AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **63(1)** (2015), Pages 1–11

Fractional and circular 1-defective colorings of outerplanar graphs

Zuzana Farkasová   Roman Soták*

Institute of Mathematics
Faculty of Science, P.J. Šafárik University
Jesenná 5, 041 54 Košice
Slovak Republic
zuzana.farkasova@student.upjs.sk   roman.sotak@upjs.sk

Abstract

A graph is called *fractional* $\left(\frac{r}{s}, d\right)$*-defective colorable* if its vertices can be colored with r colors in such a way that each vertex receives s distinct colors and has at most d defects (a defect corresponds to the situation when two adjacent vertices are assigned with non-disjoint sets).

We show that each outerplanar graph having no triangle faces sharing a vertex is fractional $\left(\frac{7}{3}, 1\right)$-defective colorable; moreover, this bound is tight also in the case when the graph has no touching triangles. These results correct the claim in [W. Klostermeyer, Defective circular coloring, *Australas. J. Combin.* 26 (2002), 21–32] on circular $\left(\frac{5}{2}, 1\right)$-defective colorability of outerplanar graphs having no adjacent triangles.

Further, we show that if one allows overlapping triangles then one cannot improve on the upper bound of 3 given by the 3-colorability of outerplanar graphs.

1 Introduction

Throughout this paper, we consider connected and simple graphs; for terms undefined here, readers are referred to [3]. Let S be a nonempty set of colors and i be a positive integer. Then $\binom{S}{i}$ denotes the collection of all i-element subsets of S. Further, instead of a color set $\{a, b, c\}$ we will write abc, for short. The set $\{1, 2, \ldots, r\}$ will be denoted by $[1, r]$.

Among various generalization of proper vertex colorings, we mention here two approaches. The first one relaxes the condition to be proper by allowing, for each vertex, a fixed number of color conflicts. This gives the notion of d-improper coloring

* This work was supported by Science and Technology Assistance Agency under the contract No. APVV-0023-10, by Slovak VEGA Grant 1/0652/12, and by the Agency of the Slovak Ministry of Education for the Structural Funds of the EU under project ITMS:26220120007.

(see [9]) or (k,d)-defective coloring (see [1], [2]). The second approach relies on assigning vertices of a graph with sets of colors with the aim to minimize the ratio between total number of used colors and set size. Corresponding invariants are circular chromatic number (see [6], [10], [4]) and fractional chromatic number (see [7], [8]). Both these approaches can be combined to apply to circular/fractional defective colorings.

Definition 1.1. *([5]) A circular $\left(\frac{r}{s}, d\right)$-defective coloring of a simple graph G is a function $f\colon V(G) \to \{\{1,2,\ldots,s\},\{2,3,\ldots,s+1\},\ldots,\{r,1,\ldots,s-1\}\}$ such that each vertex u is adjacent to at most d vertices v with $f(u) \cap f(v) \neq \emptyset$.*

So, in a circular $\left(\frac{r}{s}, d\right)$-defective coloring, each vertex is assigned with s consecutive integers from $[1,r]$ (integers 1 and r are consecutive via wrap-around). By coloring any vertex of a graph with an arbitrary s distinct elements subset of $[1,r]$ we get the fractional version.

Definition 1.2. *A fractional $\left(\frac{r}{s}, d\right)$-defective coloring of a simple graph G is a function $f\colon V(G) \to \binom{[1,r]}{s}$ such that each vertex u is adjacent to at most d vertices v with $f(u) \cap f(v) \neq \emptyset$.*

The fractional d-defective chromatic number of G, denoted by $\chi_F^d(G)$, is defined by

$$\chi_F^d(G) = \inf\left\{\frac{r}{s} \colon G \quad \textit{can be fractional} \quad \left(\frac{r}{s}, d\right) - \textit{defective colored}\right\}.$$

For a coloring $f\colon V \to \binom{[1,r]}{s}$, an edge xy is *defective* if $f(x) \cap f(y) \neq \emptyset$; a vertex has d defects if it is incident with d defective edges. In Definitions 1.1 and 1.2, if $d = 1$, we refer to circular/fractional 1-defective colorings, and $s = 1, d = 0$ yields the proper vertex coloring. Of course, if a graph G cannot be fractional $\left(\frac{r}{s}, d\right)$-defective colored then it fails to be circular $\left(\frac{r}{s}, d\right)$-defective colorable, too. By Definitions 1.1 and 1.2, a circular $\left(\frac{r}{s}, d\right)$-defective coloring for a graph G is also fractional $\left(\frac{r}{s}, d\right)$-defective.

In this paper we deal with circular and fractional defective colorings of outerplanar graphs. Note that an outerplanar graph is a planar graph that can be embedded in the plane in such a way that all vertices lie on the boundary of the outer face. In the next we shall assume such embeddings for all outerplanar graphs.

It is known that, in outerplanar graphs, the fractional and circular chromatic numbers are equal — particularly, for outerplanar bipartite graphs, this common value is 2 while, for the non-bipartite ones, the common value is $2+\frac{1}{n}$ where $2n+1$ is the *odd girth* (that is, the length of the shortest odd cycle) (see e.g. [6, 4, 8]). Analogous result holds also for circular/fractional $\left(\frac{r}{s}, d\right)$-defective coloring of outerplanar graphs: the question of finding the optimal defective colorings is precisely equivalent to the question of maximizing the odd girth by removing edges (at most d edges incident to each vertex).

It has been shown (see [1]) that each outerplanar graph is properly 3-colorable, and $(2,2)$-defective colorable, too. Hence by Definitions 1.1 and 1.2, each outerplanar graph is circular/fractional $\left(\frac{3}{1}, 0\right)$-defective $\left(\left(\frac{2}{1}, 2\right)\text{-defective}\right)$ colorable. In [5], W. Klostermeyer proved that each outerplanar graph without triangles sharing an edge can be circular $\left(\frac{5}{2}, 1\right)$-defective colorable. However this result fails to hold, as we show in Section 2. In Section 3 we consider outerplanar graphs without triangles sharing a vertex and prove that each such graph can be circular/fractional $\left(\frac{7}{3}, 1\right)$-defective colorable (see Theorem 3.2) and $\frac{7}{3}$ is the tight upper bound. For purposes of proofs of these results, we introduce some particular notation. Having an outerplanar graph G with at least one chord, an inner face of G which is incident with one chord is called an *end-face.* In the other words an end-face of an outerplanar graph is adjacent with exactly one inner face. Two cycles of a graph are *adjacent* if they share a common edge, and *touching* if they share a common vertex.

2 Outerplanar graphs without adjacent triangles

In [5] (pp. 27), the following theorem was stated:

Theorem 2.1. *Let G be an outerplanar graph without adjacent triangles. Then G is circular $\left(\frac{5}{2}, 1\right)$-defective colorable.*

In this section we give a counterexample to show that the above theorem does not hold.

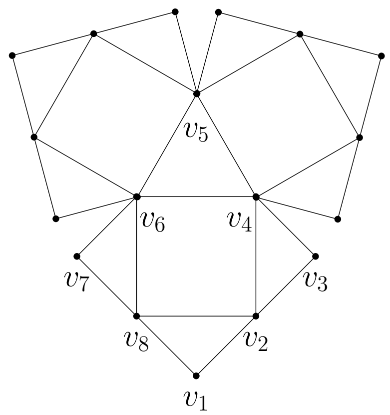

Figure 1: Graph G without adjacent triangles contradicting Theorem 2.1 of [5]

Lemma 2.2. *There exists an outerplanar graph G without adjacent triangles such that G fails to be fractional $\left(\frac{r}{s}, 1\right)$-defective colorable for all $\frac{r}{s} < 3$.*

Proof. Consider the outerplanar graph G on Figure 1 and assume that there exists a fractional $\left(\frac{r}{s}, 1\right)$-defective coloring f with $\frac{r}{s} < 3$. It follows that each triangle of G contains a unique defective edge. By the symmetry, suppose that v_4v_6 is the defective edge of the triangle $v_4v_5v_6$. This implies that v_2v_3 and v_7v_8 are defective edges in triangles $v_2v_3v_4$ and $v_6v_7v_8$. Thus, the triangle $v_1v_2v_8$ either contains no defective edge or contains a vertex with two defects, a contradiction. □

This lemma shows that Theorem 2.1 of [5] does not hold. From 3-colorability of outerplanar graphs we have that each outerplanar graph without adjacent triangles can be circular (also, fractional) $\left(\frac{3}{1}, 1\right)$-defective colorable (circular $\left(\frac{3}{1}, 0\right)$-defective colorable).

3 Outerplanar graphs without touching triangles

Before proving the main theorem on circular defective colorability of particular outerplanar graphs, we present the following auxiliary result on certain edge 3-colorings of these graphs (called 012-*colorings* in the sequel):

Lemma 3.1. *Let G be an outerplanar graph without touching triangles. Then there is a* 012*-coloring* $\varphi\colon E(G) \to \{0, 1, 2\}$ *of G such that*

1. *each vertex of G is incident with at most one nonzero edge, and*
2. *for each face α of G, the following holds:*
   (a) *If* $\deg_G(\alpha) = 3$ *then colors of edges incident with α do not form sequences* $0, 0, 0$ *or* $1, 0, 0$.
   (b) *If* $\deg_G(\alpha) = 4$ *then colors of edges incident with α do not form sequence* $2, 0, 0, 0$.
   (c) *If* $\deg_G(\alpha) = 5$ *then colors of edges incident with α do not form sequence* $0, 0, 0, 0, 0$.

Proof. First suppose that G is 2-connected and let $r = |F(G)| - 1$ be the number of inner faces in G. The result is obvious for $r = 1$ since, in this case, G is a (chordless) cycle. Let $r > 1$ and assume that the assertion holds for every graph containing fewer inner faces than G. Observe that either G contains an end-face of degree greater than three or all end-faces are triangles. In each of the cases of the subsequent analysis, we proceed in a common manner: we take a suitable subgraph of G, color it by induction and describe the extension of the so obtained coloring to the coloring of the whole graph G (the fact that the conditions (1), (2) are always satisfied can be checked easily).

Case 1: Assume that there exists an end-face α of degree greater than three. Let $E(\alpha) = \{v_1v_2, v_2v_3, \ldots, v_kv_1\} = \{e_1, e_2, \ldots, e_k\}$ where $e_k = v_kv_1$ be a chord in G. Consider a subgraph H of G induced by the vertex set $V(G) \setminus \{v_2, \ldots, v_{k-1}\}$. Note that H is outerplanar and 2-connected with fewer inner faces as G. By induction, color H and denote by φ_H its 012-coloring. We claim that φ_H can be extended to a coloring φ_G of G, i.e., to a coloring including the remaining edges $e_1, \ldots, e_{k-1}$. This is done in the following manner: put $\varphi_G(e) = \varphi_H(e)$ for each $e \in E(H)$. Whenever $k > 5$ color all edges $e_1, e_2, \ldots, e_{k-1}$ with the color 0. If $k = 5$ then color e_2 with 1 and other remaining edges with 0. Finally, if $k = 4$, color e_2 with the color 1 and edges e_1, e_3 with 0.

Case 2: Now assume that all end-faces are triangles in G. Denote by V^* the set of all vertices of G of degree two which are incident with an end-face of G. Let H be the subgraph of G induced by $V(G) \setminus V^*$. Note that H is 2-connected, too. Since G contains no touching triangles, either H is a (chordless) cycle of the length greater than three or each of its end-faces has degree greater than three. Let us consider the following two cases according to H:

Subcase 2.1: Suppose that H is a (chordless) cycle of the length $l > 3$. Next, let α be a triangle end-face in G with the edges e_1, e_2 and e_3 where e_3 be a chord. Denote by u the vertex incident with e_1 and e_2, and take a subgraph G_1 of G induced by $V(G) \setminus \{u\}$. By the inductive hypothesis, there exists a 012-coloring φ_{G_1} of G_1. Extend φ_{G_1} to a coloring φ_G of G as follows: put $\varphi_G(e) = \varphi_{G_1}(e)$ for each $e \in E(G_1)$. Consider, after this coloring, three subcases depending on $\varphi_{G_1}(e_3)$.

2.1.1: If $\varphi_{G_1}(e_3) = 2$ then color e_1 and e_2 with 0.

2.1.2: Let $\varphi_{G_1}(e_3) = 1$. Whenever $l = 4$ and the colors of edges incident with H (which is an four element cycle) create a sequence $1, 0, 2, 0$ or $l > 4$, then switch the color 1 of the chord e_3 to the color 2, and then assign the color 0 to the edges e_1, e_2. Now suppose that $l = 4$ and the colors of edges incident with H create a sequence $1, 0, 0, 0$ or $1, 0, 1, 0$ with $\varphi_{G_1}(e_3) = 1$. Then switch the color of e_1 to 0, and color e_2 with 0 and e_3 with 2.

2.1.3: Let $\varphi_{G_1}(e_3) = 0$. Denote by f_i $(i = 1, 2)$, the unique edge in $E(H) \setminus \{e_3\}$ which is adjacent to e_i. If $\varphi_{G_1}(f_i) = 0$ for some $i \in \{1, 2\}$, color e_i with 2 and e_1 with 0. Otherwise recolor f_1 with 0, f_2 with 1 and color e_1 with 2 and e_2 with 0.

Subcase 2.2: Assume that H is not a (chordless) cycle. Choose an end-face β of H (see Figure 2). Recall that each end-face of H has degree greater than three. Obviously, there exists a positive integer p such that β is adjacent with p end-triangles β_i $(i = 1, \ldots, p)$ in G. Let $E(\beta) = \{e_1, e_2, \ldots, e_k\}$ where e_k be a chord in H. For a chord $e_j \in E(\beta) \cap E(\beta_i)$, $i = 2, \ldots, p$, we denote by e_j^- and e_j^+ the edge of $E(\beta_i)$ which is adjacent with e_{j-1} and e_{j+1}, respectively. Next, cut G along e_k into two subgraphs G_1 and G_2 (both containing the chord e_k) and without loss of generality, suppose that $\beta \in F(G_2)$. By induction, color G_1. We extend φ_{G_1} to a coloring φ_G of G as follows. Put $\varphi_G(e) = \varphi_{G_1}(e)$ for each $e \in E(G_1)$. Let us examine seven subcases according to $k = \deg_G(\beta)$ and p.

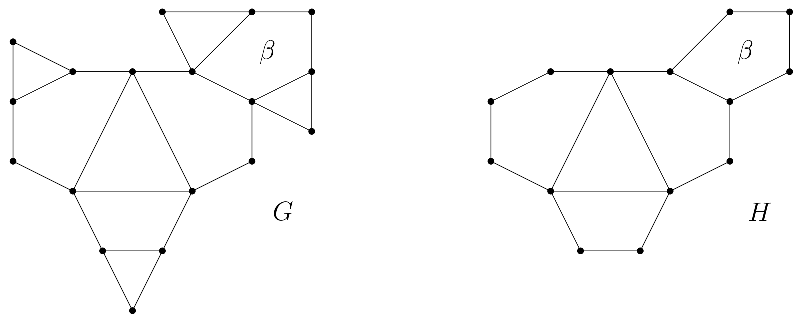

Figure 2: Construction of graph H and the choice an end-face β in the Subcase 2.2

2.2.1: $k = 4$, $p = 1$ and e_1 or e_3 is a chord in G_2. If $\varphi_G(e_4) = 2$, then recolor e_4 with 1. Now $\varphi_G(e_4) \in \{0, 1\}$ and we can color e_1^+, e_3^- (if they exist) with 2 and the remaining edges with 0.

2.2.2: $k = 4$, $p = 1$ and e_2 is a chord in G_2. If $\varphi_G(e_4) \in \{0, 1\}$ then we can color edges e_1, e_2, e_3, e_2^- with 0 and the edge e_2^+ with 2. Otherwise, $\varphi_G(e_4) = 2$ and color e_2 with 2 and other remaining edges with 0.

2.2.3: $k = 5$, $p = 1$ and e_1 is a chord. Then we color edges e_3 and e_1^+ with 2 and the remaining edges with 0. The case where $k = 5$, $p = 1$ and e_4 is a chord, is symmetric with this subcase.

2.2.4: $k = 5$, $p = 1$ and e_i is a chord for some $i \in \{2, 3\}$. Then we color e_i with 2 and other remaining edges with 0.

2.2.5: $k = 5$, $p = 2$ and e_1, e_4 are chords in G. Then add a vertex w and edges v_1w, v_5w into outer face of G_1. Denote this outerplanar graph by G_1^*. By the structure of G, we can see that G_1^* contains no touching triangles. According to the induction hypothesis, there exists a 012-coloring $\varphi_{G_1^*}$ of G_1^*. Put $\varphi_G(e) = \varphi_{G_1^*}(e)$ for each $e \in E(G_1) = E(G_1^*) \setminus \{v_1w, v_5w\}$. If $\varphi_G(e_5) = 2$ then color edges e_1^+, e_4^- with 2 and the remaining edges of G with 0. Now, if $\varphi_G(e_5) = 0$ then either $\varphi_{G_1^*}(v_1w) = 2$ or $\varphi_{G_1^*}(v_5w) = 2$, say $\varphi_{G_1^*}(v_1w) = 2$, then color e_1, e_4^- with 2 and the remaining edges of G with 0.

2.2.6: $k = 5$, $p = 2$ and edges e_1, e_3 are chords in G. Then we color e_3, e_1^+ with the color 2 and the remaining edges with 0.

2.2.7: $k > 5$. Then color either e_i^+ or e_i^- with 2 ($i = 1, \ldots, p$) and other edges with 0. Note that if v_1 (v_k) has defect in G_1 and the edge e_1^- (e_{k-1}^+) exists then e_1^- (e_{k-1}^+) has to be colored with 0.

Finally, assume that G has a cut-vertex. If G is a tree then we can color all its edges with 0. Now suppose that G is not a tree, and let us construct a 2-connected graph H from G as follows:
Take an arbitrary cut-vertex v. Denote by u, w adjacent vertices of v such that u, v, w are consecutive vertices on a boundary trail of outer face. Then add new vertex v^* into outer face and add new edges uv^*, wv^*. The resulting graph is outerplanar without touching triangles. Moreover, the number of cut-vertices decreased by one. By repeating this process we construct 2-connected graph H with required properties (see Figure 3). Now we can use induction for 2-connected graphs, so, there exists

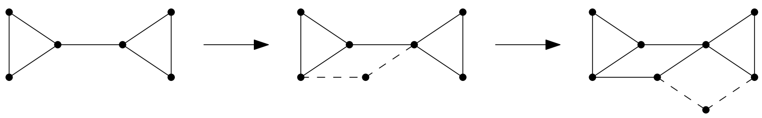

Figure 3: A construction of 2-connected outerplanar graph by adding vertices and edges

a 012-coloring for H. Note that, in this moment, no 5-cycle (not only faces) of H is colored with only 0. Moreover, each 3-cycle and 4-cycle in H bounds a face. Therefore, this edge 012-coloring is a suitable 012-coloring of G, too. $\square$

Theorem 3.2. *Let G be an outerplanar graph without touching triangles. Then G is circular $\left(\frac{7}{3}, 1\right)$-defective colorable.*

Proof. First, if G is a tree, then G is bipartite, hence, there exists $(2, 0)$-defective coloring and therefore circular $\left(\frac{2}{1}, 0\right)$-defective coloring, too. If we replace the color set $\{1\}$ by $\{1, 2, 3\}$ and $\{2\}$ by $\{4, 5, 6\}$, we get circular $\left(\frac{6}{3}, 0\right)$-defective coloring. Hence, for any tree there exists a circular $\left(\frac{6}{3}, 1\right)$-defective coloring which is circular $\left(\frac{7}{3}, 1\right)$-defective coloring, too. So suppose that G is not a tree. Whenever G contains a cut-vertex then, as in the proof of Lemma 3.1, we can construct from G a 2-connected outerplanar graph H. Obviously, if there exists a desired vertex coloring of H then also G is circular $\left(\frac{7}{3},1\right)$-defective colorable. Thus, in the proof, it suffices to consider only 2-connected outerplanar graphs. Moreover, if G contains an inner face of degree greater than five then we can construct from G (by adding new chords to G) a new outerplanar graph H such that H has no touching triangles and the degree of each inner face is less than six. Clearly, if H is circular $\left(\frac{7}{3},1\right)$-defective colorable then so is G. Therefore, it is sufficient to prove the theorem for such outerplanar graphs (without touching triangles) which are 2-connected and have no inner k-face for $k > 5$. Now, let G be a graph with the above properties and r be the number of inner faces in G.

We show by induction on r that G can be circular $\left(\frac{7}{3},1\right)$-defective colorable. Suppose that φ is the edge 012-coloring of G (as defined in Lemma 3.1). Using the coloring φ, we show that there exists a required vertex coloring f of G such that $|f(u) \cap f(v)| = \varphi(e)$ for each $e = uv \in E(G)$. The result is obvious for

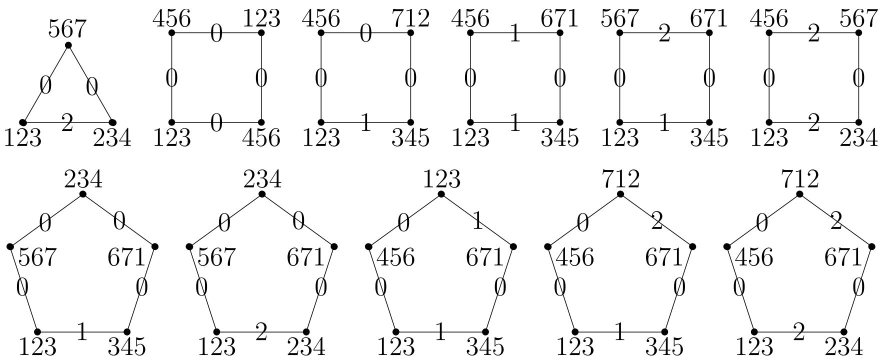

Figure 4: The 012-colorings and derived circular $\left(\frac{7}{3},1\right)$-defective colorings of the graphs C_3, C_4 and C_5.

$r = 1$ (see Figure 4). Next, suppose $r > 1$. Let α be an end-face of G and let $k = \deg_G \alpha \in \{3,4,5\}$, $E(\alpha) = \{v_1v_2, v_2v_3, \dots, v_kv_1\}$ where v_1v_k be the unique chord of α. Let us consider two cases depending on k.

Case 1: Suppose that $k \in \{4,5\}$ and consider a subgraph H of G induced by vertex set $V(G) \setminus \{v_2, \dots, v_{k-1}\}$. Obviously, H is 2-connected and has $r-1$ inner faces. By induction, color H and denote f_H its circular $\left(\frac{7}{3},1\right)$-defective coloring. We show that f_H can be extended to a coloring f_G of G as follows: put $f_G(v) := f_H(v)$ for each $v \in V(H)$. While coloring the vertices $v_2, \dots, v_{k-1}$, let us consider the following subcases:

1.1: If $k = 4$ and $\varphi(v_1v_4) = 0$, then without loss of generality suppose that $f_G(v_1) = 123$ and $f_G(v_4) = 456$. If edges v_1v_2, v_2v_3 and v_3v_4 are colored with 0 in the edge coloring φ (i.e., $\varphi(e_1) = \varphi(e_2) = \varphi(e_3) = 0$), then we color v_2 with 456 and v_3 with 123. Other cases how to color v_2, v_3 depending on $\varphi(e_1), \varphi(e_2), \varphi(e_3)$, are illustrated in Table 1. The cases 001 and 102 are symmetric to 100 and 201, therefore we omit it.

1.2: If $k = 4$ and $\varphi(v_1v_4) = 1$, then without loss of generality suppose that

$\varphi(e_1)\varphi(e_2)\varphi(e_3)$	100	010	101	102	202
$f_G(v_2), f_G(v_3)$	$345, 712$	$567, 712$	$345, 671$	$671, 345$	$712, 345$

Table 1: Subcase 1

$f_G(v_1) = 123$ and $f_G(v_4) = 345$. Since $\varphi(v_1v_4) = 1$, edges v_1v_2 and v_3v_4 have the color 0. Thus we get three cases described in Table 2.

$\varphi(e_1)\varphi(e_2)\varphi(e_3)$	000	010	020
$f_G(v_2), f_G(v_3)$	$456, 712$	$456, 671$	$567, 671$

Table 2: Subcase 2

1.3: If $k = 4$ and $\varphi(v_1v_4) = 2$, then suppose that $f_G(v_1) = 123$ and $f_G(v_4) = 234$. Obviously, $\varphi(v_1v_2) = \varphi(v_3v_4) = 0$ and we get two cases illustrated in Table 3.

$\varphi(e_1)\varphi(e_2)\varphi(e_3)$	010	020
$f_G(v_2), f_G(v_3)$	$456, 671$	$456, 567$

Table 3: Subcase 3

1.4: If $k = 5$ and $\varphi(v_1v_5) = 0$, then suppose that $f_G(v_1) = 123$ and $f_G(v_5) = 456$. Now we color vertices v_2, v_3, v_4 depending on colors of edges e_1, e_2, e_3, e_4 (up to symmetry) as given in Table 4.

1.5: If $k = 5$ and $\varphi(v_1v_5) = 1$, then suppose that $f_G(v_1) = 123$ and $f_G(v_5) = 345$. Since $\varphi(v_1v_5) = 1$, the edges v_1v_2, v_4v_5 are colored with 0. The coloring of vertices v_2, v_3, v_4, depending on colors of edges e_1, e_2, e_3, e_4, (up to symmetry) is given by Table 5.

1.6: Let $k = 5$ and $\varphi(v_1v_5) = 2$. Without loss of generality suppose that $f_G(v_1) = 123$ and $f_G(v_5) = 234$. Observe that we can color remaining vertices as in the above Subcase 1.5.

Case 2: Assume that $k = 3$ and consider a subgraph H of G induced by $V(G) \setminus \{v_2\}$. By induction, color H and extend, without loss of generality, its circular $\left(\frac{7}{3}, 1\right)$-defective coloring f_H to G as follows: put $f_G(v) = f_H(v)$ for each $v \in V(H)$. First let $\varphi(v_1v_3) = 2$ ($\varphi(v_1v_2) = \varphi(v_2v_3) = 0$). Without loss of generality suppose that $f_G(v_1) = 123$ and $f_G(v_3) = 234$. Then we can color v_2 with 567. Otherwise, $\varphi(v_1v_3) = 0$ and suppose that $f_G(v_1) = 123$, $f_G(v_3) = 456$. Note that then $\varphi(v_iv_{i+1}) = 2$ for some $i \in \{1, 2\}$. Therefore, color v_2 either with 712 or 567 depending on $\varphi(v_1v_2)$. □

The following assertion holds immediately, as a circular $\left(\frac{r}{s}, d\right)$-defective coloring is fractional, too.

Corollary 3.3. *Let $G = (V, E)$ be an outerplanar graph without touching triangles. Then G is fractional $\left(\frac{7}{3}, 1\right)$-defective colorable.*

$\varphi(e_1)\varphi(e_2)\varphi(e_3)\varphi(e_4)$	1000	0100	2000
$f_G(v_2), f_G(v_3), f_G(v_4)$	$671, 345, 712$	$567, 345, 712$	$712, 345, 712$
$\varphi(e_1)\varphi(e_2)\varphi(e_3)\varphi(e_4)$	0200	1010	1020
$f_G(v_2), f_G(v_3), f_G(v_4)$	$456, 345, 712$	$345, 671, 123$	$345, 671, 712$
$\varphi(e_1)\varphi(e_2)\varphi(e_3)\varphi(e_4)$	1001	1002	2010
$f_G(v_2), f_G(v_3), f_G(v_4)$	$345, 671, 234$	$345, 671, 345$	$234, 567, 712$
$\varphi(e_1)\varphi(e_2)\varphi(e_3)\varphi(e_4)$	2020	2002	
$f_G(v_2), f_G(v_3), f_G(v_4)$	$234, 671, 712$	$234, 671, 345$	

Table 4: Subcase 4

$\varphi(e_1)\varphi(e_2)\varphi(e_3)\varphi(e_4)$	0000	0100	0200
$f_G(v_2), f_G(v_3), f_G(v_4)$	$567, 234, 671$	$456, 234, 671$	$456, 345, 671$

Table 5: Subcase 5

Lemma 3.4. *There exists an outerplanar graph G without touching triangles such that $\chi_F^1(G) = \frac{7}{3}$.*

Proof. We prove that $\chi_F^1(G) = \frac{7}{3}$ for the graph G in Figure 5. According to Theorem 3.3, $\chi_F^1(G) \leq \frac{7}{3}$. By way of contradiction, suppose that $\chi_F^1(G) = \frac{r}{s} < \frac{7}{3}$ and that f is the corresponding coloring of G. Since $\frac{r}{s} < 3$, the triangle $u_1u_2u_3$ contains a defective edge, say u_1u_2 (thus, $f(u_1) \cap f(u_2) \neq \emptyset$). Next, $|f(u_1) \cup f(u_2)| \leq r - s$, because $f(u_3) \cap (f(u_1) \cup f(u_2)) = \emptyset$. Hence,

$$\begin{aligned} |f(u_1) \cap f(u_2)| &= |f(u_1)| + |f(u_2)| - |f(u_1) \cup f(u_2)| \\ &\geq s + s - (r - s) = 3s - r > 0. \end{aligned}$$

To color the vertices v_1, v_2, we can use only colors from the set $S_1 = \{1, 2, \ldots, r\} \setminus \{f(u_1) \cap f(u_2)\}$ because the vertices v_1, v_2 have defects neither with u_1 nor u_2. Therefore $|S_1| \leq r - (3s - r) = 2r - 3s$ and

$$\begin{aligned} |f(v_1) \cap f(v_2)| &= |f(v_1)| + |f(v_2)| - |f(v_1) \cup f(v_2)| \\ &\geq s + s - (2r - 3s) = 5s - 2r > 0. \end{aligned}$$

The last inequality follows from the fact that $\frac{r}{s} < \frac{5}{2}$. So, v_1 has defect with v_2. Similarly, to color the vertices w_1, w_2 we can use only colors from the set $S_2 = \{1, 2, \ldots, r\} \setminus \{f(v_1) \cap f(v_2)\}$. Thus $|S_2| \leq r - (5s - 2r) = 3r - 5s$ and, since $\frac{r}{s} < \frac{7}{3}$, we obtain

$$\begin{aligned} |f(w_1) \cap f(w_2)| &= |f(w_1)| + |f(w_2)| - |f(w_1) \cup f(w_2)| \\ &\geq s + s - (3r - 5s) = 7s - 3r > 0, \end{aligned}$$

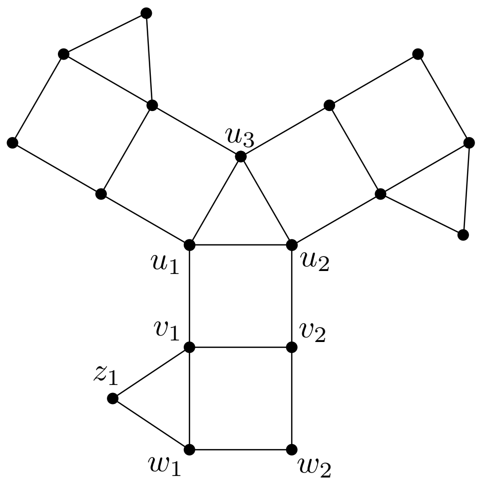

Figure 5: The outerplanar graph G with $\chi_F^1(G) = \frac{7}{3}$.

i.e., w_1w_2 is defective. Observe, one of the edges of the triangle $v_1w_1z_1$ is defective because $\frac{r}{s} < 3$. Then some vertex in $\{v_1, w_1\}$ has two defects, which is a contradiction. □

Acknowledgements

The authors are indebted to one of the anonymous referees for pointing out the equality on fractional and circular chromatic number for outerplanar graphs.

References

[1] L.J. Cowen, R.H. Cowen and D.R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, *J. Graph Theory* **10** (1986), 187–195.

[2] L.J. Cowen, W. Goddard and C.E. Jesurum, Defective coloring revisited, *J. Graph Theory* **24** (1997), 205–219.

[3] R. Diestel, *Graph Theory*, 4th ed., Graduate Texts in Mathematics 173, Springer, Heidelberg, 2010.

[4] A. Kemnitz and P. Wellman, Circular chromatic numbers of certain planar graphs, *Congr. Numer.* **169** (2004), 199–209.

[5] W. Klostermeyer, Defective circular coloring, *Australas. J. Combin.* **26** (2002), 21–32.

[6] W. Klostermeyer and C.Q. Zhang, $(2 + \epsilon)$-coloring of planar graphs with large odd-girth, *J. Graph Theory* **33** (2000), 109–119.

[7] E.R. Scheinerman and D.H. Ullman, *Fractional Graph Theory*, John Wiley & Sons, 1997.

[8] G. Wang, Girth and fractional chromatic number of outer planar graphs, *J. Shandong Univ. Nat. Sci.* **38** (2003), 61–64,69.

[9] D.R. Woodall, Improper colourings of graphs, in: R. Nelson and R.J. Wilson (eds.), *Graph Colourings*, Pitman Research Notes in Math. 218, Longman, Harlow, Essex, 1990, 45–63.

[10] X. Zhu, Circular chromatic number: a survey, *Discrete Math.* **229** (2001), 371–410.

(Received 5 Dec 2013; revised 14 Oct 2014, 23 June 2015)