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The non-singularity of looped-trees and complement of trees with diameter 5

Nirutt Pipattanajinda

Program of Mathematics, Faculty of Sciences and Technology
Kamphaeng Phet Rajabhat University
Kamphaeng Phet
Thailand
nirutt.p@gmail.com

Yangkok Kim

Department of Mathematics
Dongeui University, Busan 614-714
Korea
ykkim@deu.ac.kr

Abstract

A graph G is said to be singular if its adjacency matrix is singular; otherwise it is said to be non-singular. In this paper, we introduce a class of graphs called looped-trees, and find the determinant and the non-singularity of looped-trees. Moreover, we determine the singularity or non-singularity of the complement of a certain class of trees with diameter 5 by using the results for looped-trees.

1 Introduction and Preliminaries

Non-singular trees were completely characterized by Gervacio and Rara [2]. Furthermore, the singularity or non-singularity of the complement of a tree with diameter less than 5 was completely determined by Gervacio [1]. Recently, Pipattanajinda and Kim [7] obtained the determinant of the complement of a tree with diameter 5, and determined the singularity or non-singularity of the complement of a certain class of trees with diameter 5. In this paper we shall introduce a class of graphs called looped-trees and determine the singularity of looped-trees with diameter less than or equal to 5. Moreover, we shall solve the singularity or non-singularity problem of the complement of a certain class of trees with diameter 5. In Section 2, we shall give the formula for the determinant of looped-trees with diameter less than 5. We note that an adjacency matrix of a looped-tree is also a neighborhood matrix of a tree (for details, see [5]). We then determine the singularity or non-singularity of the complement of a looped-tree with diameter 4. Furthermore, the determinant and the non-singularity of looped-trees with diameter 5 will be solved in Section 3. In final section we find some relation between determinants of a looped-tree and the complement of a tree with diameter 5 by using the results in [4], and determined the singularity or non-singularity of the complement of a certain class of trees with diameter 5.

By a graph G we mean a pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of elements called vertices and $E(G)$ is a set of 2-subsets of $V(G)$ whose elements are called edges. In particular, G is a simple graph if it has no loops (edges connected at both ends to the same vertex) and no more than one edge between any two different vertices. For a simple graph $G = (V(G), E(G))$, a graph $G^o = (V(G^o), E(G^o))$ with $V(G^o) = V(G)$ and $E(G^o) = \{\{u, v\} | \{u, v\} \in E(G)\} \cup \{\{u, u\} | u \in V(G^o)\}$ is called a looped-graph of G. In particular, if G is a tree, G^o is called a looped-tree.

If G is a graph with vertices $x_1, x_2, \ldots, x_n$, we define the adjacency matrix of G to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $\{x_i, x_j\} \in E(G)$ and $a_{ij} = 0$ otherwise. The graph G is said to be singular if $A(G)$ is singular, i.e., $detA(G) = 0$; otherwise G is said to be non-singular. If $S \subset V(G)$, then $G \backslash S$ denotes the graph obtained from G by deleting all the vertices $x \in S$. The complement $\overline{G}$ of G is a graph such that $V(\overline{G}) = V(G)$ and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$ for any $u, v \in V(G)$ and $u \neq v$. The loop complement $\overline{G}^o$ of G is a graph such that $V(\overline{G}^o) = V(G)$ and $\{u, v\} \in E(\overline{G}^o)$ if and only if $\{u, v\} \notin E(G)$ for any $u, v \in V(G)$. When G is a simple graph, the loop complement of G is the complement of G^o, that is, $\overline{G}^o = \overline{G^o}$, and the loop complement of G^o is the complement of G, that is, $\overline{G^o}^o = \overline{G}$. Other terms whose definitions are not given here may be found in many graph theory books, e.g., [3].

For non-negative integers $m, r, s, m_1, m_2, \cdots, m_r, n_1, n_2, \cdots, n_s$, we define a series of looped-trees, $T^o_{2:m}, T^o_{3:r,s}, T^o_{4:m_1,m_2,\ldots,m_r}$, and $T^o_{5:m_1,\cdots,m_r;n_1,\cdots,n_s}$ as follows: by $T^o_{2:m}$ we mean a looped-tree of a tree with diameter ≤ 2, which is depicted in Figure 1 where w is called the central vertex, and m is the number of vertices but the central vertex w. For two disjoint looped-trees $T^o_{2:r}, T^o_{2:s}$, with central vertices x_0, y_0 respectively, we form a looped-tree $T^o_{3:r,s}$ by joining two central vertices as shown in Figure 1, where x_0, y_0 are called central vertices of $T^o_{3:r,s}$. For disjoint looped-trees $T^o_{2:m_1}, T^o_{2:m_2}, \ldots, T^o_{2:m_r}$, with central vertices $x_1, x_2, \ldots, x_r$ respectively, we form a looped-tree $T^o_{4:m_1,m_2,\ldots,m_r}$ by joining all central vertices x_i to a new vertex z (see Figure 2 where z is called the central vertex of $T^o_{4:m_1,m_2,\ldots,m_r}$). Similarly for two disjoint looped-trees $T^o_{4:m_1,m_2,\ldots,m_r}, T^o_{4:n_1,n_2,\ldots,n_s}$, with central vertices x_0, y_0 respectively, we form a looped-tree $T^o_{5:m_1,\cdots,m_r;n_1,\cdots,n_s}$ by joining two central vertices (see Figure 3 where x_0, y_0 are called central vertices of $T^o_{5:m_1,\cdots,m_r;n_1,\cdots,n_s}$).

From the construction, we have the following:

(i) if $m \geq 2$, then $T^o_{2:m}$ is a looped-tree with diameter 2;

(ii) if $rs \neq 0$, then $T^o_{3:r,s}$ is a looped-tree with diameter 3;

(iii) if $r > 2$ and $m_i m_j \neq 0$ for two distinct i, j, then $T^o_{4:m_1,m_2,\ldots,m_r}$ is a looped-tree with diameter 4; and

(iv) if $r, s > 1$ and $m_i n_j \neq 0$ for some i, j, then $T^o_{5:m_1,\cdots,m_r;n_1,\cdots,n_s}$ is a looped-tree with diameter 5.

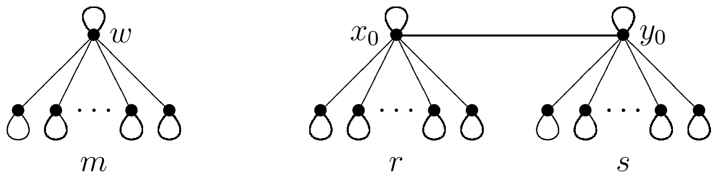

Figure 1: $T^{\circ}_{2:m}$ and $T^{\circ}_{3:r,s}$

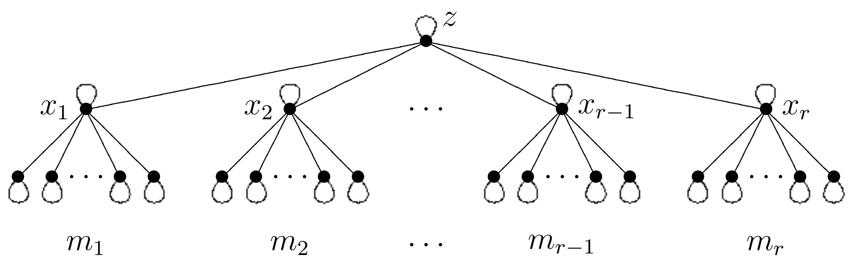

Figure 2: $T^{\circ}_{4:m_1,m_2,\cdots,m_r}$

Moreover, we note that $T^o_{4:m} = T^o_{2:m+1}, T^o_{3:r,0} = T^o_{2:r+1}, T^o_{4:m,\underbrace{0,\cdots,0}_{j}} = T^o_{3:m,j}$ and $T^o_{5:m_1,\cdots,m_r;\underbrace{0,\cdots,0}_{j}} = T^o_{4:m_1,m_2,\ldots,m_r,j}$.

From now on, for the simplicity of expressions, we denote the determinant of an adjacency matrix of a graph G by $|G|$ or $|A(G)|$, whenever there is no margin for confusion. For example, $|T^o_{4:m_1,m_2,\ldots,m_r}|$ means the determinant of an adjacency matrix of $T^o_{4:m_1,m_2,\ldots,m_r}$. Furthermore, we use $T^o_{4:\ldots,m_r}$ and $T^o_{5:\ldots,m_r;\ldots,n_s}$ for $T^o_{4:m_1,m_2,\ldots,m_r}$ and $T^o_{5:m_1,\ldots,m_r;n_1,\ldots,n_s}$. We now recall a definition of some graph and a lemma in [6] crucial for our further arguments. For any graph G with $x \in V(G)$ and $y \notin V(G)$, $G_{x\sim y^0}$ means the graph with $V(G_{x\sim y^0}) = V(G) \cup \{y\}$ and $E(G_{x\sim y^0}) = E(G) \cup \{\{x,y\},\{y,y\}\}$.

Lemma 1.1 *[6] Let $G = (G(V), G(E))$ be a graph, $x \in G(V)$ and $y \notin G(V)$. Then*

$$|G_{x\sim y^o}| = |G| - |G\backslash\{x\}|$$

Lemma 1.2 *(i) $|T^o_{2:m}| = 1 - m$; (ii) $|T^o_{3:r,s}| = rs - r - s$.*

Proof. (i) We can write $T^o_{2:m} = (T^o_{2:m-1})_{x\sim y^o}$ (see Figure 4) and apply Lemma 1.1 to get

$$|T^o_{2:m}| = |T^o_{2:m-1}| - 1.$$

By applying the same arguments repeatedly, we have

$$|T^o_{2:m}| = |T^o_{2:m-1}| - 1 = |T^o_{2:m-2}| - 2 = \cdots = |T^o_{2:2}| - (m-2) = |T^o_{2:1}| - (m-1) = 1 - m.$$

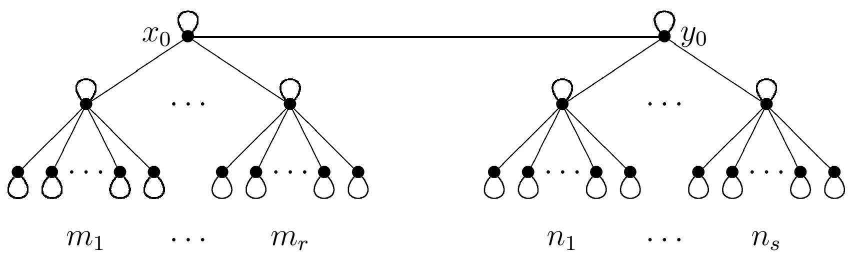

Figure 3: $T^{o}_{5:m_1,m_2,\cdots,m_r;n_1,n_2,\cdots,n_s}$

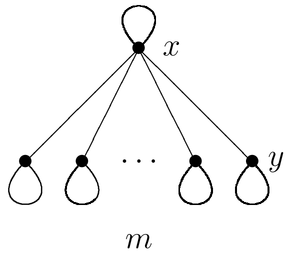

Figure 4: $T^o_{2:m} = (T^o_{2:m-1})_{x\sim y^o}$

(ii) By the same argument for (i),

$$\begin{aligned} |T^o_{3:r,s}| &= |T^o_{3:r,s-1}| - |T^o_{2:r}| = |T^o_{3:r,s-2}| - 2|T^o_{2:r}| \\ &= |T^o_{3:r,s-3}| - 3|T^o_{2:r}| = \cdots = |T^o_{3:r,0}| - s|T^o_{2:r}| \\ &= |T^o_{2:r+1}| - s|T^o_{2:r}| = 1 - (r+1) - s(1-r) = rs - r - s. \end{aligned}$$

□

From Lemma 1.2, we have the following.

Corollary 1.3 $T^o_{2:r}$ *is singular if and only if* $r = 1$.

Corollary 1.4 $T^o_{3:r,s}$ *is singular if and only if* $r = s = 2$.

2 Looped-Trees with diameter 4

Lemma 2.1 *For non-negative integers* $m_1, m_2, \cdots, m_r (r \geq 2)$,

$$|T^o_{4:\cdots,m_r}| = -(1-m_1)(1-m_2)\cdots(1-m_{r-1}) + (1-m_r)|T^o_{4:\ldots,m_{r-1}}|$$

Proof. We can write $T^o_{4:\ldots,m_r} = (T^o_{4:\ldots,m_r-1})_{x\sim y^o}$ (Figure 5, where x is adjacent to the central vertex of $T^o_{4:\ldots,m_{r-1}}$) and apply Lemma 1.1 to get

$$|T^o_{4:\ldots,m_r}| = |T^o_{4:\ldots,m_r-1}| - |T^o_{4:\ldots,m_{r-1}}|.$$

By applying the same arguments repeatedly, we have

$$|T^o_{4:\dots,m_r}| = |T^o_{4:\dots,m_{r-1},0}| - m_r|T^o_{4:\dots,m_{r-1}}|.$$

Now we write $T^o_{4:\dots,m_{r-1},0} = (T^o_{4:\dots,m_{r-1}})_{x\sim y^o}$ (see Figure 6, where y is adjacent to the central vertex x of $T^o_{4:\cdots,m_{r-1}}$) and apply Lemma 1.1 to get

$$|T^o_{4:\dots m_{r-1},0}| = |T^o_{4:\dots,m_{r-1}}| - |T^o_{2:m_1}||T^o_{2:m_2}|\dots|T^o_{2:m_{r-1}}|$$

and so

$$|T^o_{4:\dots,m_r}| = -(1-m_1)(1-m_2)\dots(1-m_{r-1}) + (1-m_r)|T^o_{4:\dots,m_{r-1}}|.$$

□

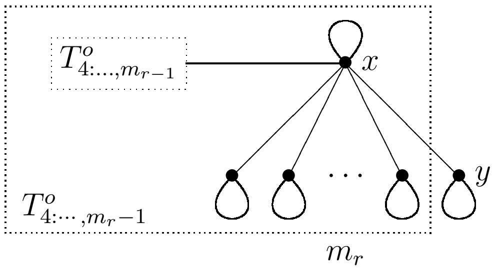

Figure 5: $T^o_{4:\cdots,m_r} = (T^o_{4:\cdots,m_r-1})_{x\sim y^o}$

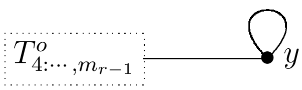

Figure 6: $T^o_{4:\cdots,m_{r-1},0} = (T^o_{4:\cdots,m_{r-1}})_{x\sim y^o}$

Theorem 2.2 *For non-negative integers* $m_1, m_2, \cdots, m_r$,

$$|T^o_{4:\cdots,m_r}| = \prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{1-m_i}.$$

Proof. By mathematical induction on r. Let $r=1$. By applying Lemma 1.2, we have

$$|T^o_{4:m_1}| = |T^o_{2,m_1+1}| = 1-(m_1+1) = (1-m_1)-1.$$

We assume that the formula works for $r-1$. Then by Lemma 2.1 and induction hypothesis,

$$\begin{aligned}
& |T^o_{4:\cdots,m_r}| \\
= \; & -(1-m_1)\cdots(1-m_{r-1}) + (1-m_r)|T^o_{4:\cdots,m_{r-1}}| \\
= \; & -(1-m_1)\cdots(1-m_{r-1}) \\
& +(1-m_r)\left((1-m_1)\cdots(1-m_{r-1}) - \sum_{i=1}^{r-1}\frac{(1-m_1)\cdots(1-m_{r-1})}{1-m_i}\right) \\
= \; & -(1-m_1)\cdots(1-m_{r-1}) + (1-m_1)\cdots(1-m_{r-1})(1-m_r) \\
& -(1-m_r)\sum_{i=1}^{r-1}\frac{(1-m_1)\cdots(1-m_{r-1})}{1-m_i} \\
= \; & \prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{1-m_i}.
\end{aligned}$$

□

Corollary 2.3 *For non-negative integers $m_{1,}\cdots,m_r,j$,*

$$|T^o_{4:\dots,m_r,\underbrace{0,\cdots,0}_{j}}| = (1-j)\prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}.$$

Theorem 2.4 *For positive integers $m_{1,}\cdots,m_r$, $T^o_{4:m_1,m_2,\cdots,m_r}$ is a singular graph if and only if at least two distinct m_i are* 1.

Proof. If at least two distinct m_i are 1, then $\prod_{i=1}^{r}(1-m_i)$ and $\sum_{i=1}^{r}\frac{(1-m_1)\cdots(1-m_r)}{1-m_i}$ of $|T^o_{4:\cdots,m_r}|$ in Theorem 2.2 are zero and so $|T^o_{4:\cdots,m_r}|=0$. For the converse, if only one m_i is 1, say $m_r=1$, then

$$|T^o_{4:\dots,m_{r-1},1}| = \prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)\cdots(1-m_r)}{(1-m_i)} = -(1-m_1)\cdots(1-m_{r-1})$$

which is clearly non-zero. If $m_i \neq 1$ for all i, then

$$\begin{aligned}
|T^o_{4:\cdots,m_r}| &= \prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)} \\
&= \prod_{i=1}^{r-1}(1-m_i) - \prod_{i=1}^{r-1}(1-m_i)m_r - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)} \\
&= -\prod_{i=1}^{r-1}(1-m_i)m_r - \sum_{i=1}^{r-1}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}
\end{aligned}$$

where $\prod_{i=1}^{r-1}(1-m_i)m_r$ and $\sum_{i=1}^{r-1}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}$ have the same sign, $(-1)^{r+1}$. Hence, $|T^o_{4:\cdots,m_r}|$ cannot be zero. □

Theorem 2.5 *For positive integers* $m_1, m_2, \cdots, m_r$, $T^o_{4:\cdots,m_r,0}$ *is a singular graph if and only if at least two distinct* m_i *are* 1.

Proof. By Corollary 2.3,

$$|T^o_{4:\cdots,m_r,0}| = -\sum_{i=1}^{r} \frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}$$

which is clearly zero if and only at least two distinct m_i are 1. □

We cannot extend Theorem 2.5 to the general case that more than one m_i are zeros as we see in the following Example.

Example 2.1 *For positive integers* m, r, j, $T^o_{4:\underbrace{m,\cdots,m}_{r},\underbrace{0,\cdots,0}_{j}}$ *is a singular graph if* $1 - m - j + jm - r = 0$.

Proof. By Corollary 2.3 with $m_1 = m_2 = \cdots = m_r = m$ and j times 0, we have

$$\begin{aligned} & |T^o_{4:\underbrace{m,\cdots,m}_{r},\underbrace{0,\cdots,0}_{j}}| \\ = \; & (1-j)\prod_{i=1}^{r}(1-m) - \sum_{i=1}^{r} \frac{(1-m)(1-m)\dots(1-m)}{(1-m)} \\ = \; & (1-j)(1-m)^r - r(1-m)^{r-1} = (1-m)^{r-1}((1-j)(1-m)-r) = 0 \end{aligned}$$

In particular, $T^o_{4:3,3,0,0}$ is singular. □

3 Looped-Trees with diameter 5

Lemma 3.1 *For non-negative integers* $m_1, \cdots, m_r, n_1, \cdots, n_s (s \geq 2)$,

$$|T^o_{5:\cdots,m_r;\cdots,n_s}| = (1-n_s)|T^o_{5:\cdots,m_r;\cdots,n_{s-1}}| - |T^o_{4:\cdots,m_r}|(1-n_1)(1-n_2)\cdots(1-n_{s-1})$$

Proof. We can write $T^o_{5:\cdots,m_r;\cdots,n_s} = (T^o_{5:\cdots,m_r;\cdots,n_s-1})_{x\sim y^o}$ (see Figure 7, where x is adjacent to the central vertex of $T^o_{4:\cdots,n_{s-1}}$) and apply Lemma 1.1 to get

$$|T^o_{5:\cdots,m_r;\cdots,n_s}| = |T^o_{5:\cdots,m_r;\cdots,n_s-1}| - |T^o_{5:\cdots,m_r;\cdots,n_{s-1}}|.$$

By applying the same argument repeatedly, we have

$$|T^o_{5:\cdots,m_r;\cdots,n_s}| = |T^o_{5:\cdots,m_r;\cdots,n_{s-1},0}| - n_s|T^o_{5:\cdots,m_r;\cdots,n_{s-1}}|.$$

Now we note that $T^o_{5:\cdots,m_r;\cdots,n_{s-1},0} = (T^o_{5:\cdots,m_r;\cdots,n_{s-1}})_{x\sim y^o}$ (see Figure 8, where y is adjacent to the central vertex x of $T^o_{4:\cdots,n_{s-1}}$) and apply Lemma 1.1 to get

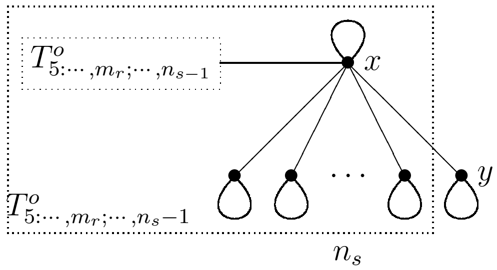

Figure 7: $T^o_{5:\cdots,m_r;\cdots,n_s} = (T^o_{5:\cdots,m_r;\cdots,n_s-1})_{x\sim y^o}$

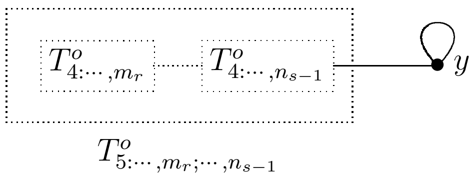

Figure 8: $T^o_{5:\cdots,m_r;\cdots,n_{s-1},0} = (T^o_{5:\cdots,m_r;\cdots,n_{s-1}})_{x\sim y^o}$

$$|T^o_{5:\cdots,m_r;\cdots,n_{s-1},0}| = |T^o_{5:\cdots,m_r;\cdots,n_{s-1}}| - |T^o_{4:\cdots,m_r}||T^o_{2,n_1}||T^o_{2,n_2}|\cdots|T^o_{2,n_{s-1}}|.$$

Hence,

$$|T^o_{5:\cdots,m_r;\cdots,n_s}| = (1-n_s)|T^o_{5:\cdots,m_r;\cdots,n_{s-1}}| - |T^o_{4:\cdots,m_r}|(1-n_1)(1-n_2)\cdots(1-n_{s-1}).$$

□

Theorem 3.2 *For non-negative integers $m_1,\cdots,m_r,n_1,\cdots,n_s$,*

$$|T^o_{5:\cdots,m_r;\cdots,n_s}| = -\prod_{j=1}^{s}(1-n_j)\prod_{i=1}^{r}(1-m_i) + |T^o_{4:\cdots,m_r}||T^o_{4:\cdots,n_s}|$$

Proof. By induction on s. Let $s = 1$. Then, by the same argument as in Lemma 3.1,

$$\begin{aligned}
& |T^o_{5:\cdots,m_r;n_1}| = |T^o_{5:\cdots,m_r;n_1-1}| - |T^o_{4:\cdots,m_r,0}| = |T^o_{5:\cdots,m_r;n_1-2}| - 2|T^o_{4:\cdots,m_r,0}| \\
= & \cdots = |T^o_{5:\cdots,m_r,0}| - n_1|T^o_{4:\cdots,m_r,0}| = |T^o_{4:\cdots,m_r,1}| - n_1|T^o_{4:\cdots,m_r,0}| \\
= & \left(|T^o_{4:\cdots,m_r,0}| - |T^o_{4:\cdots,m_r}|\right) - n_1|T^o_{4:\cdots,m_r,0}| = (1-n_1)|T^o_{4:\cdots,m_r,0}| - |T^o_{4:\cdots,m_r}| \\
= & (1-n_1)\left(|T^o_{4:\cdots,m_r}| - \prod_{i=1}^{r}(1-m_i)\right) - |T^o_{4:\cdots,m_r}| \\
= & -(1-n_1)\prod_{i=1}^{r}(1-m_i) - n_1|T^o_{4:\cdots,m_r}| \\
= & -(1-n_1)\prod_{i=1}^{r}(1-m_i) + |T^o_{4:\cdots,m_r}||T^o_{4:n_1}|.
\end{aligned}$$

We assume that the formula works for $s-1$. Then by Lemma 3.1 and induction hypothesis, we have

$$\begin{aligned}
& |T^o_{5:\cdots,m_r;\cdots,n_s}| \\
= & (1-n_s)|T^o_{5:\cdots,m_r;\cdots,n_{s-1}}| - |T^o_{4:\cdots,m_r}|(1-n_1)(1-n_2)\cdots(1-n_{s-1}) \\
= & (1-n_s)\left(-\prod_{j=1}^{s-1}(1-n_j)\prod_{i=1}^{r}(1-m_i) + |T^o_{4:\cdots,m_r}||T^o_{4:\cdots,n_{s-1}}|\right) \\
& -|T^o_{4:\cdots,m_r}|(1-n_1)(1-n_2)\cdots(1-n_{s-1}) \\
= & -\prod_{j=1}^{s}(1-n_j)\prod_{i=1}^{r}(1-m_i) \\
& +|T^o_{4:\cdots,m_r}|\left((1-n_s)|T^o_{4:\cdots,n_{s-1}}| - (1-n_1)(1-n_2)\cdots(1-n_{s-1})\right) \\
= & -\prod_{j=1}^{s}(1-n_j)\prod_{i=1}^{r}(1-m_i) + |T^o_{4:\cdots,m_r}||T^o_{4:\cdots,n_s}|.
\end{aligned}$$

where the last formula is obtained by applying Lemma 2.1. □

Theorem 3.3 *For non-negative integers $m_1, \cdots, m_r, n_1, \cdots, n_s$,*

$$\begin{aligned}
|T^o_{5:\cdots,m_r;\cdots,n_s}| = & \sum_{j=1}^{s}\sum_{i=1}^{r}\frac{(1-m_1)\cdots(1-m_r)(1-n_1)\cdots(1-n_s)}{(1-m_i)(1-n_j)} \\
& -\prod_{j=1}^{s}(1-n_j)\sum_{i=1}^{r}\frac{(1-m_1)\cdots(1-m_r)}{1-m_i} \\
& -\prod_{i=1}^{r}(1-m_i)\sum_{j=1}^{s}\frac{(1-n_1)\cdots(1-n_s)}{1-n_j}
\end{aligned}$$

Proof. By simple application of Theorems 3.2 and 2.2. □

Theorem 3.4 *For positive integers $m_1, \cdots, m_r, n_1, \cdots, n_s$, $T^o_{5:\cdots,m_r;\cdots,n_s}$ is a singular graph if and only if at least two distinct m_i are 1 or at least two distinct n_i are 1.*

Proof. If at least two distinct m_i are 1, or at least two distinct n_i are 1, then each term of $|T^o_{5:m_1,\cdots,m_r;n_1,\cdots,n_s}|$ in Theorem 3.2 is zero and so $|T^o_{5:\cdots,m_r;\cdots,n_s}| = 0$. For the converse, we need to consider two cases. (i) If none of m_i or n_j is 1, then we just note that three terms $\sum_{j=1}^s \sum_{i=1}^r *$, $-\prod_{j=1}^s(1-n_j)\sum_{i=1}^r *$, $-\prod_{i=1}^r(1-m_i)\sum_{j=1}^s *$ in the expression of $|T^o_{5:\cdots,m_r;\cdots,n_s}|$ have the same sign $(-1)^{r+s}$. Hence, the sum cannot be zero unless each of three terms is zero, which cannot happen. (ii) For the other case, when only one m_i or n_j is 1, or only one m_i and only one n_j are 1, then

$$\begin{aligned} |T^o_{5:\cdots,m_r;\cdots,n_s}| &= -\prod_{j=1}^s(1-n_j)\prod_{i=1}^r(1-m_i) + |T^o_{4:\cdots,m_r}||T^o_{4:\cdots,n_s}| \\ &= |T^o_{4:\cdots,m_r}||T^o_{4:\cdots,n_s}| \end{aligned}$$

which cannot be zero by Theorem 2.4. □

We cannot get the similar version of Theorem 2.5 for $T^o_{5:\cdots,m_r;\cdots,n_s}$ as we see in the following Examples.

Corollary 3.5 *For non-negative integers $m_1, m_2, \cdots, m_r, n_1, \cdots, n_s$,*

$$\begin{aligned} & |T^o_{5:\cdots,m_r;\cdots,n_s,0}| \\ = & -\prod_{i=1}^r(1-m_i)\left(\prod_{j=1}^s(1-n_j) + \sum_{j=1}^s \frac{(1-n_1)(1-n_2)\cdots(1-n_{s-1})}{1-n_j}\right) \\ & + \left(\sum_{i=1}^r \frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{1-m_i}\right)\sum_{j=1}^s \frac{(1-n_1)(1-n_2)\cdots(1-n_{s-1})}{1-n_j} \end{aligned}$$

Example 3.1 $T^o_{5:\underbrace{m,m,\cdots,m}_{r};\underbrace{n,n,\cdots,n}_{s},0}$ *is a singular graph if $n+m-s-mn+ms+rs-1=0$.*

Proof. By Corollary 3.5 and simple calculation gives

$$\begin{aligned} & |T^o_{5:\underbrace{m,m,\cdots,m}_{r};\underbrace{n,n,\cdots,n}_{s},0}| \\ = & -(1-m)^r\left((1-n)^s + s(1-n)^{s-1}\right) + r(1-m)^{r-1}s(1-n)^{s-1} \\ = & -(1-n)^{s-1}(1-m)^{r-1}\left((1-m)(1-n) + s(1-m) - rs\right) = 0 \end{aligned}$$

In particular, $T^o_{5:3;2,0}$ is singular. □

Corollary 3.6 *For non-negative integers $m_1, m_2, \cdots, m_r, n_1, \cdots, n_s$,*

$$\begin{aligned} & |T^o_{5:\cdots,m_r,0;\cdots,n_s,0}| \\ = & -\prod_{j=1}^s(1-n_j)\prod_{i=1}^r(1-m_i) \\ & + \left(\sum_{i=1}^r \frac{(1-m_1)(1-m_2)\ldots(1-m_r)}{(1-n_i)}\right)\sum_{j=1}^s \frac{(1-n_1)(1-n_2)\cdots(1-n_s)}{1-n_j}. \end{aligned}$$

Example 3.2 $T^o_{5:\underbrace{m+1,\cdots,m+1}_{m},0;\underbrace{n+1,\cdots,n+1}_{n},0}$ *is a singular graph.*

Proof. By Corollary 3.6 and simple calculation, we have

$$\begin{aligned} & |T^o_{5:\underbrace{m+1,m+1,\cdots,m+1}_{m},0;\underbrace{n+1,n+1,\cdots,n+1}_{n},0}| \\ = & -(-n)^n(-m)^m + n(-n)^{n-1}m(-m)^{m-1} = 0. \end{aligned}$$

In particular, $T^o_{5:2,0;2,0}$ is singular. □

Corollary 3.7 *Let* $T^o_{4:\cdots,m_r}$ *and* $T^o_{5:\cdots,m_r;\cdots,n_s}$ *be non-singular where* $m_1,m_2,\cdots,m_r$, $n_1,n_2,\cdots,n_s$ *are positive integers. Then*
(i) $|T^o_{4:\cdots,m_r}|$ *is positive if and only if* r *is even and,*
(ii) $|T^o_{5:\cdots,m_r;\cdots,n_s}|$ *is positive if and only if* r *and* s *have the same parity.*

4 The complement of a tree with diameter 5

We now find the determinant of a tree complement with diameter 5 in terms of determinants of looped-trees. Let G be a graph whose vertices are $v_1, v_2, \ldots$ and let every edge be associated with the variable w_i. Then we can construct a variable adjacency matrix $A(G,w)$ for the graph G as follows: the (i,j) entry is w_k if and only if $\{v_i,v_j\} \in E(G)$ and the variable w_k is associated with edge $\{v_i,v_j\}$, and this entry is 0 if $\{v_i,v_j\} \notin E(G)$. We note that the ordinary adjacency matrix $A(G)$ is obtained from $A(G,w)$ by substituting $w_k = 1$ for each of the variables for the edges of G. Let G be a graph. An (ordinary) linear subgraph of G is a spanning subgraph whose components are lines or cycles. Further, let n be the number of linear subgraphs of G and let G_i be the i^{th} linear subgraph. In [4], Harary showed the following theorem. We note that a simple observation gives that the theorem works for our case in which the components of a linear subgraph contain loops.

Theorem 4.1 *[4] Let* G *be a graph. Then*

$$|A(G,w)| = \sum_{i=1}^{n} |A(G_i,w)|,$$

and

$$|A(G,w)| = \sum_{i=1}^{n} (-1)^{e_i} 2^{c_i} \prod_{w_k \in L_i} w_k^2 \prod_{w_j \in M_i} w_j$$

where (1) e_i *is the number of even components of* G_i*, (2)* c_i *is the number of components of* G_i *containing more than two points, and thus consisting of a single undirected cycle, (3)* L_i *is the set of components of* G_i *consisting of two points and the line joining them, and (4)* M_i *is the remaining components of* G_i *each of which is a cycle.*

For the complete graph $K_\ell^{(1)}$ of order $\ell(\geq 1)$ with 1 loop, and a graph G of order n, the following property was shown in [6], where $K_\ell^{(1)} + \overline{G}^0$ means the join of $K_\ell^{(1)}$ and $\overline{G}^0$.

Lemma 4.2 *[6] Let G be a graph of order n. Then $|A(G)| = (-1)^{n+\ell-1}|A(K_\ell^{(1)} + \overline{G}^0)|$.*

Let G be a graph, $x_0, y_0 \in V(G)$ and $z \notin V(G)$. By $G\overset{x}{\underset{y}{\rangle}} z^o$, we mean the graph with $V(G\overset{x}{\underset{y}{\rangle}} z^o) = V(G) \cup \{z\}$ and $E(G\overset{x}{\underset{y}{\rangle}} z^o) = E(G) \cup \{\{x,z\},\{y,z\},\{z,z\}\}$.

Lemma 4.3 *For non-negative integers $m_1, m_2, \cdots, m_r, n_1, n_2, \cdots, n_s$,*

$$|\overline{T_{5:\cdots,m_r;\cdots,n_s}}| = (-1)^t |A(T^o_{5:\cdots,m_r;\cdots,n_s}\overset{x}{\underset{y}{\rangle}} z^o, w)|,$$

where the values associated with a loop at z, the edge $\{x,z\}$ and the edge $\{y,z\}$ are $1-(r+s), 1-r$ and $1-s$ respectively, and every other edge has the value 1, and t is the order of $\overline{T_{5:\cdots,m_r;\cdots,n_s}}$.

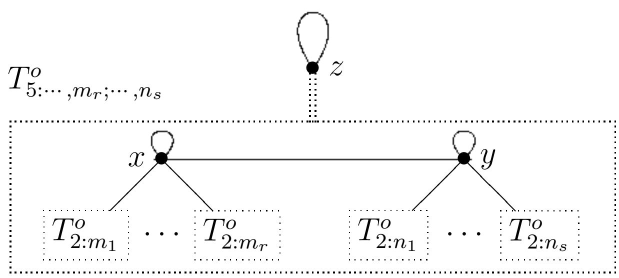

Figure 9: $T^o_{5:\cdots,m_r;\cdots,n_s} + z^o$

Proof. From Lemma 4.2, $|\overline{T_{5:\cdots,m_r;\cdots,n_s}}| = (-1)^t |T^o_{5:\cdots,m_r;\cdots,n_s} + z^o|$, where t is the order of $T_{5:\cdots,m_r;\cdots,n_s}$. (See Figure 9, where the double-dotted line between z and $T^o_{5:\cdots,m_r;\cdots,n_s}$ means that z is adjacent to every point of $T^o_{5:\cdots,m_r;\cdots,n_s}$. We note that the adjacency matrix of $T^o_{5:\cdots,m_r;\cdots,n_s} + z^o$ is of the following form:

$$A(T^o_{5:\cdots,m_r;\cdots,n_s}+z^o)=\begin{array}{c}\\ x\\ y\\ x_1\\ \vdots\\ x_r\\ y_1\\ \vdots\\ y_s\\ \vdots\\ z\end{array}\begin{array}{c}\begin{array}{cccccccccc} x & y & x_1 & \cdots & x_r & y_1 & \cdots & y_s & \cdots & z\end{array}\\ \left(\begin{array}{ccccccccc|c} 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 1\\ 1 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 1\\ 1 & 0 & & & & & & & & 1\\ \vdots & \vdots & & & & & & & & \vdots\\ 1 & 0 & & & & & & & & 1\\ 0 & 1 & & & & & & & & 1\\ \vdots & \vdots & & & & & & & & \vdots\\ 0 & 1 & & & & & & & & 1\\ \vdots & \vdots & & & & & & & & \vdots\\ \hline 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1\end{array}\right)\end{array}$$

By subtracting rows corresponding to $x_1,\ldots,x_r$ from the last row corresponding to z, we have

$$|T^o_{5:\cdots,m_r;\cdots,n_s}+z^o|=\det\begin{pmatrix}\begin{array}{ccccc|c} x & y & \cdots & \cdots & \cdots & z\\ 1 & 1 & \cdots & \cdots & \cdots & 1\\ 1 & 1 & & & & 1\\ \vdots & \vdots & & & & \vdots\\ \vdots & \vdots & & & & \vdots\\ \hline 1-r & 1 & w_3 & \cdots & w_t & 1-r\end{array}\end{pmatrix}$$

where $w_i=0$ (resp. 1) if w_i is an element of a column corresponding to a vertex in $T^o_{2:m_i}$ (resp. $T^o_{2:n_i}$). Similarly, by subtracting rows corresponding to $y_1,\ldots,y_s$ from the last row corresponding to z, we have

$$|T^o_{5:\cdots,m_r;\cdots,n_s}+z^o|=\det\begin{pmatrix}\begin{array}{ccccc|c} x & y & \cdots & \cdots & \cdots & z\\ 1 & 1 & \cdots & \cdots & \cdots & 1\\ 1 & 1 & & & & 1\\ \vdots & \vdots & & & & \vdots\\ \vdots & \vdots & & & & \vdots\\ \hline 1-r & 1-s & 0 & \cdots & 0 & 1-(r+s)\end{array}\end{pmatrix}$$

We now subtract columns corresponding to $x_1,\ldots,x_r,y_1,\ldots,y_s$ from the last column to get

$$
\begin{aligned}
|T^o_{5:\cdots,m_r;\cdots,n_s} + z^o| &= \det \left(\begin{array}{ccccc|c}
x & y & \cdots & \cdots & \cdots & z \\
1 & 1 & \cdots & \cdots & \cdots & 1-r \\
1 & 1 & & & & 1-s \\
\vdots & \vdots & & & & 0 \\
\vdots & \vdots & & & & \vdots \\
\vdots & \vdots & & & & 0 \\
\hline
1-r & 1-s & 0 & \cdots & 0 & 1-(r+s)
\end{array} \right) \\
&= |A(T^o_{5:\cdots,m_r;\cdots,n_s} \overset{x}{\underset{y}{\rangle}} z^o, w)|
\end{aligned}
$$

where the corresponding graph $T^o_{5:\cdots,m_r;\cdots,n_s} \overset{x}{\underset{y}{\rangle}} z^o$ is depicted in Figure 10. □

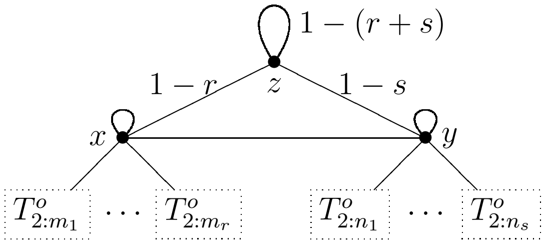

Figure 10: $T^{\circ}_{5:\cdots,m_r;\cdots,n_s} >^x_y z^o$

Theorem 4.4 *For non-negative integers $m_1, m_2, \cdots, m_r, n_1, n_2, \cdots, n_s$, then*

$$
\begin{aligned}
(-1)^t|\overline{T_{5:\cdots,m_r;\cdots,n_s}}| &= 2(1-r)(1-s)\prod_{i=1}^{r}(1-m_i)\prod_{i=1}^{s}(1-n_i) \\
&\quad -(1-r)^2\prod_{i=1}^{r}(1-m_i)|T^o_{4:\cdots n_s}| - (1-s)^2\prod_{i=1}^{s}(1-n_i)|T^o_{4:\cdots m_r}| \\
&\quad +(1-(r+s))|T^o_{5:\cdots,m_r;\cdots,n_s}|.
\end{aligned}
$$

where t is the order of $\overline{T_{5:\ldots,m_r;\ldots,n_s}}$.

Proof. By applying Lemma 4.3, we have

$$|\overline{T_{5:\cdots,m_r;\cdots,n_s}}| = (-1)^t|A(T^o_{5:\cdots,m_r;\cdots,n_s} \overset{x}{\underset{y}{\rangle}} z^o, w)|,$$

where t is the order of $\overline{T_{5:\ldots,m_r;\ldots,n_s}}$. We partition the set of all linear subgraphs of $T^o_{5:\cdots,m_r;\cdots,n_s} \overset{x}{\underset{y}{\rangle}} z^o$ into 4 classes $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, which consists of all linear subgraphs containing a cycle $\{x, y, z\}$, a line $\{x, z\}$, and a line $\{y, z\}$ respectively, and $\mathcal{G}_4$ consisting of all linear subgraphs containing neither $\{x, z\}$ or $\{y, z\}$ nor a cycle $\{x, y, z\}$. Thanks to Theorem 4.1, we have

$$(-1)^t |A(T^o_{5:\cdots,m_r;\cdots,n_s} \overset{x}{\underset{y}{\rangle}} z^o)| = \sum_{i=1}^{4} \left(\sum_{H \in \mathcal{G}_i} |A(H, w)| \right).$$

Let $H \in \mathcal{G}_1$. We note that the determinant of H is independent of the ordering of the vertices of $T^o_{5:\cdots,m_r;\cdots,n_s} \overset{x}{\underset{y}{\rangle}} z^o$, and so we may separate the vertices of a cycle $\{x, y, z\}$ so that the variable adjacency matrix is decomposed into diagonal block submatrices as follows:

$$A(H, w) = \begin{array}{c} \\ x \\ y \\ z \\ \\ \end{array} \begin{pmatrix} x & y & z & \\ 0 & 1 & 1-r & \\ 1 & 0 & 1-s & \\ 1-r & 1-s & 0 & \\ & & & D_H \end{pmatrix}$$

where D_H is a variable adjacency matrix of the complement of a cycle $\{x, y, z\}$ in H. Moreover, $\sum_{H \in \mathcal{G}_i} |D_H|$ is the determinant of $T^o_{m_1} \cup \cdots \cup T^o_{m_r} \cup T^o_{n_1} \cup \cdots \cup T^o_{n_s}$. Hence, we have

$$\begin{aligned} \sum_{H \in \mathcal{G}_1} |A(H, x)| &= 2(1-r)(1-s)|T^o_{m_1}| \ldots |T^o_{m_r}||T^o_{n_1}| \ldots |T^o_{n_s}| \\ &= 2(1-r)(1-s) \prod_{i=1}^{r} (1-m_i) \prod_{i=1}^{s} (1-n_i). \end{aligned}$$

We apply the same argument for $\mathcal{G}_2, \mathcal{G}_3$, and $\mathcal{G}_4$ to get

$$\begin{aligned} \sum_{H \in \mathcal{G}_2} |A(H, x)| &= -(1-r)^2 \prod_{i=1}^{r} (1-m_i) |T^o_{4:\ldots,n_s}|, \\ \sum_{H \in \mathcal{G}_3} |A(H, x)| &= -(1-s)^2 \prod_{i=1}^{s} (1-n_s) |T^o_{4:\ldots,m_r}|, \end{aligned}$$

and

$$\sum_{H \in \mathcal{G}_4} |A(H, x)| = (1 - (r+s)) |T^o_{5:\ldots,m_r;\ldots,n_s}|.$$

□

Theorem 4.5 *For positive integers* $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s$, $\overline{T_{5:\ldots,m_r;\ldots,n_s}}$ *is singular if and only if* $T^o_{5:\ldots,m_r;\ldots,n_s}$ *is singular, that is, at least two distinct* m_i *are* 1 *or at least two distinct* n_i *are* 1.

Proof. For the simplification, we suppress $(1-m_i)$ and $(1-n_j)$ in $|\overline{T_5}| = |\overline{T_{5:\cdots,m_r;\cdots,n_s}}|$. We note that by Theorems 4.4, 2.2 and 3.2,

$$\begin{aligned}
&(-1)^t|\overline{T_5}| \\
=\ & 2(1-r)(1-s)\prod^r\prod^s * - (1-r)^2\prod^r *|T^o_{4:n_s}| - (1-s)^2\prod^s *|T^o_{4:m_r}| \\
& +(1-(r+s))|T^o_{5:\cdots,m_r;\cdots,n_s}| \\
=\ & 2(1-r)(1-s)\prod^r\prod^s * \\
& -(1-r)^2\prod^r * \left\{\prod^s * - \sum^s *\right\} - (1-s)^2\prod^s * \left\{\prod^r * - \sum^r *\right\} \\
& +(1-(r+s))\left\{\sum^s * \sum^r * - \prod^s * \sum^r * - \prod^r * \sum^s *\right\} \\
=\ & -(r-s)^2\prod^r\prod^s * + (r^2-r+s)\prod^r * \sum^s * + (s^2-s+r)\prod^s * \sum^r * \\
& +(1-(r+s))\sum^s * \sum^r *.
\end{aligned}$$

If $T^o_{5:\cdots,m_r;\cdots,n_s}$ is singular, then at least two distinct m_i are 1 or at least two distinct n_i are 1. Therefore, in the expression of $|\overline{T_5}|$, block terms $\prod^r\prod^s *$, $\prod^r * \sum^s *$, $\prod^s * \sum^r *$, and $\sum^s * \sum^r *$ clearly vanish and so $\overline{T_{5:\cdots,m_r;\cdots,n_s}}$ is singular. For the converse, we assume that $T^o_{5:\cdots,m_r;\cdots,n_s}$ is non-singular. We need to consider three cases: (i) only one m_i or n_j is 1, (ii) only one m_i and only one n_j are 1, (iii) neither m_i nor n_j is 1. If only one m_i is 1, then

$$\begin{aligned}
(-1)^t|\overline{T_5}| &= -(r-s)^2\prod^r * \prod^s * + (r^2-r+s)\prod^r * \sum^s * \\
&\quad +(s^2-s+r)\prod^s * \sum^r * + (1-(r+s))\sum^s * \sum^r * \\
&= (s^2-s+r)\prod^s * \sum^r * + (1-(r+s))\sum^s * \sum^r *
\end{aligned}$$

where two block terms have the same sign $(-1)^{s+r+1}$ and so the sum can not be zero. The same argument can be applied for the case that only one n_j is 1. If only one m_i and only one n_j are 1, then

$$\begin{aligned}
(-1)^t|\overline{T_5}| &= -(r-s)^2\prod^r\prod^s * + (r^2-r+s)\prod^r * \sum^s * \\
&\quad +(s^2-s+r)\prod^s * \sum^r * + (1-(r+s))\sum^s * \sum^r * \\
&= (1-(r+s))\sum^s * \sum^r *
\end{aligned}$$

which is nonzero. If none of m_i nor n_j is 1, then

$$\begin{aligned}(-1)^t|\overline{T_5}| &= -(r-s)^2\prod^{r}\prod^{s}* + (r^2-r+s)\prod^{r}*\sum^{s}* \\ &\quad +(s^2-s+r)\prod^{s}*\sum^{r}* + (1-(r+s))\sum^{s}*\sum^{r}*\end{aligned}$$

where each block term has the same sign $(-1)^{r+s+1}$. Hence, $|\overline{T_5}|$ does not vanish and so $\overline{T_{5:\cdots,m_r;\cdots,n_s}}$ is non-singular. □

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