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Edge contraction and cop-win critical graphs*

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Abstract

The problem is to determine the number of 'cops' needed to capture a 'robber' where the game is played with perfect information, the different sides moving alternately. The cops capture the robber when one of them occupies the same vertex as the robber at any time in the game. A cop-win graph is one in which one cop can always capture the robber. A graph is cop-win edge-critical with respect to edge contraction (CECC) when the original graph is not cop-win, but the contraction of any edge results in a cop-win graph. In this paper, classes of CECC graphs are determined, and k-regular CECC are characterized for $k \leq 4$.

1 Introduction

The game of cops and robber is a pursuit-evasion game in which a set of k cops ($k \geq 1$) is trying to capture a single robber on a graph. We will assume it is played on a finite, connected, reflexive graph. The game begins with each of the k cops choosing a vertex to occupy. The robber then chooses a vertex. The cops and robber then alternate moves. On the cops' move, each cop can slide along an edge to an adjacent vertex or stay at his current location. The robber can do the same on his move. The cops win if at least one cop occupies the same vertex as the robber at the same time. When this occurs, we say that the robber has been captured. The robber wins if he can avoid capture indefinitely. Note that more than one cop may

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occupy the same vertex at any given time, and both the cops and robber know the others position at all times. The *copnumber* of a graph G, denoted $c(G)$, is the least number of cops required to guarantee a win for the cops. If $c(G) \leq k$, then we say that G is k-cop-win. When $k = 1$, we say that G is cop-win.

Cop-win graphs were completely characterized by Nowakowski and Winkler [9] and independently by Quillot [10]. The notion of copnumber was introduced by Aigner and Frommie [1]. In the time since these early publications, the problem has become increasingly well known, with many papers written on the subject. The reader is directed to the book *The Game of Cops and Robbers on Graphs* by Bonato and Nowakowski [3] for additional background.

The most important open problem on the subject is Meyniel's conjecture [6] which states that $c(n) = O(\sqrt{n})$, where $c(n)$ is the maximum of $c(G)$ taken over all connected graphs on n vertices. The best known bound is due to Lu and Peng [8] (found independently in Frieze, Krivelevich and Loh [7]). In this paper, they also verify Meyniel's conjecture for graphs with diameter two and bipartite graphs of diameter three. Among other results on this conjecture, it has also been shown by Bollob*á*s, Kun and Leader [2] that Meyniel's conjecture essentially holds for sparse random graphs.

In this paper, we examine further the class of *cop-win edge critical graphs*, which were introduced by Clarke, Fitzpatrick, Hill and Nowakowski in [4]. A graph is considered to be cop-win edge critical if the graph itself is not cop-win, but an operation on any edge of the graph results in a cop-win graph. In this paper, we are interested in the operation of edge contraction. Specifically, if a graph G is not cop-win, but the contraction of *any* edge of G results in a cop-win graph, then we say that G is *Cop-win Edge Critical with respect to Contraction* (CECC). Similar graphs have been considered under the operations of edge addition [4, 5] and edge deletion [5]. In [4], the authors give a characterization of all regular graphs that are Cop-win Edge Critical subject to Edge Addition (CECA). In [5], all planar CECA and CECD (Cop-win Edge Critical with respect to Edge Deletion) graphs are characterized. We direct the reader to these papers for further information, as there are many commonalities between CECC and CECA graphs.

We let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We write $u \sim_G v$ if $u \neq v$ and u is adjacent to v in G. We also refer to v as a *neighbour* of u. For any $u \in V(G)$, let $N_G(u) = \{v \in V(G) : u \sim_G v\}$, and $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of u, denoted $\deg_G(u)$, is defined as $\deg_G(u) = |N_G(u)|$. (Note that although the graphs are reflexive, and therefore, have a loop at each vertex, the loop does not contribute to the degree of the vertex.) We let $\delta(G) = \min\{\deg_G(u) : u \in V(G)\}$ and $\Delta(G) = \max\{\deg_G(u) : u \in V(G)\}$. A graph G is said to be *k-regular* if $\delta(G) = \Delta(G) = k$. For two distinct vertices u and v in $V(G)$, we say that u is a *corner* that is *dominated* by v if $N_G[u] \subseteq N_G[v]$. The complement of G is the graph $\overline{G}$ such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G) \text{ and } u \not\sim_G v\}$. Given graphs F and H such that $V(F) \cap V(H) = \emptyset$, the *graph join* of F and H, denoted $F \vee H$, is a graph such that $V(F \vee H) = V(F) \cup V(H)$ and $E(F \vee H) = E(F) \cup E(H) \cup \{xy : x \in V(F) \text{ and } y \in V(H)\}$.

Given a graph G, by *contracting* an edge $uv \in E(G)$ we can obtain a new graph

G/uv. We can think of this as associating the vertices u and v to create a new vertex u_v. Formally, we can define G/uv as follows: $V(G/uv) = (V(G) \cup \{u_v\}) \setminus \{u, v\}$, and

$$N_{G/uv}(x) = \begin{cases} (N_G(u) \cup N_G(v)) \setminus \{u, v\} & : x = u_v \\ N_G(x) & : x \neq u_v \text{ and } N_G(x) \cap \{u, v\} = \emptyset \\ (N_G(x) \setminus \{u, v\}) \cup \{u_v\} & : x \neq u_v \text{ and } N_G(x) \cap \{u, v\} \neq \emptyset \end{cases}$$

The set $E(G/uv)$ follows from the neighbourhoods described above, together with the fact that G/uv is reflexive and has no multi-edges.

It has been observed that every cop-win graph contains a corner ([9], [10]). This observation, together with the following lemma, provides a structural characterization of cop-win graphs.

Lemma 1 *[9] Suppose x is a corner in a graph G. Then G is cop-win if and only if $G - \{x\}$ is cop-win.*

Therefore, the set of cop-win graphs is exactly the set of *dismantlable* graphs. A graph is *dismantlable* if it is either an isolated vertex, or its vertices can be ordered $v_1, v_2, \ldots, v_n$ so that for each $i = 1, \ldots, n-1$, there is a $j > i$ such that $v_i \sim v_j$ and $N_{G_i}[v_i] \subseteq N_{G_i}[v_j]$ where $G_1 = G$ and $G_i = G - \{v_1, \ldots, v_{i-1}\}$ for each $i = 2, \ldots, n$. In other words, for each $i = 1, \ldots, n-1$, v_i is a *corner* in G_i *dominated by* v_j. We refer to the ordering of the vertices as a *cop-win ordering*. Therefore, a graph G is cop-win if and only there exists some cop-win ordering of its vertices.

Note that the removal of corners can also be expressed in terms of contracting edges; if u is a corner in G with dominating vertex v, then $G - \{u\} \cong G/uv$. Therefore, the dismantling scheme for a cop-win graph can be expressed as a sequence of edge contractions. It is also true that for any connected graph G, there is a series of edge contractions that will result in a cop-win graph (simply contract all of the edges to obtain a single vertex). In Section 2, we find an upper bound on the copnumber of the original graph in terms of the number of edge contractions required to obtain a cop-win graph.

In Section 3, we describe some classes of graphs that are CECC. We also show how the join operation can be used to generate new CECC graphs from known CECC graphs. In Section 4 we characterize graphs that are CECC and have at least one of the following properties: (1) minimum degree 2, (2) minimum degree 3, (3) bipartite, (4) 4-regular or (5) the complement of a 2-regular graph.

2 Edge Contraction and Edge Critical Graphs

Let $\mathbb{S}$ be the set of all graphs that are CECC. Keeping in mind that the removal of a corner is equivalent to an edge contraction, it is straightforward to show that no graph in $\mathbb{S}$ has a corner. In the next lemma,we verify that fact, as well as the location of the corner created via edge contraction.

Lemma 2 *If $G \in \mathbb{S}$, then G has no corners, and $\delta(G) \geq 2$. Furthermore, for any $uv \in E(G)$, every corner of G/uv is in $N_{G/uv}[u_v]$.*

Proof. If G is in $\mathbb{S}$, then G/uv is cop-win for any $uv \in E(G)$. If u were a corner in G, and v its dominating vertex, then $G - \{u\} = G/uv$ and $G - \{u\}$ is cop-win. By Lemma 1, this implies G is cop-win , which is a contradiction. Therefore, G has no corner. Since any vertex of degree one is a corner, it follows that $\delta(G) \geq 2$.

For some $G \in \mathbb{S}$ and $uv \in E(G)$, let $G' = G/uv$. Then $c(G') = 1$ and G' has at least one corner. Suppose x is that corner, but $x \notin N_{G'}[u_v]$. Then x is dominated in G' by a vertex $y \neq u_v$. That is, $N_{G'}[x] \subseteq N_{G'}[y]$. Since $N_{G'}[x] = N_G[x]$, this implies that x is also dominated by y in G, which is a contradiction. □

We now show that every CCEC graph has copnumber 2. In fact, we prove a stronger result: if a series of k edge contractions results in a cop-win graph, then the original graph has copnumber at most $k + 1$. Specifically, we consider edges $e_1, e_2, \ldots, e_k$ in graphs $G_1, \ldots, G_{k-1}$, respectively, where $G_1 = G$, $G_{i+1} = G_i/e_i$ for $i = 1, \ldots, k-1$, and G_k is cop-win.

We note that the contraction of e_i in G_i to obtain graph G_{i+1} is associated with a graph homomorphism. Specifically, for graph G and edge uv in $E(G)$ (assume $u \neq v$), the associated homomorphism would be $f : G \to G/uv$ where

$$f(x) = \begin{cases} x & : x \notin \{u, v\} \\ u_v & : x \in \{u, v\} \end{cases}$$

Lemma 3 *If G is a graph such that a series of k edge contractions yields a cop-win graph G', then $c(G) \leq k + 1$.*

Proof. Assume we perform the minimum number of edge contractions required to obtain a cop-win graph, G', from original graph G. Assume k edge contractions were required, and let $K = \{e_1, e_2, \ldots, e_k\}$ be the set of contracted edges, listed in the order in which they were contracted. The series of contractions gives a corresponding series of graphs, $G_1, G_2 \ldots, G_{k+1}$ such that $G_1 = G$, $G_{i+1} = G_i/e_i$ for $i = 1, \ldots k$. Note that $G_{k+1} = G'$.

Let $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(G') = \{u_1, u_2, ..., u_l\}$. There exist graph homomorphisms $f_1, f_2, ..., f_k$ where, for each $i = 1, \ldots, k$, $f_i : G_i \longrightarrow G_{i+1}$ is defined according to the contraction of edge e_i. Let $f = f_k \circ f_{k-1} \circ ... \circ f_2 \circ f_1$.

Let G_K be the subgraph of G such that $E(G_K) = K$ and $V(G_K)$ consists of the endpoints of K. Note that if we consider any vertex $x \in V(G')$ then $\{v \in V(G) | f(v) = x\}$ are exactly the vertices in some component of G_K. Let T_x denote that component of G_K.

For each $e_i \in K$, place a cop on one endpoint of that edge. Since G_K is a forest, each cop can choose a different vertex to occupy. Place an additional cop on some unoccupied vertex, x, of G. We have $k + 1$ cops on G. Simultaneously, a cop C is placed on vertex $f(x)$ on G'. As play begins on G, the robber R chooses a vertex. We will associate R with the vertex he occupies.

We now play simultaneous games on G and G'. Since G' is cop-win, a single cop, C, will capture the robber's image, $f(R)$. We will use C's strategy in G' to form a winning strategy for the $k+1$ cops on G. To do this, suppose that it is the cops' turn in both games. On G', C will move from u to v as part of its winning strategy. Assume, by induction, that there are cops on every vertex of T_u, and there are cops on all except one vertex of T_v. If $u \sim_{G'} v$, then, in G, there is some vertex u' in T_u that is adjacent to a vertex v' in T_v. Move a cop from u' to v', and redistribute the cops currently in T_v so that each vertex of T_v is occupied by some cop.

When C occupies the same vertex as $f(R)$ in G', then there are cops on all possible preimages of $f(R)$ in G, and therefore, R is captured in G. Thus $c(G) \leq k+1$. □

Corollary 4 *If $G \in \mathbb{S}$, then $c(G) = 2$.*

In Lemma 3, we perform edge contractions so that the resulting graph is cop-win. Can a similar result of $c(G) \leq k+\ell$ be obtained if the resulting graph has copnumber ℓ ? In general, the proof for Lemma 3 does not hold if we replace $c(G') = 1$ with $c(G') = \ell$. The best we could do, if we followed a similar argument, is $c(G) \leq 2\ell$. However, we can improve the result if the cops move in such a way that no two cops ever occupy the same vertex at the same time. Let's refer to this method of play as *non-overlapping.*

Corollary 5 *Suppose G is a graph such that a series of k edge contractions yields graph G'. If ℓ cops can always capture the robber on G' using non-overlapping play, then $c(G) \leq k+\ell$.*

3 Classes of CECC Graphs

We begin by considering the graph join operation and how it can be used to generate CECC graphs. We will then give the characterization of CECC graphs that result from the graph join. We conclude this section by proving that the complements of cycles on $3k+1$ ($k \geq 2$) vertices are CECC and, since they have connected complements, cannot be constructed using the graph join operation. Before beginning, we note that if a graph G has a vertex, u, such that $N[u] = V(G)$, then G is obviously cop-win. We call such a vertex, u, a *universal vertex* in G.

Lemma 6 *For any graphs F and H, $c(F \vee H) = 1$ if and only if $c(F) = 1$ or $c(H) = 1$.*

Proof. ($\Rightarrow$) Assume $c(F \vee H) = 1$ but $c(F) \geq 2$ and $c(H) \geq 2$. Since $c(F) \geq 2$ and $c(H) \geq 2$, one cop cannot win in G if her strategy only involves vertices in the copy of H, or only vertices from F. Without loss of generality, assume that the single cop, C, starts the game on some vertex x in the copy of H. We know that at some point C must move to some vertex y in the copy of F or else the robber can remain safe indefinitely. Once C moves to y, R can move to some safe vertex w in the copy of F and remain safe as long as C is on some z in the copy of F since $c(F) \geq 2$. We see that this can continue for infinitely many rounds so $c(F \vee H) \geq 2$, which contradicts our assumption. So $c(F) = 1$ or $c(H) = 1$.

($\Leftarrow$) Without loss of generality, assume $c(F) = 1$. Now consider $F \vee H$ for some graph H. Since $c(F) = 1$, there is a cop-win strategy in F. We let one cop play according to a cop-win strategy for F on the induced F in $F \vee H$. If at any time the robber moves to the induced H he is captured immediately since $x \sim y$ for all $x \in V(F)$ and $y \in V(H)$. This new strategy will therefore work as a cop-win strategy in $F \vee H$, and so $c(F \vee H) = 1$. □

We now present the result that allows us to generate classes of CECC graphs using the graph join operation.

Theorem 7 *Let* $\mathbb{K} = \{\overline{K_l} : l \geq 2\}$. *The graph* $F \vee H$ *is in* $\mathbb{S}$ *if any only if* $H \in \mathbb{S} \cup \mathbb{K}$ *and* $F \in \mathbb{S} \cup \mathbb{K}$.

Proof. ($\Rightarrow$) Let $G = F \vee H$. Assume $G \in \mathbb{S}$, and either $H \notin \mathbb{S} \cup \mathbb{K}$ or $F \notin \mathbb{S} \cup \mathbb{K}$. If F or H is cop-win, then G will be cop-win by Lemma 6. So we suppose $c(F) \geq 2$ and $c(H) \geq 2$. By assumption, $c(H/uv) \geq 2$ or $c(F/uv) \geq 2$ for some uv in their respective edge sets. Assume without loss of generality $c(H/uv) \geq 2$. We note that uv must exist or else $H \in \mathbb{K}$, contradicting our assumption, or H is the singleton graph, but H cannot be cop-win. Since $G \in \mathbb{S}$, one cop can win in G/uv for any $uv \in E(H)$, but $c(H/uv) \geq 2$ and $c(F) \geq 2$ so $c(G/uv) \neq 1$ by Lemma 6. Therefore $G \notin \mathbb{S}$ which contradicts our assumption. A similar argument shows that if $uv \in E(F)$ and $c(F/uv) \geq 2$, then $c(G/uv) \neq 1$ and so $G \notin \mathbb{S}$. Therefore if $G \in \mathbb{S}$, then $H \in \mathbb{S} \cup \mathbb{K}$ *and* $F \in \mathbb{S} \cup \mathbb{K}$.

($\Leftarrow$) Assume $H \in \mathbb{S} \cup \mathbb{K}$ and $F \in \mathbb{S} \cup \mathbb{K}$. Since $c(I) \geq 2$ for all $I \in \mathbb{S} \cup \mathbb{K}$, $c(G) \neq 1$ by Lemma 6. We now consider G/uv for some $uv \in E(G)$. If uv is an edge between the induced H and the induced F, then the vertex u_v is a universal vertex in G/uv so $c(G/uv) = 1$. If H and $F \in \mathbb{K}$, then we have no more edges to consider and the result holds so we suppose without loss of generality $H \in \mathbb{S}$. Let $uv \in E(H)$. Since $H \in \mathbb{S}$, $c(H/uv) = 1$ and $c(G/uv) = 1$ by Lemma 6. □

Corollary 8 *If* $G \cong K_{\ell_1,\ell_2,\ldots,\ell_m}$ *where* $m \geq 2$ *and* $\ell_i \geq 2$ *for each* $i = 1, 2, \ldots m$, *then* $G \in \mathbb{S}$.

Proof. If $G \cong K_{\ell_1,\ell_2,\ldots,\ell_m}$ where $m \geq 2$ and $\ell_i \geq 2$ for each $i = 1, 2, \ldots m$, then $G \cong \overline{K_{\ell_1}} \vee \overline{K_{\ell_2}} \vee \cdots \vee \overline{K_{\ell_m}}$. □

We now show that there exists a family of CECC graphs that cannot be formed by the join operation as they have connected complements.

Theorem 9 *The complement of the cycle on* j *vertices,* $\overline{C}_j$, *is CECC if and only if* $j = 3k + 1$ *for some* $k \in \mathbb{N} \setminus \{1\}$.

Proof. Let $V(\overline{C_j}) = \{0, 1, 2, ..., j-1\}$ and $i \sim l$ for $l \neq i$ and $l \neq i \pm 1 \mod j$. The reader is directed to Lemma 24 of [4] for a proof that $c(\overline{C_j}) \geq 2$. We now examine graphs resulting from a single edge contraction. Without loss of generality, assume the edge $0i$ is contracted, where $2 \leq i \leq j-2$.

Let $G = \overline{C_j}$ to simplify notation. If $0i \in E(G)$ such that $i \neq 2$ and $i \neq j-2$, then $N_{G/0i}[0_i] = (N_G[0] \cup N_G[i] \cup \{0_i\}) \setminus \{0, i\} = V(G/0i)$. Therefore, 0_i is a universal vertex in $G/0i$, and $c(G/0i) = 1$. It now remains to show that $G/02$ and $G/0(j-2)$ are cop-win.

We claim that $5, 8, 11, ..., 3k-1$ are corners in the graphs $G/02, G/02 - \{5\}, G/02 - \{5, 8\}, \ldots, G/02 - \{5, 8, ..., 3k-4\}$ respectively where $3k-1$ is the largest number congruent to 2 mod 3 of all vertex labels. To prove this claim, we proceed by induction on n to show that 5 is a corner in $G/02$ and $3n-1$ is a corner in the subgraph $G/02 - \{5, \ldots, 3k-4\}$ for all $n \geq 3$. We see that $N_{G/02}[5] = V(G/02) \setminus \{4, 6\} \subset N_{G/02}[3] = V(G/02) \setminus \{4\}$, therefore 5 is a corner and can be the first vertex added to the cop-win ordering. Now suppose that the vertices with labels congruent to 2 modulo 3 between 5 and $3l-4$ are the first $l-2$ vertices in the cop-win ordering for some $l \geq 3$ and let H be the subgraph of $G/02$ obtained from removing these vertices. We see that $N_H[3l-1] = N_H[3l-3] \setminus \{3l\}$ by our induction hypothesis and therefore $3l-1$ is a corner dominated by $3l-3$ and can be added to the cop-win ordering. It follows by induction that $5, 8, 11, ..., 3k-1$ are the first $l-1$ entries in some cop-win ordering for $G/02$. We now consider two cases to show that $j = 3k+1$.

Case 1 $j = 3k+1$ for some $k \geq 2$. Let $V(\overline{C_{3k+1}}) = \{0, 1, 2, ..., 3k\}$ and $l \sim m$ for all $m \neq l \pm 1 \mod 3k+1$. We claim that $3k$ is a universal vertex in $G/02 \setminus \{5, 8, 11, ..., 3k-1\}$, Let $F = G/02 - \{5, 8, 11, \ldots, 3k-1\}$. Since $N_{G/02}[3k] = V(G/02) \setminus \{3k-1\}$, and since $3k-1 \notin V(F)$, it follows that $N_F[3k] = V(F)$, making $3k$ universal in F and $c(G/02) = 1$. Therefore $G \in \mathbb{S}$.

Case 2 $j = 3k$ or $j = 3k+2$ for some $k \geq 2$. Let $I = G/02 - \{5, 8, 11, \ldots, 3k-1\}$. Now every vertex in I has degree $|V(I)| - 2$ and therefore $I \cong K_{2,2,..,2}$ so $I \in \mathbb{S}$ by Corollary 8. Therefore $c(I) = 2$. Therefore $G \notin \mathbb{S}$.

Therefore $j = 3k+1$ for some $k \geq 2$. □

4 CECC Graphs and Degree

In this section we characterize CECC graphs with minimum degree at most 3 and present a structural characterization for CECC graphs with minimum degree at least 4.

We begin with a characterization of CECC graphs with minimum degree 2, which requires verifying that no CECC graph has a cut-edge. We will also make use of the following lemma from [4].

Lemma 10 *[4] If G contains an induced cycle C with length at least* 4 *and* $\deg_G(u) = 2$ *for some $u \in C$, then $c(G) \geq 2$.*

Lemma 11 *If $G \in \mathbb{S}$, then G has no cut-edge.*

Proof. Suppose $G \in \mathbb{S}$. By Lemma 2, $\delta(G) \geq 2$. Suppose $uv \in E(G)$ is a cut-edge. Let H_1 and H_2 be components of $G - uv$ such that $u \in V(H_1)$ and $v \in V(H_2)$. Since uv is a cut-edge in G, u_v is a cut-vertex in G/uv . Therefore, u_v is not a corner in G/uv. Since G/uv is cop-win, it has at least one corner, w. By Lemma 2, $w \in N_{G/uv}(u_v)$. Without loss of generality, assume that $w \in V(H_2)$. Since w has no neighbour in $V(G/uv) \cap V(H_1)$, w must be dominated in G/uv by either u_v or a vertex in $V(H_2)$. If w is dominated by u_v in G/uv, then w is dominated by v in G, which is a contradiction. However, if w is dominated in G/uv by a vertex in $V(H_2)$, then w is dominated by that same vertex in G, which is also a contradiction. Thus, G has no cut-edge. □

We can now give the characterization of all CECC graphs with minimum degree 2.

Theorem 12 *Suppose $\delta(G) = 2$. Then $G \in \mathbb{S}$ if and only if $G \cong K_{2,m}$ for some $m \geq 2$.*

Proof. By Corollary 8, we know that $K_{2,m} \in \mathbb{S}$.

Assume $G \in \mathbb{S}$ and $\delta(G) = 2$. There is a vertex $w \in V(G)$ such that $\deg(w) = 2$. Let x and y be vertices of G such that $x \neq y$, $x \sim w$ and $y \sim w$. If $x \sim y$, then w is a corner in G, which contradicts Lemma 2. Therefore, $x \nsim y$.

Since $G \in \mathbb{S}$, it has no cut-edge, and every edge is on a cycle. Furthermore, by Lemma 10, the contraction of any edge other than wx or wy must result in a 3-cycle containing w. Therefore, we must have $y \sim_{G/uv} x$ for any $uv \in E(G) \setminus \{wx, wy\}$. It follows that every edge of G has either x or y as an endpoint. Hence, $G \cong K_{2,m}$. □

Now we work toward a characterization of graphs that are CECC with minimum degree 3. Two important result needed for this are (1) every bipartite CECC graph is complete bipartite and (2) if a CECC graph is not complete bipartite, then every edge of that graph lies on some 3-cycle. These results are given in Lemma 16 and Theorem 17. We begin by finding the maximum girth of a CECC graph.

We note that a corner has the property that it either has degree one, or it lies on a 3-cycle.

Lemma 13 *If $G \in \mathbb{S}$, then* $\text{girth}(G) \leq 4$.

Proof. Assume $\text{girth}(G) \geq 5$ and $G \in \mathbb{S}$. Consider an edge uv in G. Let $G' = G/uv$. Some vertex in $N_{G'}[u_v]$ is a corner in G'. Suppose u_v is a corner in G'. Since $N_{G'}(u_v) = N_G(u) \cup N_G(v) \setminus \{u, v\}$ and u and v have no common neighbours, $|N_{G'}(u_v)| = |N_G(u)| + |N_G(v)| - 2$. Since $\delta(G) \geq 2$, it follows that $|N_{G'}(u_v)| \geq 2$. Therefore, u_v must lie on a 3-cycle in G'. It follows that the edge uv must lie on a 4-cycle in G, which is a contradiction. Therefore, some $x \in N_{G'}(u_v)$ must be a corner in G'.

Without loss of generality, assume $x \sim_G u$. Since girth$(G) \geq 5$, $x \not\sim_G v$. Therefore, $N_{G'}(x) = (N_G(x) \setminus \{u\}) \cup \{u_v\}$. It follows that x has at least two neighbours in G'. Hence, x lies on a 3-cycle. Furthermore, this 3-cycle must also contain u_v. It follows that the edge uv is on a 4-cycle in G, which is a contradiction. Hence, girth$(G) \leq 4$. □

Lemma 14 *If G is a cop-win graph, then every edge of G that is not a cut-edge lies on a 3-cycle.*

Proof. Suppose G is cop-win, and there is an edge e such that e is not a cut-edge. It follows that e lies on a cycle. Suppose the length of the shortest cycle of G containing e is ℓ. Since G is cop-win, there is a cop-win ordering $v_1, v_2, \ldots, v_n$, along with associated graphs $G_1, \ldots\ G_n$. Let G_i be the last graph in the sequence for which e appears on an induced cycle of length ℓ. Let C be such a cycle. It follows that vertex v_i (a corner in G_i) is also on C.

Suppose $\ell \geq 4$. Since C is an induced cycle, v_i is not dominated by any vertex on C. Therefore, v_i is dominated by a vertex, u that is not on C. It follows that u is adjacent to the two neighbours of v_i on C. Call these neighbours x and y, respectively.

If v_i is an endpoint of e, then e would either be on the cycle xv_iux or the cycle yv_iuy which contradicts the fact that $\ell \geq 4$. So, we may assume that v_i is not an endpoint of e. However, this means the cycle formed by replacing xv_iy on C with xvy also contains e and has length ℓ. This cycle appears in G_{i+1}, which contradicts our choice of G_i. It follows that e lies on some 3-cycle. □

Corollary 15 *If $G \in \mathbb{S}$, then for any edge $e \in E(G)$, e is either on some 3-cycle in G, or e lies on a common 4-cycle with every other edge in G.*

Proof. Suppose $G \in \mathbb{S}$ and there is an edge e that lies on no 3-cycle. Let $e' \in E(G)$ be any edge such that $e' \neq e$. Let $G' = G/e'$. By Lemma 14, e is either on a 3-cycle in G' or is a cut-edge of G'. In the latter case, e would also be a cut-edge in G which, by Lemma 11, is a contradiction. So, it follows that e is on a 3-cycle in G'. Since e is on no 3-cycle of G, it follows that e was on a common 4-cycle with e'. The result follows. □

Lemma 16 *Suppose $G \in \mathbb{S}$. The graph G is bipartite if and only if $G \cong K_{m,n}$ with $m, n \geq 2$.*

Proof. Suppose $G \cong K_{m,n}$ with $m, n \geq 2$. Since $\delta(G) \geq 2$, and G has no 3-cycles, there are no corners in G. Therefore, G is not cop-win. Furthermore, for any non-loop edge uv in G, u_v is a universal vertex in G/uv. Therefore, G/uv is cop-win, and $G \in \mathbb{S}$.

Suppose $G \in \mathbb{S}$, G is bipartite and has bipartition (X, Y). Recall that G has no corner, so $\delta(G) \geq 2$. Furthermore, if $\delta(G) = 2$, then by Theorem 12, G is complete bipartite. Therefore, we may assume that $\delta(G) \geq 3$. Consider any vertices x and y

such that $x \in X$ and $y \in Y$. Since $\deg(x) \geq 3$, and $\deg(y) \geq 3$, x has a neighbour in $N(x) \setminus \{y\}$ and y has a neighbour in $N(y) \setminus \{x\}$. Let u and v be those neighbours, respectively. Then we have edge xu and yv in G. Since G is bipartite, it contains no 3-cycle, and by Corollary 15, xu and yv lie on a common 4-cycle. Since G is bipartite, it follows that $x \sim y$ and $u \sim v$. Since x and y were any vertices in X and Y, respectively, it follows that G is complete bipartite. □

Theorem 17 *Suppose G is a graph such that some edge of G is not on a 3-cycle. Then $G \in \mathbb{S}$ if and only if $G \cong K_{m,n}$ with $m, n \geq 2$.*

Proof. Suppose xy is an edge of G such that xy is on no 3-cycle. Then xy is on a 4-cycle with every other edge of G. Let $N(x) \setminus \{y\} = \{x_1, x_2, \ldots, x_k\}$ and $N(y) \setminus \{x\} = \{y_1, y_2, \ldots y_\ell\}$.

We claim that $N(x) \cup N(y) = V(G)$. To verify this, suppose $e \in E(G)$ is an edge that is not incident with either x or y. Since e is on a common 4-cycle with xy, it follows that e has exactly one end point in each of $N(x) \setminus \{y\}$ and $N(y) \setminus \{x\}$. Since every edge has both of its endpoints in $N(x) \cup N(y)$, it follows that every vertex is in $N(x) \cup N(y)$.

Next, we claim that $N(x)$ and $N(y)$ are independent sets. Suppose this is not the case. Since xy lies on no 3-cycle, $N(x) \cap N(y) = \emptyset$. Then there must be an edge between vertices in the set $\{x_1, x_2, \ldots, x_k\}$ or between vertices in the set $\{y_1, y_2, \ldots, y_k\}$. Without loss of generality, assume $k \geq 2$ and $x_1 \sim x_2$. The edge x_1x_2 does not lie on a common 4-cycle with xy since neither x_1 nor x_2 is adjacent to y. It follows that G is bipartite, and by Lemma 16, G is complete bipartite. □

We are now ready to give the characterization of CECC graphs with minimum degree 3.

Theorem 18 *Suppose $G \in \mathbb{S}$. If $\delta(G) = 3$, then $G \cong K_{3,m}$ for some $m \geq 3$.*

Proof. Suppose G is not complete bipartite and $\delta(G) = 3$. Then for some $t \in V(G)$, $\deg(t) = 3$. Let $N(t) = \{u, v, w\}$. By Theorem 17, every edge of G is on a 3-cycle. Then at least two of uv, vw and uw must be in $E(G)$. However, this would mean that t is a corner in G which is a contradiction. □

Now, we move onto the characterization of 4-regular CECC graphs. In order to prove the characterization, we need to verify that for any such graph G, and for any $e \in E(G)$, G/e has no cut edge.

Consider a graph $G \in \mathbb{S}$. We saw in the proof of Lemma 14 that if a cut-edge results from contracting an edge uv in G, then uv is on a 3-cycle in G. Furthermore, the third vertex of the 3-cycle, w, is a cut-vertex in G. Since G/uv is cop-win and only the neighbourhood of w is altered in the component containing w in $G - \{u, v\}$, it follows that this component must be cop-win with w as its unique corner.

Lemma 19 *If G is cop-win and has a unique corner, then $\Delta(G) > 4$.*

Proof. Suppose G is a cop-win graph of minimum order that satisfies $\Delta(G) \leq 4$ and has exactly one corner. Note that since G has a corner, $|V(G)| \geq 4$. Let v_1 be the unique corner in G. Let $G_2 = G - \{v_1\}$. Note that any corner in G_2 is in $N_G(v_1)$ in G, otherwise it would also be a corner in G. Furthermore, any corner in G_2 is not dominated by a vertex in $N_G(v_1)$. Again, this would imply that it is also a corner in G. It is also true that in any graph, a corner of degree two is dominated by both of its neighbours.

Case 1 Suppose $\deg(v_1) \leq 2$. By the choice of G, it follows that G_2 has at least two corners, v_2 and v_3. Furthermore, both v_2 and v_3 are adjacent to v_1 in G. Therefore $\deg(v_1) = 2$, and $v_2 \sim v_3$ since v_1 is dominated by both v_2 and v_3 in G.

Since v_2 is a corner in G_2, but not G, v_2 is dominated by a vertex w in G_2 where v_2, v_3 and w are all distinct. Again, since v_2 is not a corner dominated by v_3 in G, it follows that v_2 has a neighbour u_2, such that $u_2 \not\sim v_3$. Similarly, v_3 has a neighbour u_3 such that $u_3 \not\sim v_2$. Since $\Delta(G) \leq 4$, it follows that $N_G(v_2) = \{v_1, v_3, u_2, w\}$ and $N_G(v_3) = \{v_1, v_2, u_3, w\}$. Finally, since $v_2 \sim_{G_2} v_3$, they must each be dominated by a common neighbour in G_2. The only such vertex is w and it follows that $N_G(w) = \{v_2, v_3, u_2, u_3\}$. Therefore, the graph in Figure 1 is a subgraph of G.

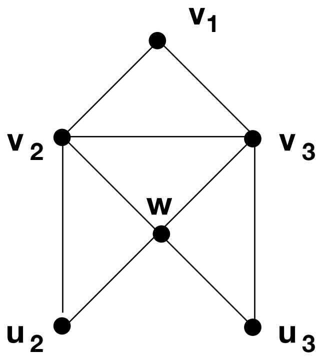


Figure 1: Resulting Subgraph of G in Case 1 of Lemma 19

Now, v_2 is a corner in G_2 and v_3 is a corner in $G_2 - \{v_2\}$. Let $G_3 = G - \{v_1, v_2\}$ and $G_4 = G - \{v_1, v_2, v_3\}$. It follows that G_4 is cop-win with $\Delta(G_4) \leq 4$. Since v_2 and v_3 were the only corners in G_2, all corners of G_4 are in the set $\{u_2, u_3, w\}$. Furthermore, at least two of u_2, u_3 and w are corners in G_4. Otherwise G_4 has a unique corner which contradicts the choice of G.

Case 1a Suppose w is a corner in G_4. Then w is dominated by u_2 or u_3 in G_4 and $u_2 \sim u_3$. Let $G_5 = G_4 - \{w\}$. Since G_5 is cop-win with at least two vertices, it has at least two corners. This means u_2 and u_3 are both corners in G_5. Since $\Delta(G) = 4$, it follows that each of u_2 and u_3 has degree at most two in G_5. Furthermore, neither can have degree one, since this would imply that they were corners dominated by w in G. Hence, u_2 and u_3 each has degree two in G_5. Since $u_2 \sim_{G_5} u_3$, u_2 is dominated in G_5 by a vertex u that is a common neighbour of u_2 and u_3 in G_5. Therefore, the graph in Figure 2 is a subgraph of G.

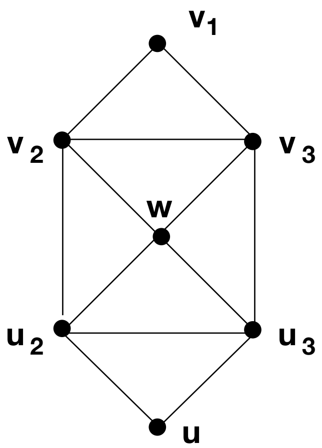


Figure 2: Resulting Subgraph of G in Case 1a of Lemma 19

Let $G_6 = G_5 - \{u_2\}$ and $G_7 = G_6 - \{u_3\}$. Since G_7 is cop-win and u is the only vertex from $V(G_7)$ that is adjacent to some vertex from $\{v_1, v_2, v_3, u_2, u_3, w\}$ in G, it follows that $V(G_7) = \{u\}$. Otherwise, u would be a unique corner in G_7. However, this means G is the graph in Figure 2, and both u and v_1 are corners in G, which is a contradiction.

Case 1b Suppose u_2 and u_3 are the only corners in G_4. Recall that the graph in Figure 1 is a subgraph of G. Since w is not a corner, it follows that $u_2 \not\sim u_3$. Since $w \sim u_2$ it follows that u_2 is dominated by a vertex in $N_{G_4}[w] - \{u_2\} = \{w, u_3\}$. Since $u_3 \not\sim v_2$, u_2 must be dominated by w in G_4. However, this implies u_2 is a corner in G, which is a contradiction.

Case 2 Suppose $\deg_G(v_1) = 3$ and $N_G(v_1) = \{v_2, v_3, v_4\}$. Since v_1 must be dominated by one of v_2, v_3, or v_4, it follows that there must be at least two edges in the subgraph of G induced by $\{v_2, v_3, v_4\}$.

Case 2a Suppose $\{v_2, v_3, v_4\}$ induces a complete graph. Without loss of generality, assume v_2 and v_3 are corners in G_2. If v_2 is dominated by either v_3 or v_4 in G_2, this would imply that v_2 is a corner in G. Therefore, v_2 is dominated by a vertex w, distinct from $v_1, \ldots, v_4$. Then w is adjacent to each of v_2, v_3 and v_4, and the graph in Figure 3 is a subgraph of G. However, this implies that v_2 and v_4 are also corners in G, dominated by v_3.

Case 2b Suppose $\{v_2, v_3, v_4\}$ induces a graph with exactly 2 edges. Without loss of generality, suppose $v_3 \sim v_2$, $v_3 \sim v_4$ and $v_2 \nsim v_4$.

Case 2b(i) Suppose v_2 and v_4 are corners in G_2. Since they are not corners in G, each is dominated by a vertex other than v_3. Say v_2 is dominated by w and v_4 is dominated by x. Since both x and y are adjacent to v_3, and $\Delta(G) \leq 4$, it follows that $w = x$ and the graph in Figure 4 is a subgraph of G.

Since none of v_2, v_4 or w is a corner in G, it follows that each has a neighbour that is

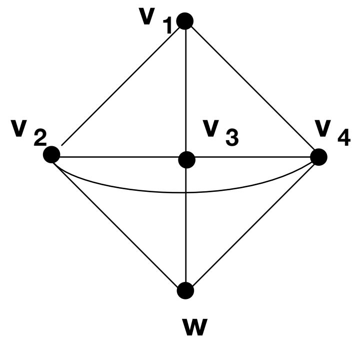


Figure 3: Resulting Subgraph of G in Case 2a of Lemma 19

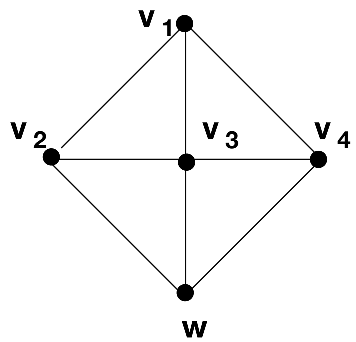


Figure 4: Resulting Subgraph of G in Case 2b of Lemma 19

not adjacent to v_3. However, each such neighbour must be adjacent to w. Therefore, for some vertex, z, $N_G(v_2) = N_G(v_4) = \{v_1, v_3, w, z\}$ and $N_G(w) = \{v_2, v_3, v_4, z\}$.

Since v_1, v_2, v_3, v_4, w can be chosen as the first five vertices in a cop-win ordering of G, $G_6 = G - \{v_1, v_2, v_3, v_4, w\}$ is cop-win. Then G_6 is either a single vertex, or has at least two corners. If G_6 is a single vertex, z, then G is the graph in Figure 5, and z is a second corner in G. However, the only vertex in G_6 that is adjacent to some vertex in $\{v_1, v_2, v_3, v_4, w\}$ is z. This implies z can be the only corner in G_6. In either case, we have a contradiction.

Case 2b(ii) Suppose exactly one of v_2 and v_4 is a corner in G_2. Without loss of generality, we may assume that v_2 and v_3 are corners in G_2. Then v_3 is dominated by a vertex w such that $w \sim v_2$, $w \sim v_3$ and $w \sim v_4$. Therefore, the graph in Figure 4 is a subgraph of G. Since v_2 is not dominated by v_3, there is a vertex x such that $x \sim v_2$ but $x \not\sim v_3$. It follows that $N_G(v_2) = \{v_1, v_3, w, x\}$ and v_2 is dominated by w in G_2. Therefore, $N_G(w) = \{v_2, v_3, v_4, x\}$. Now, since v_4 is not dominated by w in G_2, there is a vertex y such that $y \sim v_4$, but $y \not\sim w$.

Now, there is a cop-win ordering of G whose first three vertices are v_1, v_2, v_3. Let $G_4 = G - \{v_1, v_2, v_3\}$. Then G_4 is a cop-win graph. However, v_4 has degree two in G_4, and by Lemma 10, v_4 does not lie on any cycle of length at least four in G_4. Since

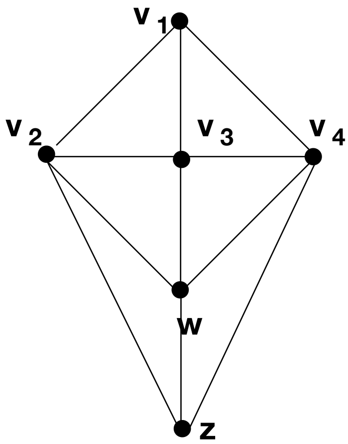


Figure 5: Resulting Subgraph of G in Case 2b(i) of Lemma 19

$y \not\sim w$, it follows that v_4 does not lie on any cycle in G_4. Therefore, G_4 has a vertex ℓ of degree one. Since ℓ is a corner of G_4, it must be adjacent to one of v_1, v_2, v_3 in G. Considering the neighbourhoods of these vertices, ℓ must be in $\{v_4, x, w\}$. However, we know that w and v_4 have two neighbours in G_4. Therefore, $\ell = x$ and $N_{G_4}(x) = \{w\}$. However, this implies x has degree two in G, and is dominated by its neighbours, v_2 and w.

Case 3 If $\deg(v_1) = 4$ and v_2 dominates v_1, then $N[v_1] = N[v_2]$ since $\Delta(G) \leq 4$. Therefore v_2 is also a corner in G which contradicts the uniqueness of v_1.

□

Lemma 20 *If $G \in \mathbb{S}$ and $\Delta(G) \leq 4$, then, for any $uv \in E(G)$, G/uv has no cut-edge.*

Proof. Suppose $G \in \mathbb{S}$, $\Delta(G) \leq 4$ and G/uv has a cut-edge for some $uv \in E(G)$. By Lemma 11, G had no cut-edge. It follows that the cut-edge e in G/uv must be incident with u_v. Let w be the other endpoint of e and let H_1 and H_2 be components of $G/uv - e$ such that $u_v \in V(H_1)$ and $w \in V(H_2)$.

If w is adjacent to exactly one of u or v in G, then G would also have a cut-edge. Therefore, $w \sim_G u$ and $w \sim_G v$. It follows that w is not the only vertex in H_2. Otherwise w would be a corner in G. Similarly, u_v is not the only vertex in H_1. Also note that for any vertex $x \in V(G/uv) - \{u_v\}$, $\deg_{G/uv}(x) \leq \deg_G(x)$ and $\deg_{G/uv}(u_v) = \deg_G(u) + \deg_G(v) - 3 \leq 5$.

Since $G \in \mathbb{S}$, G/uv must be cop-win and therefore must have at least one corner. By Lemma 2, any corners must belong to $N_{G/uv}[u_v]$. It follows that neither u_v nor w can be that corner since both vertices have neighbours in each of H_1 and H_2. Since $N_{G/uv}(u_v) - \{w\} \subseteq V(H_1)$, it follows that every corner in G/uv lies in $V(H_1) - \{u_v\}$. Furthermore, in any cop-win ordering of G/uv, every vertex in $V(H_1)$ appears before any vertex in $V(H_2)$. Since no vertex in $V(H_1)$ is a corner in G, it follows that the first corner of $V(H_1)$ in any cop-win ordering of G/uv is u_v. Therefore, u_v is the only corner in H_1. However, since u_v has degree at most five in G/uv, $\Delta(H_1) \leq 4$. This contradicts Lemma 19. □

Let H be any graph isomorphic to the graph in Figure 2.

Lemma 21 *If $G \in \mathbb{S}$ and G is 4-regular, then G does not have H as a subgraph.*

Proof. Suppose H is a subgraph of G. Assume the vertices of G are labelled as in Figure 2. Then v_1 and u denote the two vertices of degree two in H. Since G is 4-regular, we know that v_1 and u each has at least one neighbour in G that is not in $V(H)$.

If $G - V(H)$ is disconnected, then v_1 is a cut-vertex in G. It follows that G/v_2v_3 has a cut edge, namely v_1s where $s = v_{2v_3}$. This contradicts Lemma 20. Therefore, $G - V(H)$ is connected. It follows that there is a path from v_1 to u in G that is internally disjoint from $V(H)$. Let P be the shortest such path. (It may be the case that P is the edge v_1u.)

The graph G/v_2v_3 is cop-win, and vertices w and u_2 can appear as the first two vertices in a cop-win ordering (dominated by s and u_3, respectively). Let $G_3 = G/v_2v_3 - \{w, u_2\}$. Note that G_3 is a cop-win graph in which s has degree two. By Lemma 10, s can not lie on any induced cycle of length at least four in G_3. However the path v_1su_3u together P is a cycle of length at least four. Furthermore, we know this is an induced cycle since G is 4-regular. This is a contradiction. □

Theorem 22 *If $G \in \mathbb{S}$ and G is 4-regular, then $G \cong K_{4,4}$, $G \cong \overline{K_2} \vee C_4$ or $G \cong \overline{C_7}$*

Proof. Suppose $G \in \mathbb{S}$ and G is 4-regular. If G is bipartite, then G is complete bipartite by Lemma 16 and therefore $G \cong K_{4,4}$. We may, therefore, assume that G is not bipartite. By Theorem 17, every edge of G is on a 3-cycle. In other words, every pair of adjacent vertices has a common neighbour. Furthermore, no pair of adjacent vertices x and y in G can have 3 common neighbours, since this would imply $N[x] = N[y]$ and G has at least two corners, contradicting Lemma 2. We now consider the remaining two cases:

Case 1 Suppose there exists an edge $uv \in E(G)$ such that u and v have exactly 2 common neighbours. Let w_1 and w_2 be the common neighbours of u and v. Since G is 4-regular, u has one more neighbour, u_1, and v has one more neighbour, v_1, such that $u_1 \neq v_1$. Since uu_1 must be on a 3-cycle, it follows that $u_1 \sim w_1$ or $u_1 \sim w_2$. Similarly, $v_1 \sim w_1$ or $v_1 \sim w_2$. There are various cases that arise from these two previous statements. We now consider two subcases: 1a and 1b, to which all others are analogous.

Case 1a Suppose $u_1 \sim w_1$ and $v_1 \sim w_1$. Then the graph in Figure 6 is a subgraph of G. We first show that u_1, v_1, and w_2 cannot be pairwise non-adjacent in G. Consider the graph G/uw_1. Since $G \in \mathbb{S}$, G/uw_1 must be cop-win. So u_{w_1}, u_1, v_1, v, or w_2 must be a corner in G/uw_1 by Lemma 2. We see that v is a corner in G/uw_1 dominated by

u_{w_1}. We remove v from G/uw_1 to obtain the graph F. Since F is cop-win there must be at least one corner in F. Futhermore, any corner in F must belong to $N_F[u_{w_1}]$, otherwise that vertex would also be a corner in G, which contradicts $G \in \mathbb{S}$. By examining each of the possible corners and dominating vertices in F, along with the knowledge that $N_F[u_{w_1}] = \{u_1, v_1, w_2\}$, it is straightforward to see that the subgraph induced on $\{u_1, v_1, w_2\}$ has at least one edge.

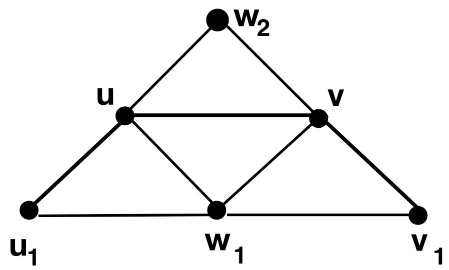
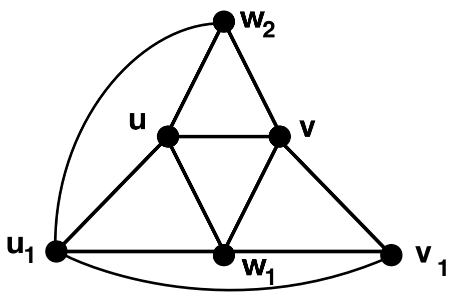


Figure 6: Subgraph assumed to exist in G for Case 1a of Theorem 22

As seen in Figure 6, we can assume $u_1 \sim v_1$ without loss of generality. Since G is 4-regular, there is a vertex x such that $N_G(u_1) = \{u, w_1, v_1, x\}$. Since u_1 and x have a common neighbour, either $x = w_2$, or $x \neq w_2$ and $x \sim v_1$. In the second case, G has a subgraph isomorphic to the graph in Figure 2 which contradicts Lemma 21. Therefore, $x = w_2$, and the graph to the left in Figure 7 is a subgraph of G. Since G is 4-regular, there is a vertex y such that $N_G(w_2) = \{u, v, u_1, y\}$. Since w_2 and y have a common neighbour it must be the case that $y = v_1$. Therefore, G is the graph to the right in Figure 7 and $G \cong C_4 \vee \overline{K_2}$.



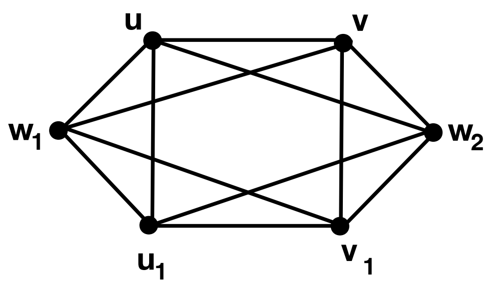


Figure 7: Resulting Subgraph of G (left) and the graph G itself (right) in Case 1a of Theorem 22

Case 1b Suppose $u_1 \sim w_1$ and $v_1 \sim w_2$. We may assume that $u_1 \not\sim w_2$ and $v_1 \not\sim w_1$, since this would result in a configuration analogous to Case 1a. We note that $w_1 \not\sim w_2$, otherwise w_1 is a corner in G dominated by u. Let x and y be vertices such that $N_G(w_1) = \{u, v, u_1, x\}$ and $N_G(w_2) = \{u, v, v_1, y\}$. Since xw_1 lies on a 3-cycle in G, $x \sim u_1$. Similarly, $y \sim v_1$. It may be the case that $x = y$, otherwise all the vertices are distinct. The resulting subgraph of G appears as the graph to left in Figure 8, where it is drawn with x and y as distinct vertices.

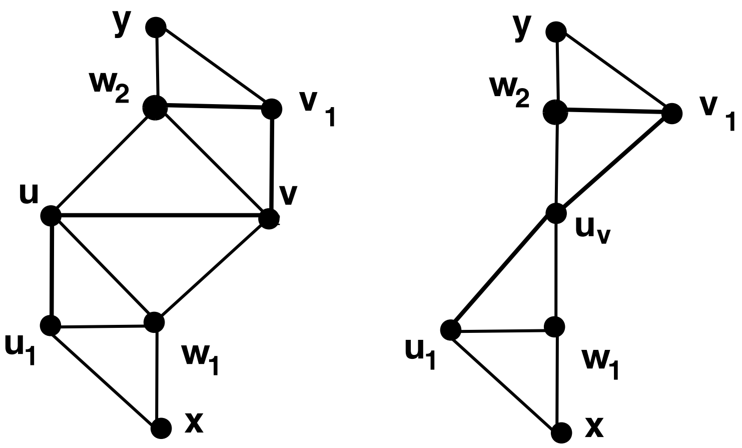


Figure 8: Subgraphs of G (left) and G/uv in Case 1b of Theorem 22

Assume that $x \neq y$. Since the graph to the left in Figure 8 is a subgraph of G, it is straightforward to see that $u_1 \not\sim v_1$. Otherwise, the edge u_1v_1 would not be on a 3-cycle, which is a contradiction.

Let $G_1 = G/uv$. We note that the graph to the right in Figure 8 is a subgraph of G_1. In G_1, w_1 is a corner dominated by u_1 and w_2 is a corner dominated by v_1. Let $G_3 = G_1 - \{w_1, w_2\}$. Since G_1 is cop-win, G_3 is cop-win. Furthermore, we note that u_v has exactly two neighbours, u_1 and v_1. Since $u_1 \not\sim v_1$, it follows from Lemma 14 that u_vu_1 and u_vv_1 are both cut-edges in G_3.

Let W be the component of $G_3 - u_vv_1$ containing u_v (and subsequently, u_1). Since G is 4-regular, it follows that $\deg_W u_v = 1$, $\deg_W u_1 = 3$ and $\deg_W x = 3$, while all other vertex in W have degree four. This is impossible, since it implies that W has an odd number of vertices of odd degree. Therefore, it must be the case that $x = y$.

By associating x and y in the graph appearing to the left in Figure 8, we obtain the required subgraph of G. Call this subgraph F. Since G is 4-regular, there is some edge e incident with u_1 that does not appear in F. Since every edge of G is on a 3-cycle, and every neighbour of u_1 in F has degree four in F, it follows that e must have v_1 as its other endpoint. Hence, G is the graph F with the added edge u_1v_1. It is straightforward to verify that $G \cong \overline{C_7}$.

Case 2 For every $uv \in E(G)$, u and v have exactly one common neighbour. Let $w \in V(G)$ be the common neighbour of u and v. Let $N_G(u) = \{v, w, u_1, u_2\}$, $N_G(v) = \{u, w, v_1, v_2\}$ and $N_G(w) = \{u, v, w_1, w_2\}$. Since no two adjacent vertices have two common neighbours, it follows that the vertices in $\{u_1, u_2, v_1, v_2, w_1, w_2\}$ are all distinct.

We know that $G_1 = G/uv$ is cop-win, and has a corner in $N_{G_1}[u_v]$. Since u_v has degree five in G_1, there is no vertex of high enough degree in G_1 to dominate it. Since $N_{G_1} = \{u_v, w_1, w_2\}$ and u_v is adjacent to neighbour w_1, nor w_2, w is not a corner in G_1. Therefore, at least one of u_1, u_2, v_1 or v_2 is a corner in G_1. Without loss of generality, assume u_1 is a corner. Since $u_1 \not\sim v$ and u_1 is not a corner in G, it must be the case that u_1 is dominated by u_v, v_1 or v_2 in G_1.

If u_1 is dominated by v_1, then $N_G[u_1] - \{u\} \subseteq N_G[v_1]$. This means u_1 and v_1 have two common neighbours in G, which is a contradiction. If u_1 is dominated by u_v, then $N_{G_1}[u_1] \subseteq \{u_v, u_1, u_2, v_1, v_2\}$. Since $u_1 \not\sim w$ and G is 4-regular, $N_G[u_1] = \{u, u_2, v_1, v_2\}$. However, this means v_1 and v_2 have both v and u_1 as a common neighbour. This is also a contradiction.

Therefore, if G is 4-regular and $G \in \mathbb{S}$, then $G \cong K_{4,4}$, $G \cong \overline{K_2} \vee C_4$ or $G \cong \overline{C_7}$. □

References

[1] M. Aigner and M. Fromme, A game of cops and robbers, *Discrete Appl. Math.* **8** (1984), 1-11.

[2] B. Bollobás, G. Kun and I. Leader, Cops and robbers in a random graph, (English summary), *J. Combin. Theory Ser. B* **103** no. 2 (2013), 226–236.

[3] A. Bonato and R. J. Nowakowski, *The Game of Cops and Robbers on Graphs*, Amer. Math. Soc., Providence, Rhode Island, 2011.

[4] N. E. Clarke, S. L. Fitzpatrick, A. Hill and R. J. Nowakowski, Edge critical cops and robber, *Discrete Math.* **310** (2010), 2299–2309.

[5] S. L. Fitzpatrick, Edge Critical Cops and Robber in Planar Graphs, *Discrete Math.* **329** (2014), 1–11.

[6] P. Frankl, Cops and robbers in graphs with large girth and Cayley graphs, *Discrete Appl. Math.* **17** (1987), 301–305.

[7] A. M. Frieze, M. Krivelevich and P. S. Loh, Variations on cops and robbers, *J. Graph Theory* **69** no. 4 (2012), 383–402.

[8] L. Lu and X. Peng, On Meyniels conjecture of the cop number, *J. Graph Theory* **71** (2012), 192–205.

[9] R. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Math.* **43** (1983), 23–239.

[10] A. Quilliot, *Thèse d'Etat,* Université de Paris VI, 1983.

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