AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **63(2)** (2015), Pages 268–296

Matching extension in toroidal quadrangulations II: the 3-extendable case

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Abstract

A graph G containing a perfect matching is said to be m-extendable if $m \leq (|V(G)| - 2)/2$ and for every matching M with $|M| = m$, there is a perfect matching F in G such that $M \subseteq F$. In a previous paper, four of the present five authors characterized those quadrangulations of the torus which are 2-extendable. In the present work a characterization of those which are 3-extendable is obtained. Since no quadrangulation of the torus can be m-extendable for any $m \geq 4$, this completes the study of m-extendability for toroidal quadrangulations. Moreover, by another previous result, it follows that we have therefore characterized all 3-extendable toroidal graphs.

1 Introduction

A connected graph G with a perfect matching is said to be *m-extendable* if $|V(G)| \geq 2m + 2$ and every matching of size m extends to (i.e., is a subset of) a perfect matching. (For a general reference on the subject of matching extension, see [17], and for three surveys on the subject, see also [14, 15, 16].)

We begin by noting that if an even toroidal quadrangulation is only 3-connected, it may not even have a perfect matching. (See Figure 2.1 of [1].) Hence we immediately focus on those even toroidal graphs which are (at least) 4-connected. Dean [5] showed that no toroidal graph is 4-extendable. Hence by Theorem 2.2 of [12], no toroidal graph is m-extendable for any $m > 4$ as well.

A 4-connected even toroidal quadrangulation need not be 3-extendable or even 2-extendable. In [1] the members of this family which are 2-extendable were characterized. We complete this direction of study in the present paper by focusing on the 3-extendability property. Although not all 4-connected even toroidal quadrangulations are 3-extendable, we will characterize precisely which ones are. In so doing we provide a complete characterization of all maximally extendable toroidal graphs. Such a characterization is unknown for any other surface.

We adopt the following standard notation. The neighborhood of vertex v will be denoted by $N(v)$ and $\{v\} \cup N(v)$ by $N[v]$. Also if $S, T \subseteq V(G)$ and $S \cap T = \emptyset$, we denote by $\delta(S,T)$ the set of edges with one endvertex in S and the other in T. Further, we let $\delta(S) = \delta(S, V(G) - S)$, $q(S,T) = |\delta(S,T)|$ and $q(S) = |\delta(S)|$. Additional notation and terminology will be introduced as needed. For all other background material, we direct the reader to [1], [4] and [6].

* Qiuli Li is supported by NSFC grant (no. 11401279).
† Heping Zhang is supported by NSFC grant (no. 11371180).

2 Preliminary Results

In order to present our results on 4-connected quadrangulations of the torus, we will require a more detailed analysis of the structure of these graphs. Although this was first accomplished by Altshuler [2], we will follow, for the most part, the presentation due to Nakamoto and Negami. (Cf. [9, 10, 11].)

The universal covering space of a torus is homeomorphic to the $x-y$ plane $\mathbb{R}^2$. Let $\tilde{G}$ be the union of the vertical and horizontal lines through the points of $\mathbb{R}$ which have integral coordinates; that is,

$$\tilde{G} = \{(x, y) \in \mathbb{R}^2 | x, y \in \mathbb{Z}\}.$$

Let $\hat{G}$ denote the infinite 4-regular and 4-face-regular graph induced by $\tilde{G}$. We will denote by $\Box(m, n, t)$ the set of all translations $T_{(\alpha,\beta)}$ of $\mathbb{R}^2$ onto $\mathbb{R}^2$ defined by

$$T_{(\alpha,\beta)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} 0 \\ m \end{pmatrix} + \beta\begin{pmatrix} n \\ -t \end{pmatrix}, \tag{2.1}$$

where m and n are positive integers, $\alpha, \beta \in \mathbb{Z}$ and t is a non-negative integer. Then $\Box$ is a group under composition of translations and all members of $\Box$ leave $\hat{G}$ invariant. The orbit space $\frac{\mathbb{R}^2}{\Box(m,n,t)}$ of the group is homeomorphic to a torus and the projection $\frac{\hat{G}}{\Box(m,n,t)}$ is a 4-regular and 4-face-regular graph on the torus. Altshuler [2] has shown that these graphs are precisely the graphs $Q(m, n, t)$ described as follows. Form a cylinder of length n and cross-sectional cycles of length m and then identify the ends of the cylinder with a "twist" of t units. (Note that in Equation 2.1 the "twist" always occurs along the y-axis (i.e. in the "m"-direction).)

As an example, see the graph $Q(4, 5, 1)$ displayed in Figure 2.1.

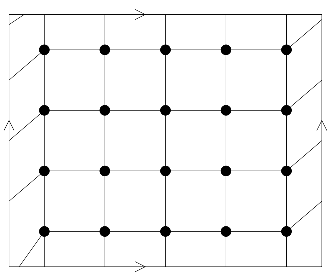

Figure 2.1. The quadrangulation $Q(4, 5, 1)$

We will always assume that $m \geq 3$ to exclude loops and parallel edges. For the same reason, we will also assume $0 \leq t \leq m-1$ and, when $n = 1$, that $m \geq 5$ and $2 \leq t \leq m-2$, while if $n = 2$, we assume $1 \leq t \leq m-1$. Finally, since we are concerned with perfect matchings throughout this paper, we will also assume that mn is even.

Note that we will sometimes want to single out graphs of the type $Q(m,n,t)$ which contain triangles. We will denote this class by $3Q$ and note that $3Q$ consists of all toroidal quadrangulations of the form $Q(3,n,t), Q(m,3,0), Q(m,2,1), Q(m,2,m-1), Q(m,1,2)$ and $Q(m,1,m-2)$.

We will have need of the following result.

Theorem 2.1. *Let G be a connected graph minimally embedded on the torus. If G_1 is a non-planar subgraph of G, then $G - V(G_1)$ is plane.*

Proof. Suppose to the contrary that $G-V(G_1)$ is not plane. Then $G-V(G_1)$ contains a non-contractible cycle C of G. Then G_1 can be embedded on a cylinder which arises from cutting the torus along C. But then G_1 is planar, a contradiction. □

Def.: A set of edges X in a graph G is said to be a *cyclic edge-cut* if $G - X$ contains at least two components each of which contains a cycle.

Def.: A graph G is said to be *cyclically k-edge-connected* if there exists no cyclic edge-cut X in G with $|X| < k$. The maximum value of k for which a graph G is cyclically k-edge-connected is called the *cyclic edge-connectivity* of G and is denoted by $c\lambda(G)$.

Lemma 2.2. *[1] $c\lambda(Q(m,n,t)) = 8$, except for the members in class $3Q$, in which case $c\lambda(Q(m,n,t)) = 6$.*

Corollary 2.3. *[1] If $G = Q(m,n,t)$ contains a cyclic edge-cut S of size 8 and if one of the components of $G - S$ is non-planar, then the other component is a quadrangle.*

Def.: If G' is a connected induced subgraph of some $G = Q(m,n,t)$ such that $q(V(G'), V(G) - V(G')) = k$, we call the subgraph G' a *k-shooter* (in G).

Corollary 2.4. *[1] If $G = Q(m,n,t)$ and G contains a 4-shooter G', then either G' or $G - G'$ is a singleton.*

Lemma 2.5. *[1] A quadrangulation G of the torus is 4-regular if and only if it is 4-connected.*

The following is a corollary of a theorem of Berge [3].

Theorem 2.6. *Every 4-connected 4-regular even graph is 1-extendable.*

The characterization of those 4-connected even toroidal quadrangulations which are 2-extendable was obtained in [1] and we state it next for the sake of completeness.

Theorem 2.7. *[1] Let $G = Q(m,n,t)$ with mn even. Then G is 2-extendable if and only if G does not belong to any of the following three classes:*

(i) $\{Q(3,2,1), Q(3,2,2), Q(m,1,2), Q(m,1,m-2)\}$*; or*

(ii) $\{Q(m,2,2), Q(m,2,m-2) | m \text{ odd and } m \geq 5\}$*; or*

(iii) $\{Q(m,1,m/2+1), Q(m,1,m/2-1) | m/2 \text{ odd and } m \geq 6\}$.

Finally, we shall need the following well-known definition from matching theory.

Def.: A graph G is said to be *factor-critical* if $G - v$ contains a perfect matching for all $v \in V(G)$.

3 The Main Result

First, we require two lemmas about the structure of a plane component which is also a k-shooter.

Lemma 3.1. *If H be a k-shooter subgraph of $G = Q(m,n,t)$, then k is even.*

Proof. Clearly $k = \sum_{v \in V(H)}(4 - \deg_H(v)) = \sum_{v \in V(H)} 4 - \sum_{v \in V(H)} \deg_H(v) = 4|V(H)| - 2|E(H)|$ and is thus even. □

Lemma 3.2. *Let H be a k-shooter of some $Q(m,n,t)$. If the embedding of H inherited from $Q(m,n,t)$ is plane, then, for $k = 6, 8, 10, 12$ and 14, H is one of the graphs listed in the proof below. Moreover, none of these graphs is factor-critical.*

Proof. First, we show that H is not annular (i.e., it has only one facial walk such that this walk does not bound a face of $Q(m,n,t)$). Suppose to the contrary that H contains at least two cycles C_1 and C_2 which are not facial cycles of $Q(m,n,t)$. Without loss of generality, assume that C_1 is the outer boundary of H. Let $k_i, i = 1, 2$, be the number of edges joining vertices of C_i and vertices of $Q(m,n,t) - V(H)$. Let ℓ_i be the length of the walk C_i. Then $k_1 = \ell_1 + 4$ and $k_2 = \ell_2 - 4$ since each C_i is a boundary walk. So $k = k_1 + k_2 = \ell_1 + \ell_2 \geq 16$ since $\ell_1, \ell_2 \geq 8$, and this contradicts the assumption that $k \leq 14$.

Since H is not annular, H has only one boundary cycle, say C_1. But then $k = k_1 = \ell_1 + 4$.

If $k = 4$, then it can be easily seen that H is a single vertex.

If $k = 6$, then H is an edge.

If $k = 8$, then H is either a quadrangle (when it contains a cycle) or a path of length 2 otherwise.

If $k = 10$, then if H does not contain a cycle, it is a path of length 3 or a $K_{1,3}$ (claw), and if it does contain a cycle, it is the union of two quadrangles which share a single edge or it is a quadrangle with a pendant edge.

If $k = 12$, then if H is cycle-free, it must be one of the three trees with four edges since the length of the boundary walk is 8. If H contains a cycle, but has a cutvertex, it is easy to check that it must be one of the six graphs shown in Figure 3.1.

If, on the other hand, H contains a cycle, but no cut vertices, then it must be one of the three graphs shown in Figure 3.2.

It is easy to check that none of the graphs found for $k = 4, 6, 8, 10$ and 12 above is factor-critical.

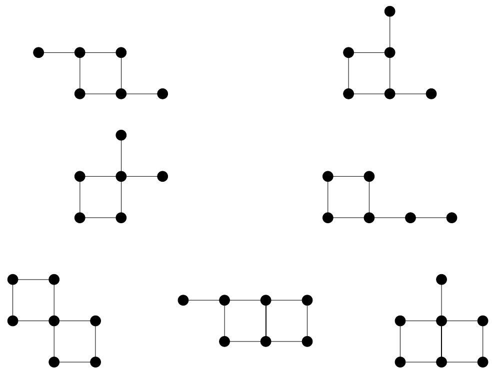

Figure 3.1.

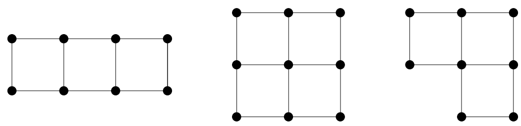

Figure 3.2.

It remains to treat the case when $k = 14$.

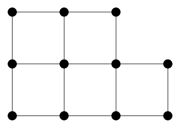

Figure 3.3.

Let $Int(C)$ be the subgraph of H induced by the vertices of H lying in the interior of C. Since $k = 14$, it follows that $l = 10$ and $q(Int(C)) \leq 6$. There are thus six possibilities for H. Of these, four have $q(Int(C)) = 0$, but all are even and hence not factor-critical. If $q(Int(C)) = 4$, we obtain the graph shown in Figure 3.3 which is not factor-critical either.

The one remaining possibility for H has $q(Int(C)) = 6$. It is a 12-vertex graph isomorphic to a 4×3 grid and is hence even. But then it is not factor-critical. □

The main theorem of this paper is now presented.

Theorem 3.3. *Let G be a member of $Q(m, n, t)$ with $|V(G)|$ even. Then G is 3-extendable if and only if G does not belong to one of the following classes:*

(1) *$3Q$;*

(2) *$\{Q(m, 2, 2), Q(m, 2, m-2) | m \geq 5\}$;*

(3) *$\{Q(m, 1, \frac{m}{2}-1), Q(m, 1, \frac{m}{2}+1) | m$ even and $m \geq 10\}$;*

(4) *$\{Q(m, 4, 0) | m$ odd and $m \geq 3\}$;*

(5) $\{Q(m, 2, \frac{m}{2}) | \frac{m}{2}$ *odd and* $m \geq 6\}$;
(6) $\{Q(m, 1, \frac{m}{4}), Q(m, 1, \frac{3m}{4}) | \frac{m}{4}$ *even and* $m \geq 16\}$;
(7) $\{Q(m, 4, 2), Q(m, 4, m-2) | m$ *odd and* $m \geq 3\}$;
(8) $\{Q(m, 2, \frac{m}{2}+1), Q(m, 2, \frac{m}{2}-1) | \frac{m}{2}$ *even and* $m \geq 8\}$;
(9) $\{Q(m, 1, \frac{m+2}{4}), Q(m, 1, \frac{3m-2}{4}) | \frac{m+2}{4}$ *even and* $m \geq 14\}$;
(10) $\{Q(m, 1, \frac{m-2}{4}), Q(m, 1, \frac{3m+2}{4}) | \frac{m-2}{4}$ *even and* $m \geq 18\}$;
(11) $\{Q(m, 4, 4), Q(m, 4, m-4) | m$ *odd and* $m \geq 5\}$;
(12) $\{Q(m, 2, \frac{m}{2}+2), Q(m, 2, \frac{m}{2}-2) | \frac{m}{2}$ *odd and* $m \geq 10\}$;
(13) $\{Q(m, 1, \frac{m}{4}+1), Q(m, 1, \frac{3m}{4}-1) | \frac{m}{4}$ *odd and* $m \geq 20\}$;
(14) $\{Q(m, 1, \frac{3m}{4}+1), Q(m, 1, \frac{m}{4}-1) | \frac{m}{4}$ *odd and* $m \geq 20\}$;
(15) $\{Q(m, 2, m-4), Q(m, 2, 4) | m$ *odd and* $m \geq 7\}$;
(16) $\{Q(m, 1, \frac{m}{2}-2), Q(m, 1, \frac{m}{2}+2) | \frac{m}{2}$ *even and* $m \geq 20\}$;
(17) $\{Q(m, 3, m-2), Q(m, 3, 2) | m \geq 6\}$;
(18) $\{Q(m, 1, \frac{m-2}{3}), Q(m, 1, \frac{2m+2}{3}) | m$ *even* , $\frac{m-2}{3}$ *an integer and* $m \geq 20\}$;
(19) $\{Q(m, 1, \frac{2m-2}{3}), Q(m, 1, \frac{m+2}{3}) | m$ *even* , $\frac{m+2}{3}$ *an integer and* $m \geq 22\}$;
(20) $\{Q(m, 2, 3), Q(m, 2, m-3) | m \geq 12\}$.

Proof. First we must show that if G belongs to any of the twenty classes, then G is not 3-extendable.

(1) If $G \in 3Q$, then G contains (non-contractible) triangles. Hence G cannot be 3-extendable from the well-known result (cf. [12]) that for a vertex v of degree $n+t$ in an n-extendable graph G, $G[N(v)]$ does not contain a matching of size t. Here we have set $n = 3$ and $t = 1$, to obtain the desired result.

The details of the remaining nineteen cases may be found in an Appendix at the end of the paper.

Now to prove the converse, let us suppose G is not 3-extendable. We must show that G belongs to one of the twenty classes listed above.

Suppose, then, that e_1, e_2 and e_3 are three independent edges which do not extend to a perfect matching. Then by a variation on Tutte's theorem (cf. Exercise 3.3.18(b) of [8] or Theorem 2.2.3 of [6]) there exists a set $S \subseteq V(G) - V(e_1) - V(e_2) - V(e_3)$ such that $G - S - V(e_1) - V(e_2) - V(e_3)$ consists of at least $|S| + 2$ components each of which is factor-critical (and hence odd). Let $K = S \cup V(e_1) \cup V(e_2) \cup V(e_3)$. Note that if G is not 2-extendable, then by Theorem 2.7, G belongs to one of the classes detailed in parts (1), (2) and (3) of the statement of the theorem. If G contains a triangle, it belongs to $3Q$. So we may assume that G is 2-extendable and triangle-free. Hence $G - K$ consists of precisely $|S| + 2$ factor-critical components.

By the 4-regularity of G we have $q(K, V(G) - K) \leq 4|S| + 18 = 4|K| - 6$, and by the 4-connectivity of G, we have $q(V(G) - K, K) \geq 4(|S| + 2) = 4|S| + 8 = 4(|K| - 6) + 8 = 4|K| - 16$.

Since G is 4-regular, $q(K, G-K)$ is even, and hence there are six cases to consider:

Case 1: $q(K, G-K) = 4|K| - 16$;

Case 2: $q(K, G-K) = 4|K| - 14$;

Case 3: $q(K, G-K) = 4|K| - 12$;

Case 4: $q(K, G-K) = 4|K| - 10$;

Case 5: $q(K, G-K) = 4|K| - 8$;

Case 6: $q(K, G-K) = 4|K| - 6$.

Case 1. In this case each factor-critical component $G_i, 1 \leq i \leq |S|+2$, of $G-K$ is a 4-shooter and hence a singleton by Corollary 2.5. Hence $|E(G[K])| = 8$.

We claim that every face which contains an edge in $G[K]$ must contain either two (adjacent) edges or four edges in $G[K]$. (This observation will be subsequently used repeatedly.) This is easy to see, for suppose that $F = abcda$ is a face and $ab \in G[K]$, but $bc \notin G[K]$. Thus vertex c must be one of the singleton odd components, call it G_i. But then the edge cd also joins G_i to $G[K]$ and hence cannot lie in $G[K]$, but vertex d does lie in $G[K]$. Hence the edge da is in $G[K]$. This proves the claim.

We distinguish cases based on the degrees of vertices in $G[K]$. We shall refer to $d_{G[K]}(v)$ as the *K-degree* of v. First note that $G[K]$ cannot have more than two vertices of K-degree 4 (since $|E(G[K])| = 8$). If $G[K]$ has precisely two vertices of K-degree 4, then there cannot be a matching of size 3 in $G[K]$. Thus we have at most one vertex of K-degree 4 in $G[K]$.

1.1: Suppose there is a vertex of K-degree 4 in $G[K]$; call it v.

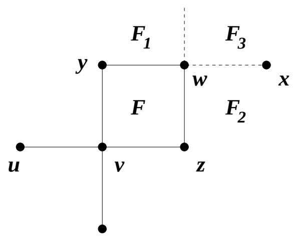

Figure 3.4.

1.1.1: Suppose first that there is exactly one face containing vertex v and having all four incident edges in $G[K]$. Call this face F. Label the vertices and faces of the graph G as in Figure 3.4 and recall that G does not contain triangles nor does it have parallel edges. This implies that face F_1 does not contain any other of the six edges of $G[K]$ shown in Figure 3.4 nor do faces F_2 and F_3.

Since $G[K]$ has exactly one vertex of K-degree 4, we claim that neither of the remaining two edges of $G[K]$ can be incident with vertex w. Indeed, suppose one of these remaining edges of $G[K]$ is incident with w, call it wx. Then both faces F_1 and F_3 contain one edge of $G[K]$ so far. There remains only one edge of $G[K]$ to consider; call it e. If e is incident with y, then F_3 contains at most one edge of $G[K]$, a contradiction. If e is incident with x, then F_1 contains at most one edge of $G[K]$, again a contradiction.

Now since neither of the remaining two edges of $G[K]$ is incident with w, one of them must be incident with vertex y and the other incident with vertex z. (See Figure 3.5 where the edges bearing arrows are to be identified in the sense of the arrows.)

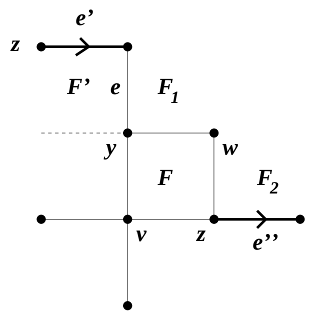

Figure 3.5.

Consider now the face F' adjacent to face F_1. Since every face containing one edge in $G[K]$ contains at least two adjacent edges in $G[K]$, face F' contains one more edge other than the edge e (where e is the boundary edge common to F' and F_1). But G does not contain triangles or parallel edges, so it must be the case that $e' = e''$.

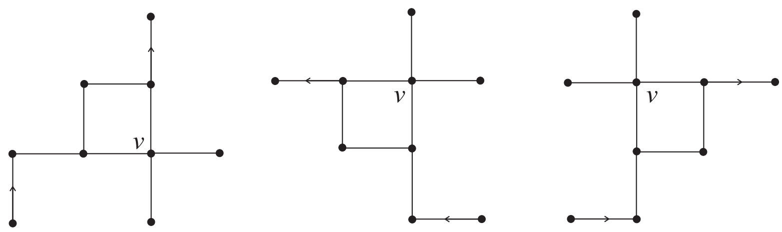

Figure 3.6.

For the other cases, (that is, the rotations of the configuration in Figure 3.4 through $90, 180$ and 270 degrees), we have the configurations of Figure 3.6 and similar analyses apply. It may be checked that the configuration obtained in Figure 3.5 can only occur in $Q(m, 1, \frac{m}{2} - 1), Q(m, 1, \frac{m}{2} + 1), Q(m, 2, 2)$ or $Q(m, 2, m - 2)$. But these classes belong to our list of twenty.

1.1.2: If there are exactly two faces containing v each with all four edges in $G[K]$, then up to symmetry, there are two cases as in Figure 3.7.

For the configuration (i) of Figure 3.7, consider the edge e in $G[K]$. Then the other face containing the edge e not shown in Figure 3.7 must contain a second edge in $G[K]$ which must be adjacent to e. But then this edge must be e_1 or e_2 as shown in (i) of Figure 3.8. However, nine edges of $G[K]$ now appear, so some two of them must be identical. No matter which pair of edges are identified, we are led to the existence of triangles or multiple edges, a contradiction.

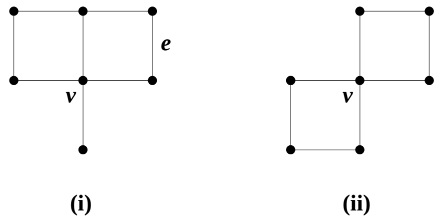

Figure 3.7.

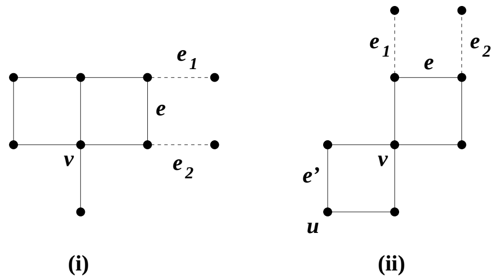

Figure 3.8.

For Case (ii) in Figure 3.7, via a discussion similar to that above, we see that $G[K]$ must contain e_1 or e_2. (See (ii) of Figure 3.8.) If $G[K]$ contains e_1, then multiple edges or a triangle must appear, a contradiction. Hence $G[K]$ contains e_2 and to avoid triangles or multiple edges, e_2 must be identical to e'. But then there are two vertices u and v in $G[K]$ each having degree 4, a contradiction.

1.1.3: If there are at least three faces containing v with all their edges in $G[K]$, we can show that triangles or multiple edges must exist, and again contradictions result.

1.1.4: Suppose there is no face containing v with all its edges in $G[K]$. Let S be the set of all edges of $G[K]$ which are not incident with v. Then $|S| = 4$. So assume that $S = \{e_1, e_2, e_3, e_4\}$. The subgraph G' consisting of these four edges does not have a K-degree 4 vertex. Moreover, G' does not have a K-degree 3 vertex either, for suppose it did. Let w be such a vertex of K-degree 3 in G' and assume that w is incident with e_1, e_2 and e_3 as shown in Figure 3.9.

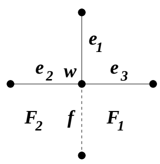

Figure 3.9.

Let f be the remaining edge incident with w which is not in $G[K]$. So $f \notin G[K]$. But then it is impossible for faces F_1 and F_2 to share the one remaining edge of the eight edges in $G[K]$.

So each vertex of G' has degree 2 or 1. Since every face containing an edge of $G[K]$ contains precisely two edges of $G[K]$, G' has no vertices of K-degree 1. Thus G' is a cycle of length 4 and since the four edges of G' cannot bound a face by the hypothesis of this subcase, G' must be one of the four configurations shown in Figure 3.10. (Here again the arrows indicate identification.)

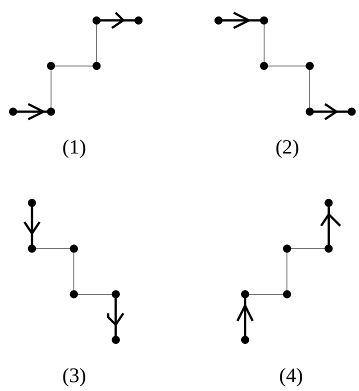

Figure 3.10.

One can easily check that such configurations can exist only in $Q(m, 1, \frac{m}{2} - 1), Q(m, 2, m-2)$, $Q(m, 1, \frac{m}{2} + 1)$ and $Q(m, 2, 2)$. Again these classes appear in our list of twenty.

1.2. Suppose then that there is no vertex of degree 4 in $G[K]$. But suppose there are vertices of degree 3 in $G[K]$. Let v be such a vertex. Then there are four possible structures at v in G, as shown in Figure 3.11. Consider Configuration (i). (The other three configurations in the figure are settled similarly.)

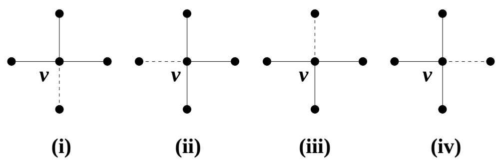

Figure 3.11.

If there is no face with all four edges in $G[K]$, then the possible structures of $G[K]$ are shown in Figure 3.12.

To show that these four structures are, in fact, the only ones possible, we have to make clear that when extending face boundary edges in $G[K]$ as in Figure 3.12, if the edge a appears, then the edge b must be present and if the edge x appears, then the edge y must be present also. These two proofs are similar; we give only the former one.

Consider Figure 3.13. First note that all solid edges in the figure are distinct; for otherwise there would be triangles or multiple edges. Suppose that the edge a is

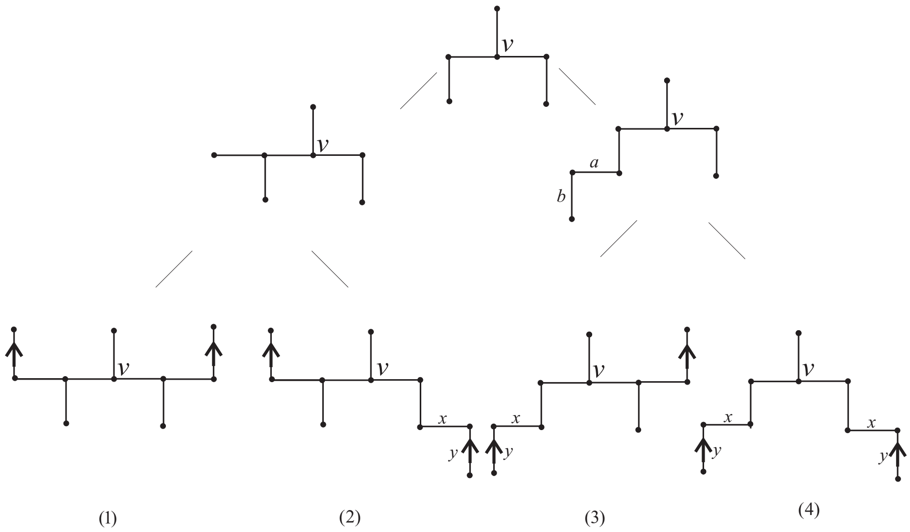

Figure 3.12.

present, but the edge b is not present. Then the edge c lies in $G[K]$ and hence the edge d also lies in $G[K]$ (since there is no vertex of K-degree 4 in $G[K]$). Consider the face F. This boundary of this face must contain two adjacent edges of $G[K]$. So either the edge h or the edge i must belong to $G[K]$.

Suppose h belongs to $G[K]$. Then either $h = a$ or $h = d$. If $h = a$, then the boundary of face F must contain the edge $c = c'$ and hence the boundary of F contains four edges from $G[K]$, a contradiction. On the other hand, if $h = d$, then vertex $u' = u$ and hence u is a vertex of K-degree 4, again a contradiction. So h does not belong to $G[K]$.

So the edge i belongs to $G[K]$. Hence either $i = a$ or $i = d$. But if $i = a$, we have a triangle, which is impossible. So $i = d$. But then vertex u' has K-degree 3 which is impossible.

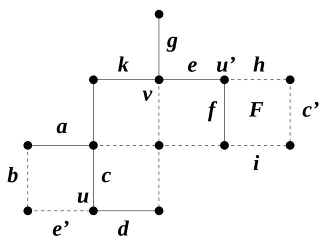

Figure 3.13.

For Case (1) in Figure 3.12, we have that

$$\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}=\alpha\begin{pmatrix}0\\m\end{pmatrix}+\beta\begin{pmatrix}n\\-t\end{pmatrix}.$$

So $n = 4/\beta$ and $t = \alpha m/\beta$ and keeping in mind that $0 \le t \le m-1$, it must be the case that $n = 4, t = 0$ or $n = 2, t = \frac{m}{2}$ or $n = 1, t = \frac{m}{4}$ or $n = 1, t = \frac{m}{2}$ or $n = 1, t = \frac{3m}{4}$. If $n = 1, t = \frac{m}{2}$, multiple edges appear, a contradiction.

Suppose first that $n = 4, t = 0$. If m is odd, $Q(m, 4, 0)$ is a member of Class (4) of our list of twenty exceptional classes.

Therefore, suppose m is even; that is, we consider $Q(m, 4, 0)$ with m even. Let e_1, e_2 and e_3 be any three disjoint edges. If $G - \cup_{i=1}^{3}(V(e_i))$ contains no perfect matching and hence contains a vertex set S such that $G - [(\cup_{i=1}^{3} V(e_i)) \cup S]$ has at least $|S| + 2$ odd components. Let $K = S \cup (\cup_{i=1}^{3} V(e_i))$. Then $G[K]$ has the structure shown in (1) of Figure 3.12. (As an example, we show $Q(6, 4, 0)$ with $G[K]$ embedded in bold lines in Figure 3.14. Note that e_1, e_2 and e_3 must be three disjoint edges in this configuration.)

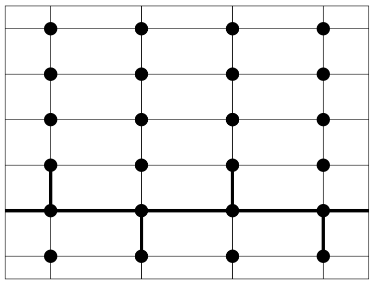

Figure 3.14.

If all three of these edges are vertical in the configuration, it is easy to see that they must extend to a perfect matching, since m is even. Otherwise, since they belong to the configuration of Case (1), it must be the case that one of them is horizontal and the other two are vertical. But again it is easy to check they extend to a perfect matching in this case as well. Thus $G = Q(m, 4, 0)$ is, in fact, 3-extendable, contrary to our initial hypothesis.

Similarly for $n = 2, t = \frac{m}{2}$, we obtain that G is not 3-extendable if and only if $\frac{m}{2}$ is odd and $m \geq 6$.

For $n = 1, t = \frac{m}{4}$, we obtain that G is not 3-extendable if and only if $\frac{m}{4}$ is even and $m \geq 16$. By symmetry, $Q(m, 1, \frac{3m}{4})$ is not 3-extendable if and only if $m \geq 16$ and $\frac{3m}{4}$ is even.

For Case (2) in Figure 3.12, we have that

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ m \end{pmatrix} + \beta \begin{pmatrix} n \\ -t \end{pmatrix}.$$

Hence $n = 4, t = 2$ or $n = 2, t = \frac{m}{2} + 1$ or $n = 2, t = 1$ or $n = 1, t = \frac{m+2}{4}$ or $n = 1, t = \frac{m+1}{2}$ or $n = 1, t = \frac{3m+2}{4}$.

Consider the case when $n = 4$ and $t = 2$. If m is odd, $Q(m, 4, 2)$ is in Class (7) of our list of twenty classes. If m is even, we claim $Q(m, 4, 2)$ is 3-extendable. To see this, consider the configuration of Case (2) embedded in $Q(m, 4, 2)$ and suppose m is even. See the example shown in Figure 3.15 in which $m = 6$. It is easy to see from

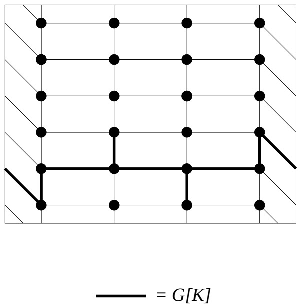

Figure 3.15.

this figure that it is impossible that three independent edges with two horizontal and one vertical in $G[K]$ can exist, so we need only check the possibilities that (i) all three edges to be extended are vertical, (ii) one is horizontal and the other two are vertical and (iii) all three are horizontal. It is straightforward to check that the three edges extend to a perfect matching in all three cases.

Consider next the case when $n = 2$ and $t = \frac{m}{2} + 1$. If $\frac{m}{2}$ is even and $m \geq 8$, the graphs belong to Class (8). If $\frac{m}{2}$ is odd, we claim that $Q(m, 2, \frac{m}{2}+1)$ is 3-extendable. We illustrate this case with $Q(10, 2, 6)$ shown in Figure 3.16 where once again $G[K]$ in shown in bold. Checking the 3-extendability is left to the reader.

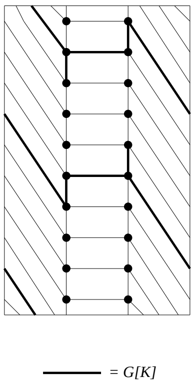

Figure 3.16.

We must also check $Q(m, 2, \frac{m}{2} + 1)$ when $m \leq 4$. But the reader can easily check that $Q(4, 2, 3)$ contains triangles and hence belongs to Class $3Q$.

In the case when $n = 2$ and $t = 1$, the graph is again in Class $3Q$.

If $n = 1$ and $t = \frac{m+2}{4}$, if $\frac{m+2}{4}$ is even and $m \geq 14$, $Q(m, n, t)$ belongs to Class (9) of our twenty classes. If, on the other hand, $\frac{m+2}{4}$ is odd, we leave it to the reader to check that the graph is 3-extendable. Similarly, it may be checked that the graph is also 3-extendable when $m = 10$.

If $n = 1$ and $t = \frac{m+1}{2}$, to assure that t is an integer, m must be odd. But a graph with odd number of vertices cannot be 3-extendable.

Finally, if $n = 1$ and $t = \frac{3m+2}{4}$, if $\frac{3m+2}{4}$ is even and $m \geq 18$, $Q(m, n, t)$ belongs to Class (10) of our twenty classes. The reader may check that $Q(m, 1, \frac{3m+2}{4})$ is 3-extendable when $\frac{3m+2}{4}$ is odd and also that $Q(10, 1, 8)$ is 3-extendable as well.

For Case (3) in Figure 3.12, we have that

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ m \end{pmatrix} + \beta \begin{pmatrix} n \\ -t \end{pmatrix}.$$

Hence $n = 4, t = m - 2$ or $n = 2, t = \frac{m}{2} - 1$ or $n = 2, t = m - 1$ or $n = 1, t = \frac{m-2}{4}$ or $n = 1, t = \frac{m-1}{2}$ or $n = 1, t = \frac{3m-2}{4}$. Similar contradictions are obtained for each possibility.

For Case (4) in Figure 3.12, we obtain the same result as in Case (1).

For the other three cases in Figure 3.11, we get as possible structures for $G[K]$ the configurations obtained from the final structures in Figure 3.12 by rotating 90, 180 and 270 degrees clockwise respectively. Then $Q(4, n, 0), Q(m, 2, m - 4), Q(m, 1, \frac{m}{2} - 2), Q(m, 1, m - 2), Q(m, 2, 4), Q(m, 1, 2)$, and $Q(m, 1, \frac{m}{2} + 2)$ are the possible graphs in which the structures may lie. Contradictions similar to those obtained previously are obtained in each case. We omit the details.

If $G[K]$ does contain a face F having all of its edges in $G[K]$, there are no vertices of degree 4, and so $G[K]$ is as in Figure 3.17. Then nine edges appear in $G[K]$, so there must be a pair of identical edges. But no matter which pair of edges are identified, we are led to the appearance of triangles or multiple edges, a contradiction.

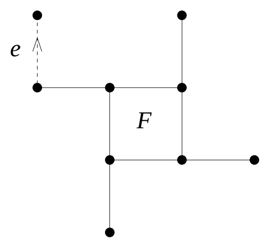

Figure 3.17.

1.3. Suppose there is no vertex of degree 3 or 4 in $G[K]$.

Since every face contains either no edges from $G[K]$ or two adjacent edges from $G[K]$, it can be easily verified that there is no vertex of degree 1 in $G[K]$. Since $|E(G[K])| = 8$, it follows that either $G[K]$ is a union of two disjoint cycles each of length 4 or a cycle of length 8 shown in Figure 3.18. (In the figure, just one case is shown; the others are the figures obtained by rotating the first figure by 90, 180, 270 degrees respectively.)

Using similar arguments, we obtain that the possibilities of $Q(m, n, t)$ are (m, n, t) $= (m, 2, 2), (m, 2, m-2), (m, 1, \frac{m}{2}-1), (m, 1, \frac{m}{2}+1), (m, 4, 4), (m, 4, m-4), (m, 2, \frac{m}{2}+$

2), $(m, 2, \frac{m}{2} - 2), (m, 1, \frac{m}{4} + 1), (m, 1, \frac{3m}{4} - 1), (m, 1, \frac{m}{4} - 1), (m, 1, \frac{3m}{4} + 1)$ and some members in $3Q$, all of which lead to contradictions.

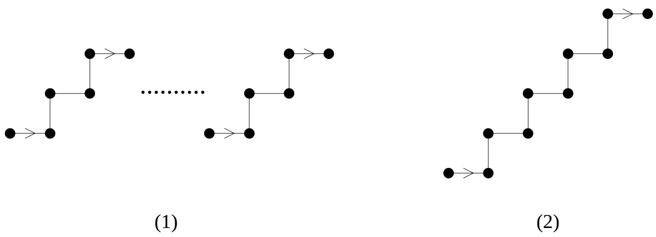

Figure 3.18.

Case 2. In this case, there must be exactly one factor-critical component, which is a 6-shooter, say G_1, with the others all 4-shooters and hence singletons. G_1 is factor-critical and $\neq K_1$, so it must contain a cycle ([7, 8]). Let $G - G_1 = T$. Then $4|T| - 2|E(T)| = 6$ which implies $|E(T)| - |T| = |T| - 3 \geq 0$. Hence T must also contain a cycle and hence $\delta(G_1)$ is a *cyclic* edge-cut of G. But G is not a member of $3Q$, so by Lemma 2.3, $c\lambda(G) = 8$. But this contradicts the fact that $\delta(G_1)$ is a cyclic edge-cut of size 6. So Case 2 cannot occur.

Before proceeding further, we now insert the following lemma about cylindrical graphs.

Lemma 3.4. *Let G be an induced cylindrical subgraph of $Q(m, n, t)$ with boundary cycles C_1 and C_2. Then for each C_i, the number of vertices contributing two shooter edges equals the number of vertices contributing no shooter edge.*

Proof. Without loss of generality, let e be an edge of C_1 oriented in the direction of increasing n. Further, without loss of generality, we may suppose that e passes through the boundary of the rectangle and that G lies below the cycle C_1. (See the example shown in Figure 3.19.)

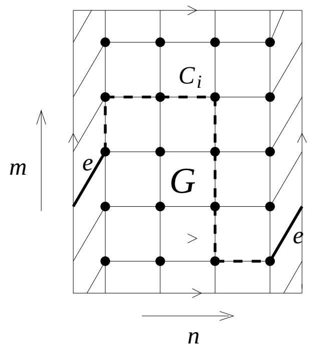

Figure 3.19.

Since C_1 is induced, starting with the edge e, one can traverse C_1 so that progress in the n-direction is non-decreasing. Note that in so doing, the number of right-hand turns equals the number of left-hand turns. But the number of right-hand turns equals the number of vertices contributing two shooter edges and the number of left-hand turns equals the number of vertices contributing no shooter edge and the proof is complete. □

We point out that the cycles C_1 and C_2 may have common vertices and edges. If $C_1 \cap C_2 \neq \emptyset$, let v a vertex in the intersection. If v emits two shooter edges e_1 and e_2 such that e_1 and e_2 do not lie on the same side of C_i, $i = 1, 2$, then we say that v *contributes* one shooter edge from each of C_1 and C_2.

Case 3. In this case, since $c\lambda(G) = 8$, there must be a factor-critical component which is an 8-shooter, say G_1, and the others are 4-shooters and hence singletons.

If G_1 is embedded in the plane, then by Lemma 3.2, G_1 cannot be factor-critical, a contradiction. If G_1 is non-planar, then $G - V(G_1)$ is embedded in the plane and hence a quadrangle by Corollary 2.4. But then there is no matching of size 3 in $G - V(G_1)$, a contradiction.

Thus G_1 is planar, but not plane, and hence it can only be a cylindrical graph. Let C_1 and C_2 be the two boundary cycles of G_1 and let $\ell_1 = |C_1|$ and $\ell_2 = |C_2|$. Since G has no multiple edges and no triangles, $\ell_1 \geq 4$ and $\ell_2 \geq 4$. Then by Lemma 3.4, $\ell_1 + \ell_2 = 8$ and it then follows that $\ell_1 = \ell_2 = 4$. Note that every face of G_1 inherited from the embedding of G is a quadrilateral. Hence G_1 is bipartite. But then G_1 is not factor-critical, a contradiction. Hence Case 3 cannot occur.

Case 4. In this case, since $c\lambda(G) = 8$, there must be a factor-critical component which is a 10-shooter, say G_1, and the other components are 4-shooters and hence singletons. Also there are five edges in $G[K]$.

If G_1 is a plane 10-shooter, then by Lemma 3.2, G_1 is not factor-critical, a contradiction. If G_1 is non-planar, then $G - V(G_1)$ is plane by Lemma 3.2 (and of course a 10-shooter as well). Then since $G - V(G_1)$ contains a matching of size 3, again using Lemma 3.2 it follows that $G - V(G_1)$ must be the 6-vertex graph consisting of two quadrilaterals which share exactly one edge and hence it is even. But G_1 is odd, since it is factor-critical, and hence so is G, again a contradiction.

Thus we need only consider the case when G_1 is a cylindrical graph. Adopting the notation used in Case 3, by Lemma 3.4 we obtain that $\ell_1 + \ell_2 = 10$. But if either $\ell_1 = 4$ and $\ell_2 = 6$ or vice-versa, then G_1 is bipartite and hence not factor-critical.

So we may assume that $\ell_1 = \ell_2 = 5$.

Let us now define σ_i to be the number of vertices of C_i contributing two shooter edges, for $i = 1, 2$. Let $\sigma = \max\{\sigma_1, \sigma_2\}$. Then by the preceding lemma, $\sigma \leq 5/2$.

Our proof for Case 4 now separates into two subcases.

4.1. Suppose $\sigma = 2$ and without loss of generality, suppose C_1 has two vertices u and v contributing two shooter edges.

Then u and v are not adjacent to each other (and neither are the two corresponding vertices which contribute no shooter edge), for otherwise a triangle would be present, a contradiction.

Let w be a vertex of C_1 adjacent to both u and v. If w is a vertex contributing one shooter edge, then C_1 must be as shown in Figure 3.20. But then again, a triangle must be present and we have a contradiction.

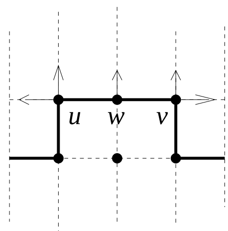

Figure 3.20.

So we may assume that w is a vertex contributing no shooter edge. Then C_1 must be as shown in one of the two configurations of Figure 3.21.

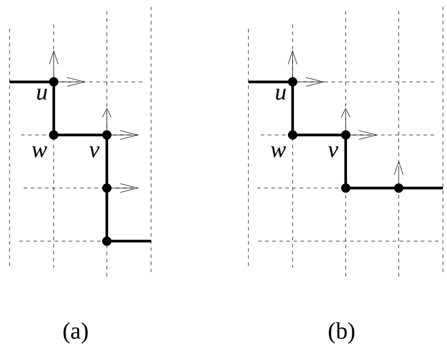

Figure 3.21.

If C_1 is as shown in (a), then $Q(m, n, t) = Q(m, 2, 3)$ which is Class (20) in our list. On the other hand, if C_1 is as shown in (b), then $Q(m, n, t) = Q(m, 3, 2)$ which is Class (17) in our list.

4.2. Suppose then that $\sigma = 1$. Let $C_1 = v_0v_1v_2v_3v_4v_0$ such that v_0 is a vertex contributing two shooter edges.

4.2.1. Suppose that v_1 is a vertex contributing one shooter edge. By symmetry, we may also assume that v_2 contributes no shooter edge. Then C_1 must be embedded as shown in bold in Figure 3.22.

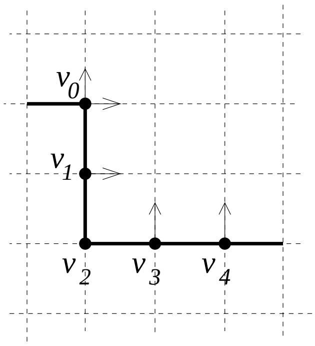

Figure 3.22.

So $Q(m, n, t) = Q(m, 3, 2)$ which is number 17 in our list of twenty classes.

4.2.2. Next suppose that v_1 emits no shooter edge.

Moreover, we may assume that each of v_2, v_3 and v_4 emits exactly one shooter edge. Let $v_j w_j$ be the shooter edge incident with vertex v_j, for $j = 2, 3, 4, 0$. Note that v_0 is adjacent to w_2 via a shooter edge as well. Then $\{w_2w_3, w_3w_4, w_4w_0\} \subseteq E(G[K])$. Let P_1 denote the path $w_2w_3w_4w_0$ which lies in $G[K]$. By symmetry, there exists a path $P_2 = w_2'w_3'w_4'w_0'$ in $G[K]$ such that every vertex of P_2 is adjacent to a vertex of cycle C_2. Then, since $|E(P_1) \cup E(P_2)| \leq |E(G[K])| = 5$, it follows that $E(P_1) \cap E(P_2) \neq \emptyset$.

Let x be the fourth vertex of the face containing path $w_0v_0w_2$ and denote this face by F_x. If $x \in V(G[K])$, then the 5-cycle $xw_2w_3w_4w_0x$ lies in $G[K]$. But since $|E(G[K])| = 5$, this 5-cycle constitutes all of $E(G[K])$ and hence $E(G[K])$ does not contain a matching of size 3, a contradiction. So $x \in V(G) - V(G[K])$.

If $x \in V(G_1)$, then it follows that $x \in C_2$. So x emits two shooter edges, namely xw_0 and xw_2. Let x_1 be the fourth vertex of the face containing boundary path w_3w_2x, let x_2 be the fourth vertex of the face containing boundary path $w_4w_3x_1$ and let x_3 be the fourth vertex of the face containing boundary path $w_0w_4x_2$. Note that x_1 is adjacent to $w_3 \in V(G[K])$, so either $x \in V(C_2)$ or $x_1 \in V(G[K])$. But we already concluded in the preceding paragraph that $x \notin V(G[K])$. Moreover, x_1 has a neighbor w_3 in $V(G[K])$, so the edge x_1w_3 is a shooter edge and hence $x_1 \in V(C_2)$. Similarly, vertices x_2 and x_3 lie in $V(C_2)$. (See Figure 3.23.)

Now recalling what we know to this point, C_1 has 2-shooter v_0 adjacent to w_0 and w_2, and each of v_2, v_3, v_4 is a 1-shooter adjacent to w_2, w_3 and w_4, respectively. Moreover, recall that vertex v_1 is a 0-shooter. Turning our attention to cycle C_2, we note that x is a 2-shooter adjacent to w_2 and w_0, whereas x_1, x_2 and x_3 are each 1-shooters adjacent to w_3, w_4 amd w_0, respectively. (The fifth vertex of C_2 must be a 0-shooter.) This leads us to conclude that $P_1 = P_2$.

So all ten shooter edges from G_1 terminate in $V(P_1)$ and these edges, together with the three edges of P_1, suffice to account for the degree of each of the four vertices in P_1 being four. But then, since G is connected, it must be that $P_1 = G[K]$, thus

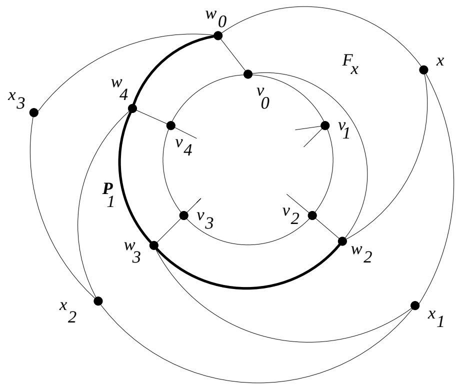

Figure 3.23.

contradicting the assumption that $|E(G[K])| = 5$.

Hence $x \notin V(G_1)$ and hence x must be a singleton 4-shooter. Let y be the fourth neighbor of w_3; that is, $xy \in E(G)$. Hence $y \in K$ and $w_3y \in E(G[K])$. So w_3 is not adjacent to any vertex of cycle C_2. It then follows that $E(P_1) \cap E(P_2) \subseteq \{w_0w_4\}$. So $|E(P_1) \cup E(P_2)| \geq 6 - |E(P_1) \cap E(P_2)| \geq 5$ and hence $|E(G[K])| \geq |E(P_1) \cup E(P_2)| + |\{w_3y\}| \geq 5 + 1 = 6$, a contradiction.

4.3. Finally, suppose that $\sigma = 0$; that is, every vertex of C_1 contributes exactly one shooter edge. But then it follows that $G[K]$ is a 5-cycle, again contradicting the assumption that $G[K]$ contains a matching of size 3.

Case 5. In this case, since $c\lambda(G) \geq 8$, there cannot be any 6-shooters among the odd components of $G - K$. Moreover, as argued in Case 3, there cannot be any 8-shooter odd components either. Hence there must be exactly one odd component which is a 12-shooter, say G_1, with all other odd components being 4-shooters and hence singletons. Also there are exactly four edges in $G[K]$.

If G_1 were plane, then by Lemma 3.2 it would have to be one of six graphs, none of which is factor-critical. Hence G_1 is not plane.

Suppose G_1 is non-planar. Then by Lemma 2.1, $G - V(G_1)$ is plane. If $G - V(G_1)$ has at least two components, then by Lemma 3.2, $G - V(G_1)$ has no matching of size 3, a contradiction. So $G - V(G_1)$ is connected and hence is a plane 12-shooter.

Suppose $G - V(G_1)$ contains no cycles. Then by Lemma 3.2, it must be one of the three trees on four edges. But none of these trees contains a matching of size 3 and we have a contradiction.

Next suppose $G - V(G_1)$ contains a cycle and also a cutvertex. Then again by Lemma 3.2, there are seven possibilities for $G - G_1$. Four of these are even and hence impossible. The other three possible graphs each have seven vertices. Now recall that $G - V(G[K])$ consists of at least two odd components. Thus six of these seven vertices make up $V(G[K])$ and the seventh vertex must be a singleton 4-shooter. But then it follows that G contains a triangle and we have a contradiction.

So finally suppose $G - V(G_1)$ contains a cycle, but no cutvertex. Again, by

Lemma 3.2, there are three possibilities for $G - V(G_1)$. But two of these are even and hence impossible. So $G - V(G_1)$ must be the 9-vertex graph displayed in Figure 3.2.

All eight boundary vertices of G_1 have neighbors in G_1, so these eight vertices must lie in K. So $|E(G[K])| \geq 8$, contradicting the fact that $|E(G[K])| = 4$.

So we may assume that G_1 is planar, but not plane, that is, it is cylindrical. As usual, we denote the two boundary walks by C_1 and C_2. For $i = 1, 2$, let ℓ_i denote the length of C_i. Since G is triangle-free, by Lemma 3.4 we have $\ell_1 + \ell_2 = 12$.

If both ℓ_1 and ℓ_2 are even, it follows that G_1 is bipartite and hence not factor-critical, a contradiction. Furthermore, G contains no triangles, so it must be the case that one of ℓ_1 and ℓ_2 is 5 and the other 7. Let us suppose, then, that $C_1 = v_0v_1v_2v_3v_4v_0$ is the 5-cycle. Then arguing as in Case 4, we obtain the configuration shown in Figure 3.23, where three of the four edges of $G[K]$ constitute the path P_1.

Consider the vertices x, x_1, x_2, x_3. None of them belong to K. Otherwise, the four edges of $G[K]$ would induce a tree which does not contain a matching of size 3, a contradiction. So these four vertices all belong to $V(G) - V(G[K])$ and it then follows immediately that the path $xx_1x_2x_3$ is part of C_2. Let z be another common neighbor (different from w_0) of x_3 and x. Since xw_0 and xw_2 are both shooter edges, the edge zx belongs to C_2. But since G_1 is an induced subgraph, it follows that C_2 is a 5-cycle $xx_1x_2x_3zx$, contradicting the original assumption that C_2 is a 7-cycle. Thus Case 5 cannot occur.

Case 6. In this final case, we first observe that by Case 2 we may suppose that there is no odd component which is a 6-shooter and by Case 3 there is no odd component which is an 8-shooter either. The only remaining possibility is that there is a 14-shooter, call it G_1, and the remaining $|K| - 5$ odd components are all 4-shooters and hence singletons. Also there are precisely three edges in $G[K]$ and these must be the original matching $\{e_1, e_2, e_3\}$.

Suppose first that G_1 is a plane graph. Then by Lemma 3.2, G_1 is the 11-vertex graph shown in Figure 3.3. But this graph is not factor-critical, a contradiction.

Suppose next that G_1 is non-planar. Then by Lemma 2.1, $G - V(G_1)$ is plane.

If $G - V(G_1)$ has at least two components, then one of them is a 10-shooter by Lemma 3.2 and the fact that $G[K] \subseteq G - V(G_1)$ contains a matching of size 3.

By Lemma 3.2 again, the 10-shooter Z is the union of two quadrangles sharing a single edge. Note that every vertex of Z has a neighbor in G_1. So $E(Z) \subseteq E(G[K])$ and hence $|E(G[K])| \geq |E(Z)| = 7$, thus contradicting the fact that $|E(G[K])| = 3$. Hence $G - V(G_1)$ is connected and hence is a plane 14-shooter.

Thus, again by Lemma 3.2, $G - V(G_1)$ must be the 11-vertex graph shown in Figure 3.3. Six of these eleven vertices must be the vertices of the original matching of size 3. The only remaining possibility is that the remaining five vertices consist of two vertices in $G[K]$ and three singleton 4-shooters. But the graph $G - V(G_1)$ shown in Figure 3.3 has only two vertices of degree 4, a contradiction.

Thus G_1 must be an annular graph. As before, let the two boundary walks be denoted by C_1 and C_2 and let ℓ_i denote the length of C_i, for $i = 1, 2$. As argued previously, $\ell_1 + \ell_2 = 14$. So without loss of generality, we may suppose that $\{\ell_1, \ell_2\}$ is one of $\{3, 11\}, \{4, 10\}, \{5, 9\}, \{6, 8\}$ or $\{7, 7\}$. First of all $\{\ell_1, \ell_2\} \neq \{3, 11\}$ since G is triangle-free. Furthermore, if $\{\ell_1, \ell_2\} = \{4, 10\}$ or $\{6, 8\}$, G_1 would be bipartite, a contradiction.

Next, suppose C_1 is a 5-cycle and C_2, a 9-cycle. Once again, fixing our attention on the 5-cycle C_1 and arguing as in Case 4, we obtain that $G[K]$ is a path of length 3, contradicting the fact that in this Case, $G[K]$ consists three independent edges.

Thus it remains to treat only the case when both C_1 and C_2 are 7-cycles. To deal with this case, we begin by stating and proving three claims.

Claim 1: For any edge $e \in E(G[K])$, let F_e^1 and F_e^2 be the two faces of G containing e. Then both F_e^1 and F_e^2 contain an edge belonging to $E(C_1) \cup E(C_2)$.

To prove this claim, suppose $e = v_1v_2 \in E(G[K])$ and denote $F_e^1 = v_1v_2v_3v_4v_1$, where v_4v_1 and v_3v_2 are shooter edges. Then v_3v_4 belongs to some odd component of $G - V(G[K])$. Note that all components of $G - V(G[K])$, except G_1, are singletons and hence 4-shooters. So $\{v_3, v_4\} \subseteq V(G_1)$ and $v_3v_4 \in E(G_1)$. Suppose that $v_3 \in V(C_i) - V(C_j)$ and $v_4 \in V(C_j) - V(C_i)$, $i \neq j$. Then all faces of G_1 containing the edge v_3v_4 in the embedding of G_1 inherited from G belong to G_1. But this contradicts the fact that $v_1v_2v_3v_4v_1$ is a face of G where $v_1v_2 \in E(G[K])$. This completes the proof of Claim 1.

Now suppose $e = xy \in E(G[K])$ and again suppose F_e^1 and F_e^2 be the faces containing e.

Claim 2: If $E(F_e^1) \cap E(C_i) \neq \emptyset$, then $E(F_e^2) \cap E(C_i) = \emptyset$.

The proof is by contradiction. Without loss of generality, let $C_i = C_1$. Suppose both $E(F_e^1) \cap E(C_1) \neq \emptyset$ and $E(F_e^2) \cap E(C_1) \neq \emptyset$. Denote C_1 by $v_0v_1v_2v_3v_4v_5v_6v_0$. Since G contains no triangles, we may assume that $E(F_e^1) \cap E(C_1) = \{v_0v_1\}$ and $E(F_e^2) \cap E(C_1) = \{v_4v_5\}$.

Note that $G_1 \cup (F_e^1 \cup F_e^2)$ contains three boundary walks, namely, $C_2, C_1^1 = v_0xv_5v_6v_0$ and $C_1^2 = v_1yv_4v_3v_2v_1$. (See Figure 3.24.)

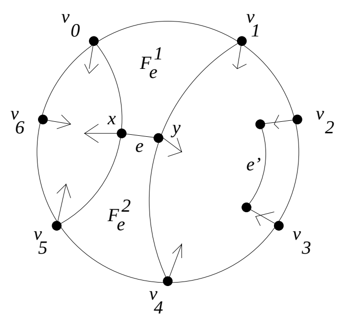

Figure 3.24.

We now assert that one of C_1^1 and C_1^2 must be contractible. To prove this, we appeal to the following topological result.

Lemma 3.5. *Let S be a compact connected subsurface of a torus T having genus zero with three boundary components. Then at least one of the three boundary components bounds a disk in T.*

Proof. [1] To see this, note that the boundary components of S subdivide T into connected subsurfaces $S_1, \dots, S_k$ with $S = S_1$, and each subsurface has a nonempty boundary consisting of at least one of the boundary components of S. The Euler characteristic of T is the sum of the Euler characteristics of $S_1, \dots, S_k$, since the Euler characteristic of the circle is 0. The Euler characteristic of T is 0 and the Euler characteristic of S is -1; hence at least one of the S_i has a positive Euler characteristic. But the only compact connected surface with nonempty boundary that has a positive Euler characteristic is the disk which has characteristic 1. Therefore, one of the boundary components of S bounds a disk as claimed. □

Now since C_2 is not contractible, one of C_1^1 and C_1^2 must be contractible by the preceding Lemma. Without loss of generality, let us assume that C_1^1 is contractible and let G' denote the plane graph bounded by C_1^1. Since G contains no triangles or multiple edges, $\delta(\{x, y\}, V(C_1)) = \emptyset$. It then follows that $G' - V(C_1^i) \neq \emptyset$. Since G' is a plane graph and G is 4-connected, at least four vertices on C_1^i send edges to $G' - V(C_1^i)$. So there are two consecutive vertices on C_1 emitting shooter edges going to $G' - V(C_1^i)$. So $G' - V(C_1^i)$ contains an edge e' from $E(G[K])$. By Claim 1, both $F_{e'}^i$ and $F_{e'}^2$ contain an edge from $E(C_1) \cup E(C_2)$. Note that $|C_1^i \cap C_1| \leq 3$ and $E(F_{e'}^i) \cap (E(C_1) \cup E(C_2)) \subseteq C_1^i$. But this implies that either G' contains a triangle or it contains multiple edges, a contradiction. This proves Claim 2.

Note that $|E(G[K])| = 3$. So let us denote $E(G[K])$ by $\{e_1, e_2, e_3\}$ and let $F_{e_i}^1$ and $F_{e_i}^2$ be the two faces containing e_i, for $i = 1, 2, 3$. By Claim 2 we may assume that $F_{e_i}^j$ contains an edge from $E(C_j)$, for $j = 1, 2$.

Claim 3: $|(\bigcup_{e \in E(G[K])} E(F_e^j)) \cap E(C_j)| = 3$, for $j \in \{1, 2\}$.

By Claim 2, $|(\bigcup_{e \in E(G[K])} E(F_e^j)) \cap E(C_j)| \leq 3$. Note that the shooter edges contained in the boundaries of $F_{e_i}^j$, $i = 1, 2, 3$, cannot cross each other in the embedding of G. So for any two faces $F_{e_i}^j$ and $F_{e_k}^j$, $i \neq k$, it follows that $E(F_{e_i}^j) \cap E(C_j) \neq E(F_{e_k}^j) \cap E(C_j)$. So $|(\bigcup_{e \in E(G[K])} E(F_e^j)) \cap E(C_j)| \geq 3$. Hence $|(\bigcup_{e \in E(G[K])} E(F_e^j) \cap E(C_j)| = 3$ and Claim 3 is proved.

Now, recalling that $C_1 = v_0v_1v_2v_3v_4v_5v_6v_0$ and $E(G[K]) = \{e_1, e_2, e_3\}$, we assert that $\bigcup_{i=1}^3 (F_{e_i}^1 \cap C_1)$ is a path of length 3. If it is not a path, then it contains at least two components, one of which is an isolated edge. Without loss of generality, assume that the isolated edge of $\bigcup_{i=1}^3 (F_{e_i}^1 \cap C_1)$ is v_0v_6 and assume that $F_{e_1}^1$ contains v_0v_6. Since v_0v_6 is an isolated edge in $\bigcup_{i=1}^3 (F_{e_i}^1 \cap C_1)$, both v_1 and v_5 contribute no shooter edges to C_1. By Claim 1 and Claim 2, both $F_{e_2}^1$ and $F_{e_3}^1$ contain an edge from $E(C_1)$. So $\bigcup_{i=2}^3 (F_{e_i}^1 \cap C_1) = v_2v_3v_4$.

[1] We are indebted to John Ratcliffe for this proof.

Since $E(G[K])$ is a matching, v_3 contributes two shooter edges from C_1 and both v_2 and v_4 contribute at least one shooter edge from C_1. (See Figure 3.25.)

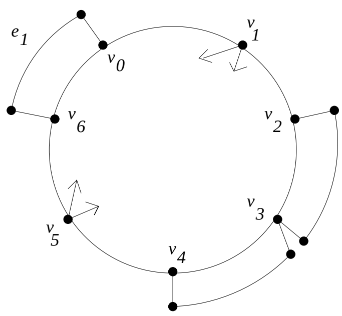

Figure 3.25.

If both v_2 and v_4 contribute one shooter edge from C_1, then we have the configuration shown in Figure 3.26. But then we have a triangle in G, a contradiction.

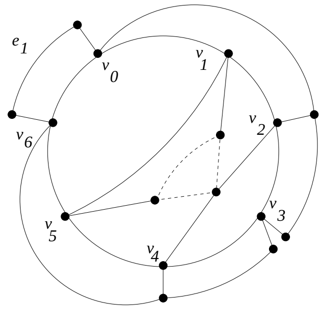

Figure 3.26.

Note that at most one of v_2 and v_4 contributes two shooter edges from C_1. For otherwise, the face containing $v_2v_3v_4$ would have size at least 5, a contradiction. So it follows that one of v_2 and v_4 contributes two shooter edges associated with C_1; suppose without loss of generality, it is v_4. Then we have one of the two configurations shown in Figure 3.27. But both of these contain a triangle and we have a contradiction. Thus our assertion is proved.

So henceforth we will assume that $\bigcup_{i=1}^{3}(F_{e_i}^1 \cap C_1) = v_0v_6v_5v_4$. It then follows that both v_6 and v_5 contribute two shooter edges from C_1, since $E(G[K])$ is a matching of size 3. Moreover, neither v_1 nor v_3 contributes any shooter edge from C_1.

Now consider the face containing $v_0v_6v_5v_4v_0$. Both v_0v_4 and v_1v_3 belong to $E(G)$ because the face containing $v_1v_0v_4v_3$ contains the edge v_1v_3 as well. (See Figure 3.28.) But then $\{v_2, v_3, v_1\}$ spans a triangle in G, a contradiction. So Case 6 cannot occur and hence the proof of Theorem 3.3 is complete. □

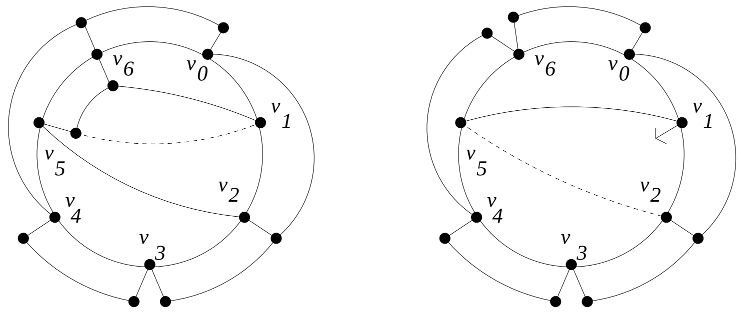

Figure 3.27.

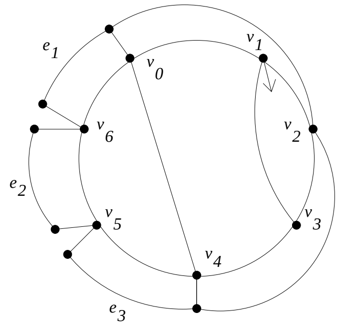

Figure 3.28.

4 A Closing Remark

In [13] it was proved that if G is a toroidal graph, then either it is *not* 3-extendable or else it is a 4-regular quadrangulation. This result, taken together with Theorem 3.3 of the present paper, shows that we have now obtained the complete list of *all* toroidal graphs which are 3-extendable.

5 Appendix

For the remaining classes, to specifically describe the three matching edges in G which do not extend to a perfect matching, we adopt the labeling of $Q(m, n, t)$ shown in Figure 5.1. Also, with the exception of cases (4) and (5), we have a pair of defining graphs in each case. As these pairs are symmetric, for brevity, we will supply a matching M of size 3 and a Tutte set S for $G - V(M)$ for only one of the two graphs in each pair. The other member of the pair has a symmetric choice for M and S which we leave to the reader to supply.

(2) If m is odd, then G is not 2-extendable by Theorem 2.7 and hence not 3-extendable. If m is even, then for the graph $Q(m, 2, 2)$ we choose $M = \{q_{2,2}q_{2,3}, q_{1,2}q_{1,4}, q_{2,5}q_{2,6}\}$ and $S = \{q_{1,6}\}$.

(3) If $\frac{m}{2}$ is odd, then it is not 2-extendable by Theorem 2.7 and hence not 3-

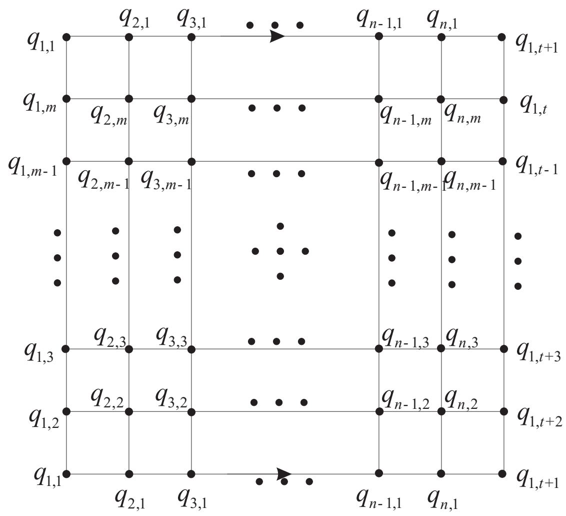

Figure 5.1.

extendable. If $\frac{m}{2}$ is even, then for the graph $Q(m, 1, \frac{m}{2} - 1)$ we choose $M = \{q_{1,1}q_{1,m}, q_{1,\frac{m}{2}}q_{1,\frac{m}{2}+1}, q_{1,3}q_{1,4}\}$ and $S = \{q_{1,\frac{m}{2}+3}\}$.

(4) Let $M = \{q_{2,1}q_{2,2}, q_{3,2}q_{3,3}, q_{4,1}q_{4,2}\}$ and $S = \{q_{1,2}, q_{1,3}\} \cup \{q_{j,i} \mid j = 1, 3, \text{and } 5 \leq i \leq m, i \text{ odd }\} \cup \{q_{j,i} \mid j = 2, 4, \text{and } 4 \leq i \leq m - 1, i \text{ even}\}$.

(5) Let $M = \{q_{1,m}q_{1,1}, q_{1,\frac{m}{2}}q_{1,\frac{m}{2}+1}, q_{2,\frac{m}{2}-1}q_{2,\frac{m}{2}}\}$ and $S = \{q_{1,i} \mid \frac{m}{2} + 3 \leq i \leq m - 2, i \text{ even}\} \cup \{q_{2,i}xi \mid |\frac{m}{2} + 2 \leq i \leq m - 1, i \text{ odd}\} \cup \{q_{1,i} \mid 3 \leq i \leq \frac{m}{2} - 2, i \text{ odd}\} \cup \{q_{2,i} \mid 2 \leq i \leq \frac{m}{2} - 3, i \text{ even}\}$.

(6) For the graph $Q(m, 1, \frac{m}{4})$, let $M = \{q_{1,\frac{m}{2}-1}q_{1,\frac{m}{2}}, q_{1,\frac{3m}{4}-2}q_{1,\frac{3m}{4}-1}, q_{1,m-1}q_{1,m}\}$ and $S = \{q_{1,\frac{m}{4}-2}, q_{1,\frac{m}{4}-1}\} \cup \{q_{1,i} \mid 2 \leq i \leq \frac{m}{4} - 4, i \text{ even}\} \cup \{q_{1,i} \mid \frac{m}{4} + 1 \leq i \leq \frac{m}{2} - 3, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{m}{2} + 2 \leq i \leq \frac{3m}{4} - 4, i \text{ even}\} \cup \{q_{1,i} \mid \frac{3m}{4} + 1 \leq i \leq m - 3, i \text{ odd}\}$.

(7) For the graph $Q(m, 4, 2)$, let $M = \{q_{1,1}q_{1,m}, q_{2m-1}q_{2m}, q_{3,1}q_{3,m}\}$ and $S = \{q_{ij} \mid i = 1, 3 \text{ and } 3 \leq j \leq m-2, j \text{ odd}\} \cup \{q_{ij} \mid i = 2, 4 \text{ and } 2 \leq j \leq m-3, j \text{ even}\} \cup \{q_{4,m-1}\}$.

(8) For the graph $Q(m, 2, \frac{m}{2} + 1)$, let $M = \{q_{1,m}q_{1,1}, q_{2,m}q_{2,m-1}, q_{1,\frac{m}{2}+1}q_{1,\frac{m}{2}+2}\}$ and $S = \{q_{2,\frac{m}{2}}, q_{2,\frac{m}{2}+1}\} \cup \{q_{1,i} \mid 3 \leq i \leq \frac{m}{2} - 1, i \text{ odd}\} \cup \{q_{2,i} \mid 2 \leq i \leq \frac{m}{2} - 2, i \text{ even}\} \cup \{q_{1,i} \mid \frac{m}{2} + 4 \leq i \leq m - 2, i \text{ even}\} \cup \{q_{2,i} \mid \frac{m}{2} + 3 \leq i \leq m - 3, i \text{ odd}\}$.

(9) For the graph $Q(m, 1, \frac{m+2}{4})$, let $M = \{q_{1,m}q_{1,1}, q_{1,\frac{m-2}{4}}q_{1,\frac{m+2}{4}}, q_{1,\frac{m}{2}+1}q_{1,\frac{m}{2}+2}\}$ and $S = \{q_{1,\frac{3m+2}{4}}, q_{1,\frac{3m+6}{4}}\} \cup \{q_{1,i} \mid 3 \leq i \leq \frac{m-2}{4} - 2, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{3m+2}{4} + 3 \leq i \leq m-2, i \text{ even}\} \cup \{q_{1,i} \mid \frac{m}{2} + 4 \leq i \leq \frac{3m}{4} - 6, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{m+2}{4} + 2 \leq i \leq \frac{m}{2} - 1, i \text{ even}\}$.

(10) For the graph $Q(m, 1, \frac{m-2}{4})$, let $M = \{q_{1,m}q_{1,1}, q_{1,\frac{3m-2}{4}}q_{1,\frac{3m+2}{4}}, q_{1,\frac{m}{2}+1}q_{1,\frac{m}{2}+2}\}$ and $S = \{q_{1,\frac{m+2}{4}}, q_{1,\frac{m+2}{4}+1}\} \cup \{q_{1,i} \mid 3 \leq i \leq \frac{m+2}{4} - 2, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{3m+2}{4} + 2 \leq i \leq m-2, i \text{ even}\} \cup \{q_{1,i} \mid \frac{m}{2} + 4 \leq i \leq \frac{3m-2}{4} - 2, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{m+2}{4} + 3 \leq i \leq \frac{m}{2} - 1, i \text{ even}\}$.

(11) For the graph $Q(m, 4, 4)$, let $M = \{q_{1,1}q_{1,2}, q_{2,2}q_{2,3}, q_{3,3}q_{3,4}\}$ and $S = \{q_{4,1}, q_{1,3}\} \cup \{q_{1,i} \mid 5 \leq i \leq m, i \text{ odd}\} \cup \{q_{2,i} \mid 4 \leq i \leq m - 1, i \text{ even}\} \cup \{q_{3,i} \mid$

$5 \leq i \leq m, i \text{ odd}\} \cup \{q_{4,i} \mid 4 \leq i \leq m-1, i \text{ even}\}$.

(12) For the graph $Q(m, 2, \frac{m}{2} - 2)$, let $M = \{q_{1,m}q_{1,1}, q_{2,m}q_{2,m-1}, q_{1,\frac{m}{2}}q_{1,\frac{m}{2}+1}\}$ and $S = \{q_{2,\frac{m}{2}-1}, q_{2,\frac{m}{2}}\} \cup \{q_{1,i} | 3 \leq i \leq \frac{m}{2} - 2, i \text{ odd}\} \cup \{q_{2,i} \mid 2 \leq i \leq \frac{m}{2} - 3, i \text{ even}\} \cup \{q_{1,i} \mid \frac{m}{2} + 3 \leq i \leq m-2, i \text{ even}\} \cup \{q_{2,i} \mid \frac{m}{2} + 2 \leq i \leq m-3, i \text{ odd}\}$.

(13) For the graph $Q(m, 1, \frac{m}{4} + 1)$, let $M = \{q_{1,m}q_{1,1}, q_{1,\frac{3m}{4}}q_{1,\frac{3m}{4}+1}, q_{1,\frac{m}{2}}q_{1,\frac{m}{2}+1}\}$ and $S = \{q_{1,\frac{m}{4}}, q_{1,\frac{m}{4}+1}\} \cup \{q_{1,i} \mid 3 \leq i \leq \frac{m}{4} - 2, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{3m}{4} + 3 \leq i \leq m-2, i \text{ even}\} \cup \{q_{1,i} \mid \frac{m}{2} + 3 \leq i \leq \frac{3m}{4} - 2, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{m}{4} + 3 \leq i \leq \frac{m}{2} - 2, i \text{ even}\}$.

(14) For the graph $Q(m, 1, \frac{m}{4} - 1)$, let $M = \{q_{1,1}q_{1,2}, q_{1,\frac{m}{4}+1}q_{1,\frac{m}{4}+2}, q_{1,\frac{m}{2}+1}q_{1,\frac{m}{2}+2}\}$ and $S = \{q_{1,\frac{m}{4}}, q_{1,\frac{m}{4}+1}\} \cup \{q_{1,i} \mid 3 \leq i \leq \frac{m}{4}, i \text{ even}\} \cup \{q_{1,i} \mid \frac{m}{4} + 3 \leq i \leq \frac{m}{2}, i \text{ odd}\} \cup \{q_{1,i} \mid \frac{m}{2} + 3 \leq i \leq \frac{3m}{4}, i \text{ even}\} \cup \{q_{1,i} \mid \frac{3m}{4} + 3 \leq i \leq m, i \text{ odd}\}$.

(15) For the graph $Q(m, 2, 4)$, let $M = \{q_{1,1}q_{2,1}, q_{2,2}q_{1,m-2}, q_{1,m-1}q_{2,m-1}\}$ and $S = \{q_{2,4}, q_{1,m}\} \cup \{q_{1,i} | 3 \leq i \leq m-4, i \text{ odd}\} \cup \{q_{2,i} \mid 6 \leq i \leq m-3, i \text{ even}\}$.

(16) For the graph $Q(m, 1, \frac{m}{2} - 2)$, let $M = \{q_{1,\frac{m}{2}-1}q_{1,1}, q_{1,\frac{m}{2}}q_{1,m-2}, q_{1,m-1}q_{1,m}\}$ and $S = \{q_{1,i} \mid \frac{m}{2} + 2 \leq i \leq m-4, i \text{ even}\} \cup \{q_{1,i} \mid 3 \leq i \leq \frac{m}{2} - 3, i \text{ odd}\}$.

(17) For the graph $Q(m, 3, 2)$, let $M = \{q_{1,1}q_{1,2}, q_{2,2}q_{2,3}, q_{3,3}q_{1,5}\}$ and $S = \{q_{1,3}, q_{2,4}, q_{3,m}\}$.

(18) For the graph $Q(m, 1, \frac{m-2}{3})$, let

$$M = \{q_{1,\frac{m-2}{3}}q_{1,m}, q_{1,\frac{m-5}{3}}q_{1,\frac{2m-7}{3}}, q_{1,\frac{2m-10}{3}}q_{1,m-4}\}, S = \{q_{1,m-2}, q_{1,\frac{m-11}{3}}, q_{1,\frac{2m-1}{3}}\}.$$

(19) For the graph $Q(m, 1, \frac{m+2}{3})$, let

$$M = \{q_{1,\frac{m-1}{3}}q_{1,m-1}, q_{1,\frac{2m-8}{3}}q_{1,m-2}, q_{1,\frac{m-13}{3}}q_{1,\frac{2m-11}{3}}\}, S = \{q_{1,\frac{2m-2}{3}}, q_{1,m-4}, q_{1,\frac{m-7}{3}}\}.$$

(20) For the graph $Q(m, 2, m-3)$, let

$$M = \{q_{1,m-4}q_{1,m-5}, q_{2,m-3}q_{2,m-4}, q_{2,m-8}q_{2,m-9}\}, S = \{q_{1,m-2}, q_{1,m-7}, q_{2,m-6}\}.$$

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(Received 2 Mar 2015; revised 22 Aug 2015)