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A counterexample to a result on the tree graph of a graph*

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Abstract

Given a set of cycles C of a graph G, the tree graph of G, defined by C, is the graph $T(G, C)$ whose vertices are the spanning trees of G and in which two trees R and S are adjacent if $R \cup S$ contains exactly one cycle and this cycle lies in C. Li et al. [*Discrete Math.* 271 (2003), 303–310] proved that if the graph $T(G, C)$ is connected, then C cyclically spans the cycle space of G. Later, Yumei Hu [*Proc. 6th Int. Conf. Wireless Communications Networking and Mobile Comput.* (2010), 1–3] proved that if C is an arboreal family of cycles of G which cyclically spans the cycle space of a 2-connected graph G, then $T(G, C)$ is connected. In this note we present an infinite family of counterexamples to Hu's result.

1 Introduction

The *tree graph* of a connected graph G is the graph $T(G)$ whose vertices are the spanning trees of G, in which two trees R and S are adjacent if $R \cup S$ contains exactly one cycle. Li et al. [2] defined the *tree graph of G with respect to a set of*

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cycles C as the spanning subgraph $T(G, C)$ of $T(G)$ where two trees R and S are adjacent only if the unique cycle contained in $R \cup S$ lies in C.

A set of cycles C of G *cyclically spans* the cycle space of G if for each cycle σ of G there are cycles $\alpha_1, \alpha_2, \ldots, \alpha_m \in C$ such that: $\sigma = \alpha_1 \Delta \alpha_2 \Delta \ldots \Delta \alpha_m$ and, for $i = 2, 3, \ldots, m$, $\alpha_1 \Delta \alpha_2 \Delta \ldots \Delta \alpha_i$ is a cycle of G. Li et al. [2] proved the following theorem:

Theorem 1. *If C is a set of cycles of a connected graph G such that the graph $T(G, C)$ is connected, then C cyclically spans the cycle space of G.*

A set of cycles C of a graph G is *arboreal* with respect to G if for every spanning tree T of G, there is a cycle $\sigma \in C$ which is a fundamental cycle of T. Yumei Hu [1] claimed to have proved the converse theorem:

Theorem 2. *Let G be a 2-connected graph. If C is an arboreal set of cycles of G that cyclically spans the cycle space of G, then $T(G, C)$ is connected.*

In this note we present a counterexample to Theorem 2 given by a triangulated plane graph G with 6 vertices and an arboreal family of cycles C of G such that C cyclically spans the cycle space of G, while $T(G, C)$ is disconnected. Our example generalises to a family of triangulated graphs G_n with $3(n + 2)$ vertices for each integer $n \geq 0$.

If α is a face of a plane graph G, we denote, also by α, the corresponding cycle of G as well as the set of edges of α.

2 Preliminary results

Let G be a plane graph. For each cycle τ, let $k(\tau)$ be the number of faces of G contained in the interior of τ. A *diagonal edge* of τ is an edge lying in the interior of τ having both vertices in τ. The following lemma will be used in the proof of Theorem 4.

Lemma 3. *Let G be a triangulated plane graph and σ be a cycle of G. If $k(\sigma) \geq 2$, then there are two faces ϕ and ψ of G, contained in the interior of σ, both with at least one edge in common with σ, and such that $\sigma \Delta \phi$ and $\sigma \Delta \psi$ are cycles of G.*

Proof. If $k(\sigma) = 2$, let ϕ and ψ be the two faces of G contained in the interior of σ. Clearly $\sigma \Delta \phi = \psi$ and $\sigma \Delta \psi = \phi$ which are cycles of G.

Assume $k = k(\sigma) \geq 3$ and that the result holds for each cycle τ of G with $2 \leq k(\tau) < k$. If σ has a diagonal edge uv, then σ together with the edge uv define two cycles σ_1 and σ_2 such that $k(\sigma) = k(\sigma_1) + k(\sigma_2)$. If σ_1 is a face of G, then $\sigma \Delta \sigma_1$ is a cycle of G and, if $k(\sigma_1) \geq 2$, then by induction there are two faces ϕ_1 and ψ_1 of G, contained in the interior of σ_1, both with at least one edge in common with σ_1, and such that $\sigma_1 \Delta \phi_1$ and $\sigma_1 \Delta \psi_1$ are cycles of G. Without loss of generality, we assume uv is not an edge of ϕ_1 and therefore $\phi = \phi_1$ has at least one edge in common with σ and is such that $\sigma\Delta\phi$ is a cycle of G. Analogously σ_2 contains a face ψ with at least one edge in common with σ and such that $\sigma\Delta\psi$ is a cycle of G.

For the remaining of the proof we may assume σ has no diagonal edges. Since $k = k(\sigma) \geq 3$, there is a vertex u of σ which is incident with one or more edges lying in the interior of σ. Let $v_0, v_1, \ldots, v_m$ be the vertices in σ or in the interior of σ which are adjacent to u. Without loss of generality we assume v_0 and v_m are vertices of σ and that $v_0, v_1, \ldots, v_{m-1}, v_m$ is a path joining v_0 and v_m, see Figure 1.

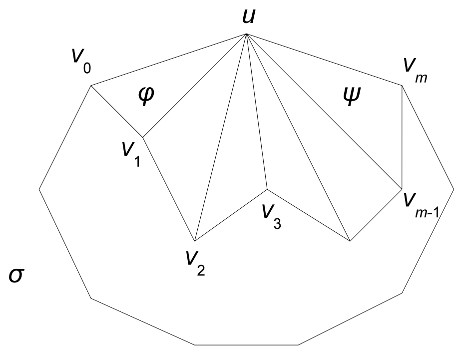

Figure 1: Cycle σ with no diagonal edges.

As σ has no diagonal edges, vertices $v_1, v_2, \ldots, v_{m-1}$ are not vertices of σ and therefore faces $\phi = uv_0v_1$ and $\psi = uv_mv_{m-1}$ are such that $\sigma\Delta\phi$ and $\sigma\Delta\psi$ are cycles of G, each with one edge in common with σ.

□

Theorem 4. *Let G be a triangulated plane graph and α and β be two internal faces of G with one edge in common. If C is the set of internal faces of G with cycle α replaced by the cycle $\alpha\Delta\beta$, then C cyclically spans the cycle space of G.*

Proof. Let σ be a cycle of G. If $k(\sigma) = 1$, then $\sigma \in C$ or $\sigma = \alpha$ in which case $\sigma = (\alpha\Delta\beta)\Delta\beta$. In both cases σ is cyclically spanned by C.

We proceed by induction assuming $k = k(\sigma) \geq 2$ and that if τ is a cycle of G with $k(\tau) < k$, then τ is cyclically spanned by C.

By Lemma 3, there are two faces ϕ and ψ of G, contained in the interior of σ, such that both $\sigma\Delta\phi$ and $\sigma\Delta\psi$ are cycles of G; without loss of generality we assume $\phi \neq \alpha$. Clearly $k(\sigma\Delta\phi) < k$; by induction, there are cycles $\tau_1, \tau_2, \ldots, \tau_m \in C$ such that: $\sigma\Delta\phi = \tau_1\Delta\tau_2\Delta\ldots\Delta\tau_m$ and, for $i = 2, 3, \ldots, m$, $\tau_1\Delta\tau_2\Delta\ldots\Delta\tau_i$ is a cycle of G. As $\sigma = (\sigma\Delta\phi)\Delta\phi = \tau_1\Delta\tau_2\Delta\ldots\Delta\tau_m\Delta\phi$, cycle σ is also cyclically spanned by C. □

3 Main result

Let G be the skeleton graph of a octahedron (see Figure 2) and C be the set of cycles that correspond to the internal faces of G with cycle α replaced by cycle $\alpha\Delta\beta$.

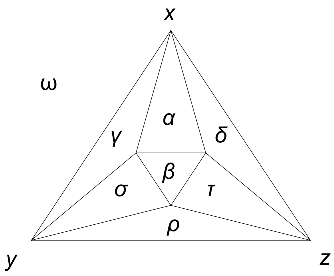

Figure 2: Graph G with internal faces $\alpha, \beta, \gamma, \delta, \sigma, \tau$ and ρ and outer face ω.

By Theorem 4, C cyclically spans the cycle space of G. Suppose C is not arboreal and let P be a spanning tree of G with none of its fundamental cycles in C. For this to happen, each of the cycles $\beta, \gamma, \delta, \sigma, \tau$ and ρ of G, must have at least two edges which are not edges of P and since P has no cycles, at least one edge of cycle α and at least one edge of cycle ω are not edgs of P.

Therefore G has at least 7 edges which are not edges of P. These, together with the 5 edges of P sum up to 12 edges which is exactly the number of edges of G. This implies that each of the cycles ω and α has exactly two edges of P and that each of the cycles $\beta, \gamma, \delta, \sigma, \tau$ and ρ has exactly one edge of P.

If edges xy and xz are edges of P, then vertex x cannot be incident to any other edge of P and therefore cycle α can only have one edge of P, which is not possible.

If edges xy and yz are edges of P, then vertex y cannot be incident to any other edge of P. In this case, the edge in cycle σ, opposite to vertex y, must be an edge of P and cannot be incident to any other edge of P, which again is not possible. The case where edges xz and yz are edges of P is analogous. Therefore C is an arboreal set of cycles of G.

Let T, S and R be the spanning trees of G given in Figure 3. The graph $T(G, C)$ has a connected component formed by the trees T, S and R since cycle ρ is the only cycle in C which is a fundamental cycle of either T, S or R. This implies that $T(G, C)$ is disconnected.

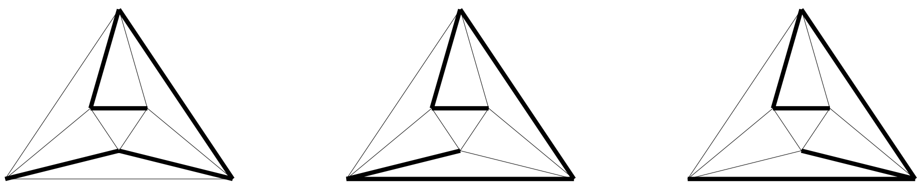

Figure 3: Trees T (left), S (center) and R (right).

We proceed to generalise the counterexample to graphs with arbitrary large number of vertices. Let $G_0 = G, x_0 = x, y_0 = y, z_0 = z$ and for $t \geq 0$ define G_{t+1} as the graph obtained by placing a copy of G_t in the inner face of the skeleton graph of an octahedron as in Figure 4.

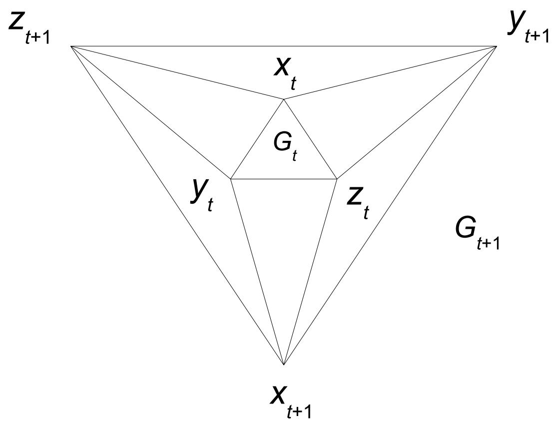

Figure 4: G_{t+1}.

Notice that each graph G_n contains a copy G' of G in the innermost layer. We also denote by $\alpha, \beta, \gamma, \delta, \sigma, \tau$ and ρ the cycles of G_n that correspond to the cycles $\alpha, \beta, \gamma, \delta, \sigma, \tau$ and ρ of G'. Let ω_n denote the cycle given by the edges in the outer face of G_n.

For $n \geq 0$ let C_n be the set of cycles that correspond to the internal faces of G_n with cycle α replaced by cycle $\alpha\Delta\beta$. By Theorem 4, C_n cyclically spans the cycle space of G_n.

We claim that for $n \geq 0$, set C_n is an arboreal set of cycles of G_n. Suppose C_t is arboreal but C_{t+1} is not and let P_{t+1} be a spanning tree of G_{t+1} such that none of its fundamental cycles lies in C_{t+1}.

As in the case of graph G and tree P, above, each cycle in C_{t+1}, other than $\alpha\Delta\beta$, has exactly one edge in P_{t+1}, while cycles α and ω_{t+1} have exactly two edges in P_{t+1}. The reader can see that this implies that the edges of P_{t+1} which are not edges of G_t form a path with length 3. Then $P_{t+1} - \{x_{t+1}, y_{t+1}, z_{t+1}\}$ is a spanning tree of G_t and by induction, one of its fundamental cycles lies in $C_t \subset C_{t+1}$ which is a contradiction. Therefore C_{t+1} is arboreal.

Let $T_0 = T$ and for $t \geq 0$ define T_{t+1} as the spanning tree of G_{t+1} obtained by placing a copy of T_t in the inner face of the skeleton graph of an octahedron as in Figure 5.

Trees S_{t+1} and R_{t+1} are obtained from S_t and R_t in the same way with $S_0 = S$ and $R_0 = R$ respectively. We claim that, for each integer $n \geq 0$, cycle ρ is the only cycle in C_n which is a fundamental cycle of either T_n, S_n or R_n. Therefore T_n, S_n and R_n form a connected component of G_n which implies that $T(G_n, C_n)$ is disconnected.

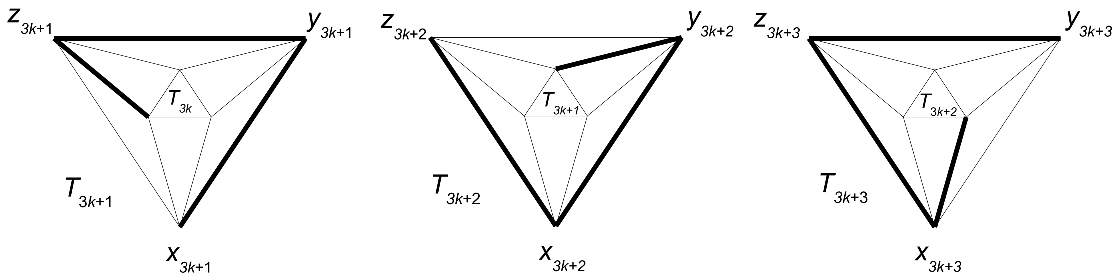

Figure 5: $t = 3k$ (left), $t = 3k + 1$ (centre) and $t = 3k + 2$ (right).

References

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[2] X. Li, V. Neumann-Lara and E. Rivera-Campo, On the tree graph defined by a set of cycles, *Discrete Math.* 271 (2003), 303–310.

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