Neighborhood total domination of a graph and its complement

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Abstract

A neighborhood total dominating set in a graph G is a dominating set S of G with the property that the subgraph induced by $N(S)$, the open neighborhood of the set S, has no isolated vertex. The neighborhood total domination number $\gamma_{nt}(G)$ is the minimum cardinality of a neighborhood total dominating set of G. Arumugam and Sivagnanam introduced and studied the concept of neighborhood total domination in graphs [S. Arumugam and C. Sivagnanam, *Opuscula Math.* 31 (2011) 519–531]. They proved that if G and $\overline{G}$ are connected, then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 2 & \text{if } \operatorname{diam}(G) \geq 3. \\ \lceil \frac{n}{2} \rceil + 3 & \text{if } \operatorname{diam}(G) = 2. \end{cases}$, where $\overline{G}$ is the complement of G. The problem of characterizing graphs attaining equality in the previous bounds was left as an open problem by the authors. In this paper, we address this open problem by studying sharpness and strictness of the above inequalities.

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1 Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* $N(v)$ of a vertex v consists of the vertices adjacent to v and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. We denote by $\Delta(G) = \Delta$ and $\delta(G) = \delta$ the *maximum degree* and the *minimum degree* of the graph G, respectively. For a set $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$, the *closed neighborhood* is $N[S] = N(S) \cup S$, and $\langle S \rangle$ is the *subgraph* induced by the vertices of S.

The *distance* between two vertices x and y, denoted by $d_G(x, y)$, is the length of a shortest path from x to y. The *diameter* of G is the maximum distance among all pairs of vertices of G, denoted by $\operatorname{diam}(G)$. The complement $\overline{G}$ of a graph G is a graph with vertex set $V(G)$ and two vertices x and y are adjacent in $\overline{G}$ if and only if $xy \notin E(G)$. Clearly, if G and $\overline{G}$ are both connected, then G has order $n \geq 4$, $\min\{\operatorname{diam}(G), \operatorname{diam}(\overline{G})\} \geq 2$ and $\min\{\delta(G), \delta(\overline{G})\} \geq 1$. The *corona* of a graph G, denoted by $\operatorname{cor}(G)$, is the graph formed from a copy of G by creating for each $v \in V$, a new vertex v' and edge vv'. The Cartesian product of two graphs G and H, $G \square H$ is a graph with vertex set $V(G \square H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 u_2 \in E(G)$ and $v_1 = v_2$ or $v_1 v_2 \in E(H)$ and $u_1 = u_2$.

A set $S \subseteq V(G)$ is a *dominating set* of G if every vertex of $V - S$ is adjacent to at least one vertex of S. The cardinality of the smallest dominating set of G is *the domination number* $\gamma(G)$ (see [4, 6]). A dominating set S of G is called a *total dominating set* if the induced subgraph $\langle S \rangle$ has no isolated vertex. The cardinality of the smallest total dominating set of G is *the total domination number* $\gamma_t(G)$.

In [1], Arumugam and Sivagnanam introduced and studied the concept of neighborhood total domination in graphs. A *neighborhood total dominating set* in a graph G is a dominating set S of G with the property that the subgraph induced by $N(S)$ has no isolated vertex. The *neighborhood total domination number* $\gamma_{nt}(G)$ is the minimum cardinality of a neighborhood total dominating set of G. We note that every total dominating set is a neighborhood total dominating set of G, while every neighborhood total dominating set is a dominating set. Hence every graph G without isolated vertices satisfies

$$\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_t(G) \tag{1}$$

It is worth mentioning that the previous inequality chain (1) may be strict as shown by the graph G in Figure 1, where $\gamma(G) = 4$, $\gamma_{nt}(G) = 5$ and $\gamma_t(G) = 6$.
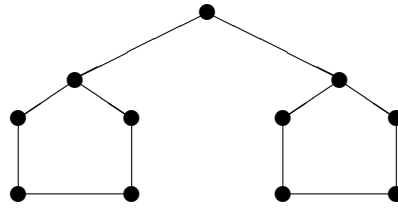
Figure 1

Arumugam and Sivagnanam [1] established the following Nordhauss-Gaddum type result for neighborhood total domination in graphs.

Theorem 1 (Arumugam and Sivagnanam [1]) *Let G be any graph such that both G and $\overline{G}$ are connected. Then*

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \left\{ \begin{array}{ll} \lceil \frac{n}{2} \rceil + 2 & \textit{if } \mathrm{diam}(G) \geq 3. \\ \lceil \frac{n}{2} \rceil + 3 & \textit{if } \mathrm{diam}(G) = 2. \end{array} \right.$$

The authors [1] concluded their paper with the following open problem.

Problem 2 *Characterize graphs which attain the bounds given in Theorem 1.*

In this paper, we consider sharpness and strictness of the inequalities in Theorem 1. By symmetry, we may always assume that $\mathrm{diam}(G) \geq \mathrm{diam}(\overline{G})$.

2   Preliminary results

In this section, we recall some important results that will be useful in our investigations. We begin by the following well known classical result that can be found for example in [3].

Observation 3 ([3]) *If G is a graph with $\mathrm{diam}(G) \geq 3$, then $\mathrm{diam}(\overline{G}) \leq 3$.*

Observation 4 (Arumugam and Sivagnanam [1]) *If G is a graph and $\mathrm{diam}(G) = 2$, then $\gamma_{nt}(G) \leq \delta(G) + 1$.*

Theorem 5 (Hellwig and Volkmann [5]) *If G is a graph of order n and diameter 2, then $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor + 1$.*

Restricted to graphs with minimum degree at least two, Dunbar et al. [2] gave an upper bound for the sum $\gamma(G) + \gamma(\overline{G})$. Moreover, they characterized graphs G attaining this upper bound. Let $K_3 \square K_3$ be the Cartesian product of K_3 by K_3. Also, the authors defined a family $\mathcal{A}$ of six graphs, each of order seven.

Theorem 6 (Dunbar et al. [2]) *If G and $\overline{G}$ are connected, with $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, then*

$$\gamma(G) + \gamma(\overline{G}) \leq \lfloor 2n/5 \rfloor + 3,$$

where equality holds if and only if G or $\overline{G}$ is in $\mathcal{A} \cup \{K_3 \square K_3\}$.

Since all graphs attaining equality in the upper bound of Theorem 6 have order at most 9, the following corollary is immediate.

Corollary 7 (Dunbar et al. [2]) *If G and $\overline{G}$ are connected of order $n \geq 10$ such that $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, then $\gamma(G) + \gamma(\overline{G}) \leq \lfloor 2n/5 \rfloor + 2$.*

The next result was first given by Dunbar et al. [2]. However, their proof contained a mistake that has been corrected by Volkmann in [8].

Theorem 8 (Volkmann [8]) *If G and $\overline{G}$ are connected graphs of order $n \neq 10$ and $n \neq 13$ with $\delta(G) \geq 3$ and $\delta(\overline{G}) \geq 3$ and with $G, \overline{G}$ different from $K_3 \square K_3$, then*

$$\gamma(G) + \gamma(\overline{G}) \leq \lfloor 3n/8 \rfloor + 2.$$

In [7], the authors defined two families of graphs: a family $\mathfrak{B}$ of eight graphs called *bad graphs* containing the five graphs $B_1, B_2, \ldots, B_5$ (see figure below), and a family F, where $F = \{F_k : k \geq 2\}$, and F_k is a graph of order $2k - 1$ obtained from the star $K_{1,k-1}$ by subdividing each edge exactly once. In particular, $F_2 = P_3$ and $F_3 = P_5$. The graph F_5 is shown as follows:
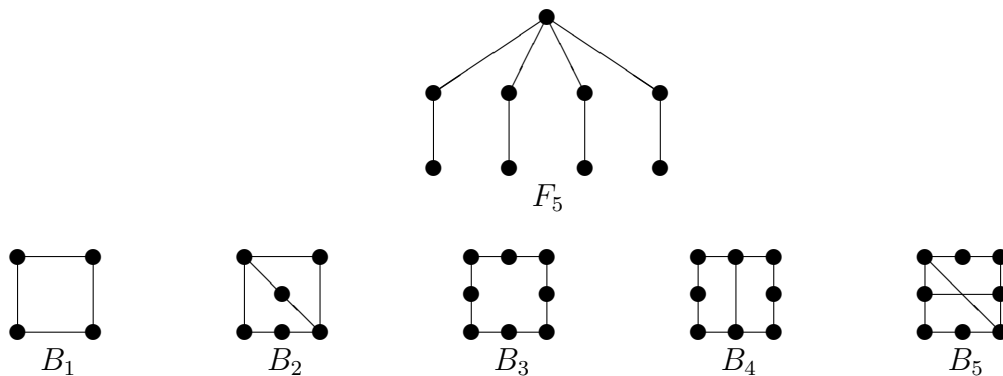

Figure 2

Theorem 9 (Henning and Jafari Rad [7]) *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{nt}(G) \leq \frac{n+1}{2}$, with equality if and only if $G = C_5$ or $G \in F$.*

Theorem 10 (Henning and Jafari Rad [7]) *Let $G \neq C_5$ be a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$. Then $\gamma_{nt}(G) \leq \frac{n}{2}$, with equality if and only if $G \in \{B_1, B_2, B_3, B_4, B_5\}$.*

Theorem 11 (Henning and Jafari Rad [7]) *If G is a connected graph with $\delta \geq 1$, then $\gamma_{nt}(G) \leq (\frac{\delta+1}{\delta})\gamma(G)$.*

3   Graphs G with diameter two

In this section we show that the inequality $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \lceil \frac{n}{2} \rceil + 3$ is strict for all connected graphs G and $\overline{G}$ of order n with diameter two, except for the cycle C_5. By Theorem 9, if $G = C_5$, then $\text{diam}(G) = \text{diam}(\overline{G}) = 2$ and $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = 6 = \lceil \frac{5}{2} \rceil + 3$. We first need the following useful observation.

Observation 12 *If G and $\overline{G}$ are connected with $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, then $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$.*

Proof. Suppose that $\delta(G) = 1$ and let v be a vertex of degree one in G. Since $\text{diam}(G) = 2$, the unique neighbor of v, say u, is adjacent to all the remaining vertices of G. Hence u is isolated in $\overline{G}$, a contradiction. Likewise we obtain $\delta(\overline{G}) \geq 2$.   □

We note that no graph of order at most 4 satisfies Observation 12.

Proposition 13 *Let G and $\overline{G}$ be connected graphs of order n with $\text{diam}(G) = \text{diam}(\overline{G}) = 2$ such that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \lceil \frac{n}{2} \rceil + 3$. Then $G = C_5$ or $n \geq 9$.*

Proof. Suppose that $G \neq C_5$. By Observation 12, $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$. By Theorem 10, $\gamma_{nt}(G) \leq \frac{n}{2}$ with equality if and only if $G \in \{B_1, B_2, B_3, B_4, B_5\}$. However, each of these graphs is excluded since B_3, B_4 and B_5 have diameter three and $\overline{B_1}$ is not connected. For the graph B_2 we have $\gamma_{nt}(\overline{B_2}) = 2$, but then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) < \lceil \frac{n}{2} \rceil + 3$. Therefore $\gamma_{nt}(G) < \frac{n}{2}$, implying that $\gamma_{nt}(\overline{G}) \geq 4$. Likewise, we obtain $\gamma_{nt}(\overline{G}) < \frac{n}{2}$ and $\gamma_{nt}(G) \geq 4$. We deduce that $n \geq 9$.   □

Corollary 14 *Let G and $\overline{G}$ be connected graphs of order $n \in \{6, 7, 8\}$ with $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) < \lceil \frac{n}{2} \rceil + 3$.*

For the special case $G = K_3 \square K_3$, we note that $\text{diam}(G) = 2$ and $\overline{G} = K_3 \square K_3$. For this graph we have $\gamma_{nt}(G) = \gamma_{nt}(\overline{G}) = 3$. Hence the following holds.

Observation 15 *For the graph $G = K_3 \square K_3$, we have $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) < \lceil \frac{n}{2} \rceil + 3$.*

Theorem 16 *Let G and $\overline{G}$ be connected graphs of order $n \geq 9$ with $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) < \lceil \frac{n}{2} \rceil + 3$.*

Proof. Recall that by Observation 12, $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$. If G or $\overline{G} = K_3 \square K_3$, then by Observation 15, the result is valid. So we can assume that neither G nor $\overline{G}$ is $K_3 \square K_3$. Assume that $\delta(G) < (n + 2)/4$ and $\delta(\overline{G}) < (n + 2)/4$. Then by Observation 4,

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \delta(G) + 1 + \delta(\overline{G}) + 1$$
$$< (n + 2)/4 + (n + 2)/4 + 2 = n/2 + 3.$$

Hence we may assume, without loss of generality, that $\delta(G) \geq (n+2)/4$. Note that since $n \geq 9$, $\delta(G) \geq 3$. By Theorem 11, $\gamma_{nt}(G) \leq (\frac{\delta(G)+1}{\delta(G)})\gamma(G)$. Suppose first that $\delta(\overline{G}) = 2$. Using Theorem 5 and the fact that $\delta(G) \geq (n+2)/4$, we arrive at $\gamma_{nt}(G) \leq (\frac{\delta(G)+1}{\delta(G)})\gamma(G) \leq \frac{n+6}{n+2}\gamma(G) \leq \frac{n+6}{n+2}(\lfloor n/4 \rfloor + 1)$. Now, applying Observation 4 on the complement of G and since $n \geq 9$, we obtain

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \frac{n+6}{n+2}(\lfloor n/4 \rfloor + 1) + (\delta(\overline{G}) + 1)$$
$$< \lceil \frac{n}{2} \rceil + 3.$$

Next, we can assume that $\delta(\overline{G}) \geq 3$. By Theorem 11, $\gamma_{nt}(\overline{G}) \leq (\frac{\delta(\overline{G})+1}{\delta(\overline{G})})\gamma(\overline{G})$. Since $\delta(G) \geq 3$ and $\delta(\overline{G}) \geq 3$, we have $\gamma_{nt}(G) \leq (\frac{\delta(G)+1}{\delta(G)})\gamma(G) \leq \frac{4}{3}\gamma(G)$ and likewise $\gamma_{nt}(\overline{G}) \leq \frac{4}{3}\gamma(\overline{G})$. Therefore $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \frac{4}{3}(\gamma(G) + \gamma(\overline{G}))$. Since G and $\overline{G}$ are different from $K_3 \square K_3$, by Theorem 8, we deduce that for all $n \geq 9$ with $n \notin \{10, 13\}$:

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \frac{4}{3}(\gamma(G) + \gamma(\overline{G})) \leq \frac{4}{3}(\lfloor 3n/8 \rfloor + 2)$$
$$< \lceil \frac{n}{2} \rceil + 3.$$

It remains to examine the cases $n = 10$ and 13, where $\delta(\overline{G}) \geq 3$ and $\delta(G) \geq (n+2)/4$.

Assume that $n = 13$. Note that $\delta(G) \geq (n+2)/4 \geq 4$. If $\delta(\overline{G}) = 3$, then Observation 4 implies that $\gamma_{nt}(\overline{G}) \leq 4$. Now, since $\gamma_{nt}(G) \leq \frac{n+6}{n+2}(\lfloor n/4 \rfloor + 1)$, it is easy to check that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \frac{n+6}{n+2}(\lfloor n/4 \rfloor + 1) + 4 < \lceil \frac{n}{2} \rceil + 3 = 10$. Hence we can assume that $\delta(\overline{G}) \geq 4$. Then $\gamma_{nt}(\overline{G}) \leq (\frac{\delta(\overline{G})+1}{\delta(\overline{G})})\gamma(\overline{G}) \leq \frac{5}{4}\gamma(\overline{G})$ and likewise $\gamma_{nt}(G) \leq \frac{5}{4}\gamma(G)$ since $\delta(G) \geq 4$. By Corollary 7 we obtain

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \frac{5}{4}(\gamma(G) + \gamma(\overline{G})) \leq \frac{5}{4}(\lfloor 2n/5 \rfloor + 2)$$
$$< \lceil \frac{n}{2} \rceil + 3.$$

Finally, assume that $n = 10$. Thus $\delta(G) \geq (n+2)/4 \geq 3$, and by Theorem 11, $\gamma_{nt}(\overline{G}) \leq \frac{4}{3}\gamma(\overline{G})$ and $\gamma_{nt}(G) \leq \frac{4}{3}\gamma(G)$. Moreover, Corollary 7 implies that $\gamma(G) + \gamma(\overline{G}) \leq \lfloor 2n/5 \rfloor + 2 = 6$. Now, if $\gamma_{nt}(G) = \gamma(G)$, then

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \gamma(G) + \frac{4}{3}\gamma(\overline{G}) < \frac{4}{3}(\gamma(G) + \gamma(\overline{G})) \leq 8,$$

and so $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) < 8 = \lceil \frac{n}{2} \rceil + 3$. Next, we can assume that $\gamma_{nt}(G) \geq \gamma(G) + 1$. If $\gamma(G) + \gamma(\overline{G}) \leq 5$, then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \frac{4}{3}(\gamma(G) + \gamma(\overline{G})) \leq \frac{20}{3} < \lceil \frac{n}{2} \rceil + 3$. Thus $\gamma(G) + \gamma(\overline{G}) \geq 5$ and by Corollary 7, $\gamma(G) + \gamma(\overline{G}) = 6$. Without loss of generality, we may assume that $\gamma(G) \leq 3$. Let S be a minimum dominating set of G. Since $\gamma_{nt}(G) > \gamma(G)$, there must be an isolated vertex z in the subgraph induced by $N(S)$. Furthermore, since $\delta(G) \geq 3$ we obtain $|S| = 3$, z is adjacent to all S and S is independent. Let x be any vertex of S. Then $\{z, x\}$ dominates the complement of G, implying that $\gamma(G) + \gamma(\overline{G}) < 6$, a contradiction. This completes the proof of the theorem. □

4   Graphs G with diameter at least three

Our aim in this section is to characterize connected graphs G of order n with diameter at least three such that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \lceil \frac{n}{2} \rceil + 2$. Since $\text{diam}(G) \geq 3$, according to Observation 3, $\text{diam}(\overline{G}) \leq 3$. Let us first consider the case when $\text{diam}(\overline{G}) = 3$.

Proposition 17 *Let G and $\overline{G}$ be connected graphs with $\text{diam}(G), \text{diam}(\overline{G}) \geq 3$ such that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \lceil \frac{n}{2} \rceil + 2$. Then $G = P_4 = \overline{G}$.*

Proof. Since $\text{diam}(G) \geq 3$, any two non-adjacent vertices of G at distance at least three total dominates $\overline{G}$. Hence $\gamma_{nt}(\overline{G}) \leq \gamma_t(\overline{G}) \leq 2$. The equality is obtained from the connectedness of G and $\overline{G}$. Thus $\gamma_{nt}(\overline{G}) = 2$. Likewise, since $\text{diam}(\overline{G}) \geq 3$, we obtain $\gamma_{nt}(G) = 2$. Therefore $4 = \gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \lceil \frac{n}{2} \rceil + 2$ implies that $\lceil \frac{n}{2} \rceil = 2$ and so $n \in \{3, 4\}$. The case $n = 3$ is excluded since $\overline{G}$ is not connected. It follows that $n = 4$, implying that G is P_4 and so is $\overline{G}$. ☐

From now on, we focus on connected graphs G of diameter at least three whose complement graphs are connected with diameter two.

Proposition 18 *Let G and $\overline{G}$ be connected graphs of order n with $\text{diam}(G) \geq 3$ and $\text{diam}(\overline{G}) = 2$ such that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \lceil \frac{n}{2} \rceil + 2$. Then:*

i)   if n is odd, then $G \in F$;

ii) if n is even and $\delta(G) \geq 2$, then $G \in \{B_3, B_4, B_5\}$.

Proof. As seen in the proof of Theorem 17, since $\text{diam}(G) \geq 3$, we obtain $\gamma_{nt}(\overline{G}) = 2$. It follows that $\gamma_{nt}(G) = \lceil \frac{n}{2} \rceil$. Now if n is odd, then $\gamma_{nt}(G) = \frac{n+1}{2}$ and so by Theorem 9, $G = C_5$ or $G \in F$. The case $G = C_5$ is excluded since $\text{diam}(C_5) = 2$. Hence (i) follows.

Suppose now that n is even and $\delta(G) \geq 2$. Then $\gamma_{nt}(G) = \frac{n}{2}$, and by Theorem 10, $G \in \{C_5, B_1, B_2, B_3, B_4, B_5\}$. Since $\text{diam}(B_1) = \text{diam}(B_2) = 2$, we conclude that $G \in \{B_3, B_4, B_5\}$ and (ii) is proved. ☐

According to Proposition 18 (ii), one can wonder whether there are connected graphs G of even order n with minimum degree $\delta(G) = 1$ such that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \frac{n}{2} + 2$. We first give a positive answer to this question and then we provide a necessary condition for such graphs.

Observation 19 *For every even integer $n \geq 6$, there is a connected graph G of order n with $\delta(G) = 1$, $\text{diam}(G) \geq 3$ and $\text{diam}(\overline{G}) = 2$ satisfying $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \frac{n}{2} + 2$.*

Proof. For every even integer $n \geq 6$, we consider the graph $G = \text{cor}(H)$, where H is any connected graph of order $n/2$. It can be seen easily that $\gamma_{nt}(G) = n/2$ and $\gamma_{nt}(\overline{G}) = 2$. ☐

In the proof of Observation 19, we gave a family of graphs (coronas graphs) that have the property given in the observation, but there may be many graphs other than coronas with this property.

Theorem 20 *If G and $\overline{G}$ are connected graphs of even order n with $\delta(G) = 1$, $\mathrm{diam}(G) \geq 3$ and $\mathrm{diam}(\overline{G}) = 2$ such that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \frac{n}{2} + 2$, then $\frac{n}{4} \leq \gamma(G)$ and $\Delta(G) \leq \frac{3n}{4}$.*

Proof. Clearly $\gamma_{nt}(\overline{G}) = 2$ since $\mathrm{diam}(G) \geq 3$. Hence $\gamma_{nt}(G) = n/2$ and by Theorem 11, we obtain $\frac{n}{4} \leq \gamma(G)$. On the other hand, it is well known that $\gamma(G) \leq n - \Delta$, which implies that $\Delta \leq \frac{3n}{4}$. $\qquad\qquad\square$

We mention that the bounds in Theorem 20 are not necessarily the best bounds. Finally, we end the paper with the following open problem:

Problem 21 *Characterize the graph G where G and $\overline{G}$ are connected of even order n with $\delta(G) = 1$, $\mathrm{diam}(G) \geq 3$ and $\mathrm{diam}(\overline{G}) = 2$ such that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \frac{n}{2} + 2$.*

Acknowledgements

References

[1] S. Arumugam and C. Sivagnanam, Neighborhood total domination in graphs, *Opuscula Math.* 31 (2011), 519–531.

[2] J.E. Dunbar, T.W. Haynes and S.T. Hedetniemi, Nordhaus-Gaddum bounds for domination sums in graphs with specified minimum degree, *Util. Math.* 67 (2005), 97–105.

[3] F. Harary and R.W. Robinson, The diameter of a graph and its complement, *Amer. Math. Monthly* 92 (1985), 211–212.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[5] A. Hellwig and L. Volkmann, Some upper bounds for the domination number, *J. Combin. Math. Combin. Comput.* 57 (2006), 187–209.

[6] M.A. Henning, Recent results on total domination in graphs: A survey, *Discrete Math.* 309 (2009), 32–63.

[7] M.A. Henning and N. Jafari Rad, Bounds on neighborhood total domination in graphs, *Discrete Appl. Math.* 161 (2013), 2460–2466.

[8] L. Volkmann, Nordhaus-Gaddum bounds for domination sums of graphs with minimum degree at least two or three, *Util. Math.* 82 (2010), 3–9.