow closely that of [2]. All graphs considered here are finite and undirected without loops or multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$, while the edge set is denoted by $E(G)$. For two graphs $G_1$ and $G_2$ with disjoint vertex sets, the *union* $G \cong G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph $G$ consists of $m$ disjoint copies of a graph $H$, then we write $G \cong mH$, where $m \geq 2$. For integers $a$ and $b$ with $a \leq b$, we will denote the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$ by simply writing $[a, b]$, where $\mathbb{Z}$ denotes the set of integers.

The type of graph labelings that have received the most attention over the years was introduced by Rosa [7] in 1967 who called them $\beta$-valuations. For a graph $G$ with $q$ edges, an injective function $f : V(G) \to [0, q]$ is called a *$\beta$-valuation* if each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such a valuation is now commonly known as a *graceful labeling* (the term was introduced by Golomb [4] in 1972) and a graph admitting a graceful labeling is called a *graceful graph*. On the other hand, a graph that is not graceful is called *nongraceful*.

The notion of a graceful labeling naturally stemmed from the study by Golomb [4] of the following problem. For a graph $G$, let grac$(G)$ denote the smallest positive integer $n$ for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels consists of distinct integers. The value grac$(G)$ is called the *gracefulness* of a graph $G$. He pointed out that if $H$ is a subgraph of a graph $G$, then $\text{grac}(H) \leq \text{grac}(G) \leq \text{grac}(K_p)$, where $K_p$ is the complete graph of order $p$. If we label the vertices of $K_p$, denoted by $v_i$, with $f(v_i) = 2^i - 1$ for each $i \in [0, p-1]$, then we obtain the inequality $\max(p-1, q) \leq \text{grac}(G) \leq \text{grac}(K_p) \leq 2^{p-1} - 1$ for every graph $G$ of order $p$ and size $q$. Hence, every graph has finite gracefulness. If $G$ is a graph of size $q$ with grac$(G) = q$, then $G$ is graceful. Thus, the gracefulness is a measure of how close $G$ is to being graceful. By the definition, it is possible to label the vertices of a graph $G$ with distinct elements of the set $[0, \text{grac}(G)]$ so that the edges of $G$ receive distinct labels. It is clear that some vertex of $G$ must be labeled grac$(G)$, but it is not known whether an edge of $G$ must be labeled grac$(G)$.

Golomb [4] determined the exact values of grac$(K_p)$ for all $p \in [1, 10]$ (for instance, grac$(K_4) = 6$, grac$(K_5) = 11$ and grac$(K_6) = 17$). Then he posed the problem of determining the exact value of grac$(K_p)$ for all $p \geq 11$. The exact value of grac$(K_p)$ is still not known in general. Along with some related results, Erdös has shown that asymptotically grac$(K_p) \sim p^2$ (see [4]).

The notion of gracefulness motivates us to define two types of new parameters measuring how close a graph is to being graceful. The *beta-number*, $\beta(G)$, of a graph $G$ with $q$ edges is defined to be either the smallest positive integer $n$ for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c + q - 1]$ for some positive integer $c$ or $+\infty$ if there exists no such integer $n$. If $c = 1$, the resulting beta-number is called the *strong beta-number* of $G$ and denoted by $\beta_s(G)$.

As an immediate consequence of the above three definitions, we have the following relations among three parameters.

**Lemma 1.1** *For every graph $G$ of order $p$ and size $q$,*

$$\max(p-1, q) \leq \text{grac}(G) \leq \beta(G) \leq \beta_s(G).$$

In Section 2, we present some necessary conditions for a graph to have finite (strong) beta-number. We also provide sufficient conditions for a graph to have infinite (strong) beta-number. In Section 3, we determine the formulas for the (strong) beta-numbers of cycles, 2-regular graphs with two isomorphic components and complete graphs. In Section 4, we prove that every forest has finite strong beta-number. This leads us to a corollary that every nontrivial tree has finite strong beta-number. In addition to these, we compute the (strong) beta-number of two kinds of forests, namely, two disjoint copies of stars, and disjoint unions of paths and stars. In Section 5, we make some remarks on bounds for the (strong) beta-number and gracefulness of graphs, and state three open problems.

For the latest developments in graph labeling, the authors refer the reader to an extensive survey by Gallian [3]. The books by Bača and Miller [1], and Marr and Wallis [6] are excellent sources of information for those who are interested in the subject of graph labeling.

## 2 Basic Results

In this section, we first present some necessary conditions for a graph to have finite (strong) beta-number. As a consequence of these, we also provide sufficient conditions for a graph to have infinite (strong) beta-number.

Golomb [4] showed that if $G$ is a graceful graph of size $q$, then there exists a partition of $V(G)$ into two subsets $V_1$ and $V_2$ such that the number of edges joining $V_1$ and $V_2$ is exactly $\lceil q/2 \rceil$. This is easily extended to the following result.

**Lemma 2.1** *If $G$ is a graph of size $q$ such that $\beta_s(G) < +\infty$, then there exists a partition of $V(G)$ into two subsets $V_1$ and $V_2$ such that the number of edges joining $V_1$ and $V_2$ is exactly $\lceil q/2 \rceil$.*

Proof: Let a vertex labeling of $G$ such that $\beta_s(G) < +\infty$ be given. Denote the set of vertices labeled with an even integer by $V_1$ and the set of vertices labeled with an odd integer by $V_2$. All edges labeled with an odd integer must then join a vertex of $V_1$ and a vertex of $V_2$. Since there are exactly $\lceil q/2 \rceil$ such edges, the result follows. □

A similar argument used in the proof of Lemma 2.1 can also be applied to establish the following result which we state next without proof.

**Lemma 2.2** *If $G$ is a graph of size $q$ such that $\beta(G) < +\infty$, then there exists a partition of $V(G)$ into two subsets $V_1$ and $V_2$ such that the number of edges joining $V_1$ and $V_2$ is either $\lfloor q/2 \rfloor$ or $\lceil q/2 \rceil$.*

If $G$ is a graph of size $q$ such that $\beta(G) < +\infty$, then there exists an injective function $f : V(G) \to [0, \beta(G)]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c + q - 1]$ for some positive integer $c$. Observe that if $c$ is even and $q$ is odd, then the set $[c, c+q-1]$ contains exactly $\lfloor q/2 \rfloor$ odd integers. This is the reason why $\lfloor q/2 \rfloor$ arises in the conclusion of Lemma 2.2.

An *Eulerian circuit* of $G$ is a circuit containing all of the edges and vertices of $G$. A graph possessing an Eulerian circuit is called an *Eulerian graph*. The necessary condition for an Eulerian graph to have finite strong beta-number is stated next. Its proof is entirely analogous to the one provided by Rosa [7].

**Theorem 2.3** *If $G$ is an Eulerian graph of size $q$ such that $\beta_s(G) < +\infty$, then $q \equiv 0$ or $3 \pmod 4$.*

An analogous argument to the proof of Theorem 2.3 can be developed to obtain the following result which we state next without proof.

**Theorem 2.4** *Let $G$ be an Eulerian graph of size $q$. If there exists an injective function $f : V(G) \to [0, n]$ such that*

$$\{|f(u) - f(v)| : uv \in E(G)\} = [c, c+q-1]$$

*for some positive integers $n$ and $c$, then $q \equiv 0 \pmod 4$, $q \equiv 1 \pmod 4$ and $c$ is even, or $q \equiv 3 \pmod 4$ and $c$ is odd.*

As we stated in Lemma 1.1, the inequality $\beta(G) \le \beta_s(G)$ holds for any graph $G$. Combining this with Lemma 2.2, we have the following result.

**Corollary 2.5** *If $G$ is a graph of size $q$ with the property that for any partition of $V(G)$ into two subsets $V_1$ and $V_2$, the number of edges joining $V_1$ and $V_2$ is neither $\lfloor q/2 \rfloor$ nor $\lceil q/2 \rceil$, then $\beta_s(G) = \beta(G) = +\infty$.*

The above result shows that relaxing the range of possible vertex labels to a graph that is nongraceful does not need to provide finite (strong) beta-number. This might be unexpected in light of the aforementioned fact that every graph has finite gracefulness. Indeed, it is not difficult to construct a graph $G$ such that $\beta(G) - \text{grac}(G) = +\infty$ (see Theorem 3.1) or $\beta_s(G) - \text{grac}(G) = +\infty$ (see Corollary 3.2).

The contrapositive of Theorem 2.3 is particularly useful to show that certain Eulerian graphs have infinite strong beta-number.

**Corollary 2.6** *If $G$ is an Eulerian graph of size $q$ such that $q \equiv 1$ or $2 \pmod 4$, then $\beta_s(G) = +\infty$.*

As a possible generalization of Corollary 2.6, we obtain the following result.

**Corollary 2.7** *If $G$ is a graph of size $q$ such that every vertex has even degree and $q \equiv 2 \pmod 4$, then $\beta_s(G) = \beta(G) = +\infty$.*

PROOF: Since every vertex of $G$ has even degree, it follows that every component of $G$ is Eulerian. Let $G_1, G_2, \ldots, G_k$ $(k \geq 1)$ be the components of $G$. For each $i \in [1, k]$, let $q_i = |E(G_i)|$, and let $C_i : v_0^i, v_1^i, \ldots v_{q_i-1}^i, v_{q_i}^i = v_0^i$ be an Eulerian circuit of $G_i$. Furthermore, assume that $\beta(G) = n$ for some positive integer $n$. Then there exists an injective function $f : V(G) \to [1, n]$ such that $\{|f(u) - f(v)| : uv \in E(G)\} = [c, c + |E(G)| - 1]$ for some positive integer $c$, and $f\left(v_j^i\right) = a_j^i$ for each $i \in [1, k]$ and $j \in [0, q_i]$, where $a_s^i = a_t^i$ if $v_s^i = v_t^i$. Thus, the label of edge $v_{j-1}^i v_j^i$ is $\left|a_j^i - a_{j-1}^i\right|$. Notice that

$$\left|a_j^i - a_{j-1}^i\right| \equiv \left(a_j^i - a_{j-1}^i\right) \pmod 2$$

for all $i \in [1, k]$ and $j \in [0, q_i]$. Thus, the sum of the labels of edges of $G$ is

$$\sum_{i=1}^{k} \sum_{j=1}^{q_i} \left|a_j^i - a_{j-1}^i\right| \equiv \sum_{i=1}^{k} \sum_{j=1}^{q_i} \left(a_j^i - a_{j-1}^i\right) \equiv 0 \pmod 2,$$

that is, the sum of the edge labels of $G$ is even. However, the sum of the edge labels is $\sum_{i=c}^{c+q-1} i = q(2c + q - 1)/2$, where $q = |E(G)|$; so $q(2c + q - 1)/2$ is even. Consequently, $4|q(2c + q - 1)$, which implies that $4|q$ or $4|2c + q - 1$ so that $q \equiv 0 \pmod 4$, $q \equiv 1 \pmod 4$ and $c$ is even, or $q \equiv 3 \pmod 4$ and $c$ is odd. This implies that $\beta_s(G) = \beta(G) = +\infty$ for $q \equiv 2 \pmod 4$. □

## 3 The Beta-Number and Strong Beta-Number of Some Graphs

As is often the case, when no general formula exists for the value of a parameter for an arbitrary graph, formulas (or partial formulas) are established for certain classes of graphs. Ordinarily, the first classes to be considered are the cycles and the complete graphs. In this section, we determine the formulas for the (strong) beta-number of graphs in these classes as well as all 2-regular graphs with two isomorphic components.

Rosa [7] proved that the cycle $C_n$ is graceful if and only if $n \equiv 0$ or $3 \pmod 4$. Combining this with Corollary 2.6, we have the following result.

**Theorem 3.1** *For every integer $n \geq 3$,*

$$\beta_s(C_n) = \begin{cases} n, & \text{if } n \equiv 0 \text{ or } 3 \pmod 4; \\ +\infty, & \text{if } n \equiv 1 \text{ or } 2 \pmod 4. \end{cases}.$$

With the aid of the results in the previous section and Theorem 3.1, we are now able to compute the beta-number for all cycles.

**Corollary 3.2** *For every integer $n \geq 3$,*

$$\beta(C_n) = \begin{cases} n, & \text{if } n \equiv 0 \text{ or } 3 \pmod 4; \\ n+1, & \text{if } n \equiv 1 \pmod 4; \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

PROOF: It follows from Lemma 1.1 and Theorem 3.1 that $\beta(C_n) = n$ for $n \equiv 0$ or $3 \pmod 4$. It also follows from Corollary 2.7 that $\beta(C_n) = +\infty$ for $n \equiv 2 \pmod 4$. As we mentioned above, the cycle $C_n$ is not graceful for $n \equiv 1 \pmod 4$. This together with Lemma 1.1 provides that $\beta(C_n) \geq \mathrm{grac}(C_n) \geq n+1$ for $n \equiv 1 \pmod 4$.

To complete the proof, we will show that $\beta(C_n) \leq n+1$ for $n \equiv 1 \pmod 4$. For this, define the cycle $C_n$ with

$$V(C_n) = \{v_i : i \in [1, n]\}$$

and

$$E(C_n) = \{v_i v_{i+1} : i \in [1, n-1]\} \cup \{v_1 v_n\},$$

and consider the vertex labeling $f : V(C_n) \to [0, n+1]$ such that

$$f(x) = \begin{cases} n+2-i, & \text{if } x = v_{2i-1} \text{ and } i \in [1, (n+1)/2]; \\ i-1, & \text{if } x = v_{2i} \text{ and } i \in [1, (n+3)/4]; \\ i, & \text{if } x = v_{2i} \text{ and } i \in [(n+3)/4+1, (n-1)/2]. \end{cases}$$

Notice that

$$\{f(v) : v \in V(C_n)\} = [0, n+1] \setminus \{(n+3)/4, (n+1)/2\},$$

which implies that $f$ is an injective function. Notice also that

$$\{|f(v_i) - f(v_{i+1})| : i \in [(n+3)/2+1, n-1]\} \cup \{|f(v_1) - f(v_n)|\} = [2, (n-1)/2]$$

and

$$\{|f(v_i) - f(v_{i+1})| : i \in [1, (n+3)/2]\} = [(n-1)/2+1, n+1].$$

This implies that

$$\{|f(u) - f(v)| : uv \in E(C_n)\} = [2, |E(C_n)|+1],$$

and therefore $\beta(C_n) \leq n+1$ for $n \equiv 1 \pmod 4$. □

Kotzig [5] proved that the 2-regular graph $2C_n$ is graceful if and only if $n \geq 4$ and $n$ is even. Combining this with Lemma 1.1 and Corollary 2.7, we have the following result.

**Corollary 3.3** *For every integer $n \geq 3$,*

$$\beta_s(2C_n) = \beta(2C_n) = \begin{cases} 2n, & \text{if } n \text{ is even;} \\ +\infty, & \text{if } n \text{ is odd.} \end{cases}$$

Recall that if $H$ is a subgraph of a graph $G$, then the inequality $\mathrm{grac}(H) \leq \mathrm{grac}(G)$ holds. However, neither $\beta_s(H) \leq \beta_s(G)$ nor $\beta(H) \leq \beta(G)$ hold in general, since we know that $\beta_s(2C_n) = \beta(2C_n) = 2n$ (see Corollary 3.3) and $\beta_s(C_n) = \beta(C_n) = +\infty$ (see Theorem 3.1 and Corollary 3.2) when $n$ is a positive integer such that $n \equiv 2 \pmod 4$.

It is easy to see that $K_p$ is graceful for each $p \in [2, 4]$. The following result of Golomb [4] shows that there are no other graceful complete graphs.

**Theorem 3.4** *The complete graph $K_p$, $p \geq 2$, is graceful if and only if $p \leq 4$.*

The above theorem allows us to compute the (strong) beta-number for all complete graphs.

**Theorem 3.5** *For every integer $p \geq 2$,*

$$\beta_s(K_p) = \beta(K_p) = \begin{cases} p(p-1)/2, & \text{if } p \in [2,4]; \\ +\infty, & \text{if } p \geq 5. \end{cases}$$

Proof: For each $p \in [2,4]$, the result follows from Theorem 3.4. In light of Lemma 1.1, it suffices to show that $\beta(K_p) = +\infty$ for all $p \geq 5$. Thus, assume that $p \geq 5$ and suppose, to the contrary, that $\beta(K_p) = n > q$ for some positive integer $n$, where $q = p(p-1)/2$. Hence, there exists an injective vertex labeling $f : V(K_p) \to [0, n]$ such that $\{|f(u) - f(v)| : uv \in E(K_p)\} = [c, c+q-1]$ for some positive integer $c$. Since such a vertex labeling of a graph of size $q$ requires 0 and $n$ to be vertex labels, it follows that $n = c + q - 1$. Thus, some edge of $K_p$ must be labeled $n-1$.

To have an edge labeled $n-1$, we must have adjacent vertices labeled either 0, $n-1$ or 1, $n$. If a vertex is labeled either 1 or $n-1$, then we have an edge labeled 1, that is, we have $c = 1$ or, equivalently, $n = q$. This implies that $f$ is a graceful labeling of $K_p$, which is impossible by Theorem 3.4. Therefore, we conclude that $\beta(K_p) = +\infty$ for all $p \geq 5$. □

## 4 The Strong Beta-Number of Forests

It has been conjectured by Kotizg (see [7]) that every nontrivial tree is graceful. In light of this conjecture, it seems natural to ask the question of whether one can compute or at least bound $\beta_s(T)$ for a nontrivial tree $T$. Indeed, we prove in the following result that every forest has finite strong beta-number. Our proof uses the concept of distance in a graph. For a connected graph $G$ and a pair of $u$, $v \in V(G)$, the *distance* $d(u,v)$ between $u$ and $v$ is the length of a shortest $u - v$ path in $G$.

**Theorem 4.1** *If $F$ is a forest, then $\beta_s(F) < +\infty$.*

Proof: Let $F \cong \bigcup_{i=1}^{n} T_i$ be a forest that is the vertex disjoint union of $n$ nontrivial trees $T_1$, $T_2$, …, $T_n$ such that each tree has been drawn in the plane as a rooted tree. For each tree $T_i$ ($i \in [1,n]$), choose a distinguished vertex to be the root of $T_i$. We will denote such a vertex by $r_i$. Of course, if a tree $T_i$ is drawn as a rooted tree in the plane, then by letting

$$S_{i,d} = \{v \in V(T_i) : d = d(r_i, v) \geq 0\},$$

each vertex of $S_{i,d}$ can be ordered from left to right using the integers 1, 2, …, $|S_{i,d}|$. Hence, $S_{i,d}$ can be written as

$$S_{i,d} = \left\{ v_{i,d}^{1}, v_{i,d}^{2}, \ldots, v_{i,d}^{|S_{i,d}|} \right\}$$

for each $d$ running from 1 up to the 'height' of $T_i$, where the upper subscript of each vertex in $S_{i,d}$ represents the order that the vertex occupies in $S_{i,d}$ starting from left to right. If we let

$$E_{i,d} = \{uv \in E(T_i) : u \in S_{i,d} \text{ and } v \in S_{i,d+1}\},$$

then the edges of $E_{i,d}$ can be also ordered from left to right in increasing order starting with 1 and ending up with $|E_{i,d}|$. Thus, we will write $e^j_{i,d}$ for the edge of $E_{i,d}$, where $j$ denotes the position of the edge in $E_{i,d}$. This means that if $e^j_{i,d}$, $e^k_{i,d} \in E_{i,d}$ with $j < k$, then $e^j_{i,d}$ is located to the left of $e^k_{i,d}$ in the drawing of $T_i$ in the plane.

From now on, for any vertex labeling $f : V(F) \to \mathbb{N} \cup \{0\}$, we will consider the edge labeling $g : E(F) \to \mathbb{N}$ defined by $g(uv) = |f(u) - f(v)|$ for any $uv \in E(F)$. At this point, consider the vertex labeling $f : V(F) \to \mathbb{N} \cup \{0\}$ defined by $f(r_1) = 0$ and $f\left(v^j_{1,1}\right) = j$ for any $j \in [1, |S_{1,1}|]$. This implies that

$$\{g(uv) : uv \in E_{1,0}\} = [1, |S_{1,1}|].$$

Next, we describe how to label the remaining vertices of $T_1$. Consider a vertex of the form $v^1_{1,d}$ and assume that any other vertex $v^1_{1,\delta}$ ($\delta \in [0, d-1]$) has been already labeled. Then let $a = \max\{g(uv) : uv \in E_{1,d-2}\}$ and label the vertex $v^1_{1,d}$ in such a way that $g\left(e^1_{1,d-1}\right) = a + 1$. The remaining vertices of $S_{1,d}$ are labeled so that

$$g\left(e^j_{1,d-1}\right) - g\left(e^{j-1}_{1,d-1}\right) = 1$$

for any $j \in [2, |S_{1,d}|]$. This establishes the vertex labeling of $T_1$.

To complete the proof, we explain how to label the vertices of $T_i$ for an arbitrary $i \in [2, n]$. Thereby, for the remainder of the proof, assume that $i \in [2, n]$. Now, let $b_{i-1} = \max\{f(v) : v \in V(T_{i-1})\}$ and label the root $r_i$ of $T_i$ with $b_{i-1} + 1$. Also, let $a_{i-1} = \max\{g(uv) : uv \in E(T_{i-1})\}$ and label the vertices of $S_{i,1}$ in such a way that

$$g\left(e^1_{i,0}\right) = a_{i-1} + 1 \text{ and } g\left(e^j_{i,0}\right) - g\left(e^{j-1}_{i,0}\right) = 1$$

for any $j \in [2, |S_{i,1}|]$. Moreover, consider a vertex of the form $v^1_{i,d}$ and assume that any other vertex $v^j_{i,\delta}$ ($\delta \in [0, d-1]$) has been already labeled. Then let $c_i = \max\{g(uv) : uv \in E_{i,d-2}\}$ and label each vertex $v^1_{i,d}$ in such a way that $g\left(e^1_{i,d-1}\right) = c_i + 1$. The remaining vertices of $S_{i,d}$ are labeled so that

$$g\left(e^j_{i,d-1}\right) - g\left(e^{j-1}_{i,d-1}\right) = 1$$

for any $j \in [2, |S_{i,d}|]$. This establishes the vertex labeling of $T_i$ for an arbitrary $i \in [2, n]$. Consequently, this completes the vertex labeling $f$ of $F$. It is now obvious that $f$ is an injective function and

$$\{g(uv) : uv \in E(F)\} = [1, |E(F)|].$$

Therefore, we conclude that $\beta_s(F) < +\infty$ for any forest $F$. □

A tree can be considered as a forest with one component. Thus, the next result immediately follows from the proof of Theorem 4.1.

**Corollary 4.2** *If $T$ is a nontrivial tree, then $\beta_s(T) < +\infty$.*

The remainder of this section concerns the (strong) beta-number of two kinds of forests. We start with the forest that consists of two disjoint copies of a star. For this, let $S_m$ denote the star with $m+1$ vertices. Before presenting our next result, we require another definition here. The *neighborhood* $N(v)$ of a vertex $v$ in a graph $G$ is the set of all vertices of $G$ that are adjacent to $v$.

**Theorem 4.3** *For every two positive integers $m$ and $n$,*

$$\beta_s(S_m \cup S_n) = \beta(S_m \cup S_n) = \begin{cases} m+n+1, & \textit{if } mn \textit{ is even;} \\ m+n+2, & \textit{if } mn \textit{ is odd.} \end{cases}.$$

Proof: Let $m$ and $n$ be positive integers, and define the forest $F \cong S_m \cup S_n$ with

$$V(F) = \{x, y\} \cup \{z_i : i \in [1, m]\} \cup \{w_i : i \in [1, n]\}$$

and

$$E(F) = \{xz_i : i \in [1, m]\} \cup \{yw_i : i \in [1, n]\}.$$

First, suppose that $mn$ is even. By Lemma 1.1, we obtain that $\beta_s(F) \geq \beta(F) \geq m+n+1$. Next, we show that $\beta(F) \leq \beta_s(F) \leq m+n+1$. To do this, without loss of generality, we may assume that $m$ is even, and consider the vertex labeling $f : V(F) \to [0, m+n+1]$ such that $f(x) = 0$; $f(y) = m+n+1$;

$$f(z_i) = \begin{cases} i, & \text{if } i \in [1, m/2]; \\ n+i, & \text{if } i \in [m/2+1, m]; \end{cases}$$

and $f(w_i) = m/2 + n + 1 - i$, if $i \in [1, n]$. Notice that $f(x) = 0$;

$$\begin{aligned} \{f(z_i) : i \in [1, m/2]\} &= [1, m/2]; \\ \{f(w_i) : i \in [1, n]\} &= [m/2+1, m/2+n]; \\ \{f(z_i) : i \in [m/2+1, m]\} &= [m/2+n+1, m+n]; \end{aligned}$$

and $f(y) = m+n+1$, which implies that $f$ is a bijective function. Notice also that

$$\begin{aligned} \{|f(x) - f(z_i)| : i \in [1, m/2]\} &= [1, m/2]; \\ \{|f(y) - f(w_i)| : i \in [1, n]\} &= [m/2+1, m/2+n]; \\ \{|f(x) - f(z_i)| : i \in [m/2+1, m]\} &= [m/2+n+1, m+n]. \end{aligned}$$

Consequently,

$$\{|f(u) - f(v)| : uv \in E(F)\} = [1, |E(F)|],$$

and thus $\beta_s(F) \leq m+n+1$. Therefore, it follows from this and Lemma 1.1 that $\beta_s(F) = \beta(F) = m+n+1$ when $mn$ is even.

For the case where $mn$ is odd, assume, to the contrary, that there exists an injective function $g : V(F) \to [0, m+n+1]$ such that $\beta(F) = m+n+1$. Since both $|V(F)| = m+n+2$ and $|E(F)| = m+n$ are even, it follows that

$$\begin{aligned} |\{v \in V(F) : g(v) \text{ is even}\}| &= |\{v \in V(F) : g(v) \text{ is odd}\}| \\ &= (m+n)/2+1 \end{aligned}$$

and

$$\begin{aligned} |\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}| &= |\{uv \in E(F) : |g(u)-g(v)| \text{ is odd}\}| \\ &= (m+n)/2. \end{aligned}$$

Now, consider the following three cases according to the parity of $g(x)$ and $g(y)$.

**Case 1:** Suppose that both $g(x)$ and $g(y)$ are odd. Then we have

$$\begin{aligned} & |\{v \in N(x) \cup N(y) : g(v) \text{ is odd}\}| \\ = \; & |\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}|, \end{aligned}$$

implying that

$$|\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}| = (m+n)/2-1.$$

This contradicts the fact that

$$|\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}| = (m+n)/2.$$

**Case 2:** Suppose that both $g(x)$ and $g(y)$ are even. Then we have

$$\begin{aligned} & |\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}| \\ = \; & |\{v \in N(x) \cup N(y) : g(v) \text{ is even}\}|, \end{aligned}$$

and the value of $|\{v \in N(x) \cup N(y) : g(v) \text{ is even}\}|$ is integer and is also equal to $(m+n)/2-1$, since both $m$ and $n$ are odd. This contradicts the fact that $|\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}| = (m+n)/2$.

**Case 3:** Suppose that either $g(x)$ or $g(y)$ is even. Without loss of generality, we may assume that $g(x)$ is even and $g(y)$ is odd. Then we have

$$\begin{aligned} & |\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}| \\ = \; & |\{v \in N(x) : g(v) \text{ is even}\}| + |\{v \in N(y) : g(v) \text{ is odd}\}|. \end{aligned}$$

If we let $l = |\{v \in N(x) : g(v) \text{ is even}\}|$, then we have

$$|\{v \in N(y) : g(v) \text{ is odd}\}| = (m+n)/2-(m-l),$$

which implies that

$$|\{uv \in E(F) : |g(u)-g(v)| \text{ is even}\}| = l+(m+n)/2-(m-l).$$

The last equation together with

$$|\{uv \in E(F) : |g(u) - g(v)| \text{ is even}\}| = (m+n)/2$$

yields the equality,

$$(m+n)/2 = l + (m+n)/2 - (m-l),$$

implying that $m = 2l$. This contradicts our assumption that $m$ is odd.

Hence, it follows from the above cases and Lemma 1.1 that $\beta_s(F) \geq \beta(F) \geq m+n+2$ when $mn$ is odd.

Next, we show that $\beta(F) \leq \beta_s(F) \leq m+n+2$ when $mn$ is odd. To do this, consider the vertex labeling $h : V(F) \to [0, m+n+2]$ such that $h(x) = m+n+2$; $h(y) = 0$;

$$h(z_i) = \begin{cases} m+n+2-i, & \text{if } i \in [1, (m+1)/2]; \\ m+2-i, & \text{if } i \in [(m+1)/2+1, m]; \end{cases}$$

and $h(w_i) = (m+1)/2 + i$, if $i \in [1, n]$. Notice that $h(y) = 0$;

$$\begin{array}{lcl} \{h(z_i) : i \in [(m+1)/2+1, m]\} & = & [2, (m+1)/2]; \\ \{h(w_i) : i \in [1, n]\} & = & [(m+1)/2+1, (m+1)/2+n]; \\ \{h(z_i) : i \in [1, (m+1)/2]\} & = & [(m+1)/2+n+1, m+n+1]; \end{array}$$

and $h(x) = m+n+2$, which implies that $h$ is an injective function. Notice also that

$$\begin{array}{lcl} \{|h(x) - h(z_i)| : i \in [1, (m+1)/2]\} & = & [1, (m+1)/2]; \\ \{|h(y) - h(w_i)| : i \in [1, n]\} & = & [(m+1)/2+1, (m+1)/2+n]; \\ \{|h(x) - h(z_i)| : i \in [(m+1)/2+1, m]\} & = & [(m+1)/2+n+1, m+n]. \end{array}$$

Consequently,

$$\{|h(u) - h(v)| : uv \in E(F)\} = [1, |E(F)|],$$

and thus $\beta_s(F) \leq m+n+2$. Therefore, it follows from this and Lemma 1.1 that $\beta_s(F) = \beta(F) = m+n+2$ when $mn$ is odd. □

Our final result in this section concerns forests that are disjoint unions of paths and stars. For this, let $P_m$ denote the path with $m$ vertices.

**Theorem 4.4** *For every two integers $m$ and $n$ with $m \geq 2$ and $n \geq 1$,*

$$\beta_s(P_m \cup S_n) = \beta(P_m \cup S_n) = \begin{cases} m+n, & \text{if } m = 2 \text{ and } n \text{ is even} \\ & \text{or } m \geq 3 \text{ and } n \geq 1; \\ m+n+1, & \text{if } m = 2 \text{ and } n \text{ is odd.} \end{cases}$$

PROOF: Let $F \cong P_m \cup S_n$, where $m$ and $n$ are positive integers. For $m = 2$ or 3 and $n \geq 1$, the result directly follows from Theorem 4.3. Thus, in light of Lemma 1.1, it suffices to show that $\beta_s(F) \leq m + n$ for $m \geq 4$ and $n \geq 1$. To do this, define the forest $F$ with

$$V(F) = \{x_i : i \in [1, m]\} \cup \{y\} \cup \{z_i : i \in [1, n]\}$$

and

$$E(F) = \{x_i x_{i+1} : i \in [1, m-1]\} \cup \{y z_i : i \in [1, n]\},$$

and consider four cases for the vertex labeling $f : V(F) \to [0, m+n]$.

**Case 1:** For $m = 4k$, where $k$ is a positive integer, let $f(x_1) = 4k + n$; $f(x_3) = 4k + n - 2$;

$$f(x_j) = \begin{cases} n - 1 + 2i, & \text{if } j = 4i - 2 \text{ and } i \in [1, k]; \\ n - 2 + 2i, & \text{if } j = 4i \text{ and } i \in [1, k]; \\ 4k + n - 2 - 2i, & \text{if } j = 4i + 1 \text{ and } i \in [1, k-1]; \\ 4k + n - 1 - 2i, & \text{if } j = 4i + 3 \text{ and } i \in [1, k-1]; \end{cases}$$

$f(y) = 4k + n - 1$; and $f(z_i) = i - 1$, if $i \in [1, n]$.

**Case 2:** For $m = 4k + 1$, where $k$ is a positive integer, let $f(x_1) = 4k + n + 1$;

$$f(x_j) = \begin{cases} n - 1 + 2i, & \text{if } j = 4i - 2 \text{ and } i \in [1, k]; \\ 4k + n - 2i, & \text{if } j = 4i - 1 \text{ and } i \in [1, k]; \\ n - 2 + 2i, & \text{if } j = 4i \text{ and } i \in [1, k]; \\ 4k + n + 1 - 2i, & \text{if } j = 4i + 1 \text{ and } i \in [1, k]; \end{cases}$$

$f(y) = 4k + n$; and $f(z_i) = i - 1$, if $i \in [1, n]$.

**Case 3:** For $m = 4k + 2$, where $k$ is a positive integer, let $f(x_1) = 4k + n + 2$; $f(x_3) = 4k + n$;

$$f(x_j) = \begin{cases} n - 1 + 2i, & \text{if } j = 4i - 2 \text{ and } i \in [1, k+1]; \\ n - 2 + 2i, & \text{if } j = 4i \text{ and } i \in [1, k]; \\ 4k + n - 2i, & \text{if } j = 4i + 1 \text{ and } i \in [1, k]; \\ 4k + n + 1 - 2i, & \text{if } j = 4i + 3 \text{ and } i \in [1, k-1]; \end{cases}$$

$f(y) = 4k + n + 1$; and $f(z_i) = i - 1$, if $i \in [1, n]$.

**Case 4:** For $m = 4k + 3$, where $k$ is a positive integer, let $f(x_1) = 4k + n + 3$; $f(x_3) = 4k + n + 1$;

$$f(x_j) = \begin{cases} n - 1 + 2i, & \text{if } j = 4i - 2 \text{ and } i \in [1, k]; \\ n - 2 + 2i, & \text{if } j = 4i \text{ and } i \in [1, k]; \\ 4k + n + 1 - 2i, & \text{if } j = 4i + 1 \text{ and } i \in [1, k]; \\ 4k + n + 2 - 2i, & \text{if } j = 4i + 3 \text{ and } i \in [1, k]; \end{cases}$$

$f(x_{4k+2}) = 2k + n$; $f(y) = 4k + n + 2$; and $f(z_i) = i - 1$, if $i \in [1, n]$.

Hence, it follows from the above cases that

$$\begin{aligned} \{f(z_i) : i \in [1, n]\} &= [0, n-1]; \\ \{f(x_i) : i \in [2, m]\} &= [n, m+n-2]; \end{aligned}$$

$f(y) = m + n - 1$; and $f(x_1) = m + n$, which implies that $f$ is an injective function. Finally, notice that

$$\begin{aligned} \{|f(x_i) - f(x_{i+1})| : i \in [1, m-1]\} &= [1, m-1]; \\ \{|f(y) - f(z_i)| : i \in [1, n]\} &= [m, m+n-1]. \end{aligned}$$

This implies that

$$\{|f(u) - f(v)| : uv \in E(F)\} = [1, |E(F)|],$$

and thus $\beta_s(F) \leq m + n$, which completes the proof. □

## 5 Conclusions

We conclude this paper with some remarks on bounds for the (strong) beta-number and gracefulness of graphs, and three open problems.

For graceful graphs $G$, we have $\beta_s(G) = \beta(G) = |E(G)|$. As we have seen in Theorems 4.3 and 4.4, there are infinitely many forests $F$ for which $\beta_s(F) = \beta(F) = |V(F)| - 1$. Thus, the lower bound presented in Lemma 1.1 is sharp. However, no good upper bounds for the (strong) beta-numbers are known. This motivates us to propose the following two problems. First, we state the problem for the strong beta-number.

**Problem 5.1** Let $G$ be a graph such that $\beta_s(G) < +\infty$. Find a good upper bound for $\beta_s(G)$.

There are infinitely many graphs $G$ for which $\beta_s(G) = +\infty$ and $\beta(G) < +\infty$ (see Theorem 3.1 and Corollary 3.2). This leads us to propose the following problem.

**Problem 5.2** Let $G$ be a graph such that $\beta(G) < +\infty$. Find a good upper bound for $\beta(G)$.

As we mentioned in the introduction, the value of grac($G$) is always finite for any graph $G$. In light of Lemma 1.1, the values of $\beta_s(G)$ and $\beta(G)$ clearly provide us upper bounds for grac($G$). However, the values of $\beta_s(G)$ and $\beta(G)$ are not necessary finite as we have seen in the previous sections. This leads us to propose the following problem.

**Problem 5.3** Let $G$ be a graph. Find a good upper bound for grac($G$).

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