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Supplementary difference sets related to a certain class of complex spherical 2-codes

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Dedicated to Professor Hiroshi Kimura on his 80th birthday

Abstract

In this paper, we study skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary difference sets related to a certain class of complex spherical 2-codes. A classification of such supplementary difference sets is completed for $v \leq 51$.

1 Introduction

Let $\Omega(d)$ denote the complex unit sphere in $\mathbb{C}^d$. For a finite set X in $\Omega(d)$, define

$$A(X) = \{x^* y \mid x, y \in X, x \neq y\},$$

where x^* is the transpose conjugate of a column vector x. A finite set X is called a *complex spherical* 2*-code* if $|A(X)| = 2$ and $A(X)$ contains an imaginary number. A complex spherical 2-code X with $A(X) = \{\alpha, \bar{\alpha}\}$ has the structure of a tournament (X, E), where $E = \{(x, y) \in X \times X \mid x^*y = \alpha\}$ [12]. We say that the tournament (X, E) is attached to the complex spherical 2-code X.

Theorem 1.1 (Nozaki and Suda [12, Theorem 4.8]). *Let X be a complex spherical 2-code in $\Omega(d)$. Let A be the adjacency matrix of the tournament G attached to X.*

(1) $|X| \leq 2d + 1$ *if d is odd, and* $|X| \leq 2d$ *if d is even.*

(2) $|X| = 2d + 1$ *for odd d if and only if G is a doubly regular tournament.*

(3) $|X| = 2d$ *for even d if and only if $I + A - A^T$ is a skew-Hadamard matrix, where I is the identity matrix and A^T denotes the transposed matrix of A.*

(4) $|X| = 2d$ *for odd d if and only if one of the following occurs:*

(a) *A is obtained as the adjacency matrix of the induced subgraph of some doubly regular tournament by deleting a certain vertex.*

(b) *There exists a permutation matrix P such that*

$$P(I + A - A^T)(I + A - A^T)^T P^T = \begin{pmatrix} \alpha I + \beta J & O \\ O & \alpha I + \beta J \end{pmatrix},$$

for some integers α, β with $\alpha \geq 2, \beta \geq 1$, where J denotes the all-one matrix and O denotes the zero matrix of appropriate size.

Doubly regular tournaments have been widely studied (see e.g., [8, 11, 13–15]). Skew-Hadamard matrices are a class of Hadamard matrices, which has been widely studied (see e.g., [2, 5, 6, 11, 13, 15, 18]). These motivate our investigation of matrices M satisfying the following conditions:

$$M \text{ is a } 2d \times 2d\ (1, -1)\text{-matrix with } d \text{ odd}, \tag{1}$$

$$M - I = -(M - I)^T, \text{ that is, } M \text{ is skew-symmetric}, \tag{2}$$

$$MM^T = \begin{pmatrix} \alpha I + \beta J & O \\ O & \alpha I + \beta J \end{pmatrix} \text{ for some integers } \alpha, \beta \text{ with } \alpha \geq 2, \beta \geq 1. \tag{3}$$

In this paper, with this motivation, we study skew-symmetry for 2-$\{v; r, k; \lambda\}$ supplementary difference sets satisfying the following conditions:

$$v \text{ is an odd positive integer}, \tag{4}$$

$$4(r + k - \lambda) \geq 2, \tag{5}$$

$$2(v - 2(r + k - \lambda)) \geq 1. \tag{6}$$

These supplementary difference sets give matrices M satisfying (1)–(3) (Proposition 3.2).

This paper is organized as follows. In Section 2, we give definitions and we recall notions on supplementary difference sets and D-optimal designs. Some basic facts on these subjects are also provided. In Section 3, we give some observations on skew-symmetric supplementary difference sets. In Section 4, we describe how to classify skew-symmetric supplementary difference sets satisfying (4)–(6). In Section 5, we give a classification of skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary differences sets satisfying (4)–(6) for $v \leq 51$ (Theorem 5.1). This is the main result of this paper. Skew-symmetric circulant D-optimal designs satisfying (9) are corresponding to a special class of supplementary difference sets. In Section 6, as a consequence of Theorem 5.1, we give a classification of skew-symmetric circulant D-optimal designs meeting (8) for orders up to 110.

2 Preliminaries

In this section, we give definitions and we recall notions on supplementary difference sets and D-optimal designs. Some basic facts on these subjects are also provided.

2.1 Supplementary difference sets

Let $\mathbb{Z}_v = \{0, 1, \ldots, v-1\}$ be the ring of integers modulo v, where $v > 2$. For $A \subset \mathbb{Z}_v$ and $i \in \mathbb{Z}_v$, define

$$P_A(i) = |\{(x, y) \in A \times A \mid y - x = i\}| \text{ and } P_A = (P_A(1), P_A(2), \ldots, P_A(v-1)).$$

Let A and B be an r-subset and a k-subset of $\mathbb{Z}_v$, respectively. If a pair (A, B) satisfies

$$P_A + P_B = (\lambda, \lambda, \ldots, \lambda),$$

then it is called a 2-$\{v; r, k; \lambda\}$ *supplementary difference set*. We refer to [3, 9, 16, 17] for basic facts on supplementary difference sets.

Lemma 2.1 (Wallis [16, Lemma 1]). *If there exists a* 2-$\{v; r, k; \lambda\}$ *supplementary difference set, then*

$$r(r-1) + k(k-1) = \lambda(v-1). \tag{7}$$

Chadjipantelis and Kounias [3, Appendix] gave a correspondence between 2-$\{v; r, k; \lambda\}$ supplementary difference sets and pairs of circulant matrices. Let A and B be an r-subset and a k-subset of $\mathbb{Z}_v$, respectively. Let R_1 and R_2 be the circulant $v \times v$ $(1, -1)$-matrices with first rows $r_1 = (r_{1,1}, r_{1,2}, \ldots, r_{1,v})$ and $r_2 = (r_{2,1}, r_{2,2}, \ldots, r_{2,v})$, respectively. The correspondence was defined as follows: $r_{1,i+1} = -1$ if $i \in A$, $r_{1,i+1} = 1$ if $i \notin A$ and $r_{2,i+1} = -1$ if $i \in B$, $r_{2,i+1} = 1$ if $i \notin B$.

Lemma 2.2 (Chadjipantelis and Kounias [3, Appendix]). *A pair* (A, B) *is a* 2-$\{v; r, k; \lambda\}$ *supplementary difference set if and only if* $R_1 R_1^T + R_2 R_2^T = 4(r + k - \lambda)I + 2(v - 2(r + k - \lambda))J$.

2.2 D-optimal designs and supplementary difference sets

A *D-optimal design* of order n is an $n \times n$ $(1, -1)$-matrix having maximum determinant. Ehlich [4] showed that for $n \equiv 2 \pmod 4$ and $n > 2$, any $n \times n$ $(1, -1)$-matrix M satisfies

$$\det M \leq (2n-2)(n-2)^{(n-2)/2}, \tag{8}$$

and that equality is possible only if $2n-2$ is a sum of two perfect squares. Moreover, if $n = 2v \equiv 2 \pmod 4$, and both R_1 and R_2 are $v \times v$ commutative $(1, -1)$-matrices such that

$$R_1 R_1^T + R_2 R_2^T = (2v-2)I + 2J, \tag{9}$$

then

$$X(R_1, R_2) = \begin{pmatrix} R_1 & R_2 \\ -R_2^T & R_1^T \end{pmatrix} \tag{10}$$

is a D-optimal design meeting the above bound (8) [4].

If the matrices R_1 and R_2 in (10) are circulant, then $X(R_1, R_2)$ is called a *circulant D-optimal design* meeting (8) [10]. Most of the known D-optimal designs meeting (8) are circulant (see e.g., [1,3,5,9,10]). If $X(R_1, R_2)$ is a circulant D-optimal design meeting (8), then it was shown in [3] that

$$(v-2r)^2 + (v-2k)^2 = 2n-2, \tag{11}$$

where r and k are the numbers of -1's in the first rows of R_1 and R_2, respectively.

By Lemma 2.2, we have the following:

Lemma 2.3 (Chadjipantelis and Kounias [3, Appendix]). *Let A and B be an r-subset and a k-subset of $\mathbb{Z}_v$, respectively. Let R_1 and R_2 be the corresponding circulant $v \times v$ $(1, -1)$-matrices described in Section 2.1. A pair (A, B) is a 2-$\{v; r, k; r+k-(v-1)/2\}$ supplementary difference set if and only if $X(R_1, R_2)$ in (10) is a circulant D-optimal design of order $2v$ meeting (8), where r and k are the numbers of -1's in the first rows of R_1 and R_2, respectively.*

3 Skew-symmetric supplementary difference sets

Let (A, B) be a supplementary difference set. Let R_1 and R_2 denote the corresponding circulant $v \times v$ $(1, -1)$-matrices described in Section 2.1. Then we consider the following matrix:

$$X(R_1, R_2) = \begin{pmatrix} R_1 & R_2 \\ -R_2^T & R_1^T \end{pmatrix}. \tag{12}$$

We call (A, B) *skew-symmetric* if the corresponding matrix $X(R_1, R_2)$ in (12) is skew-symmetric. Equivalently, (A, B) is skew-symmetric if A satisfies the condition that $0 \notin A$ and if $i \in A$ then $-i \notin A$. In [2, 15, 18], skew-symmetric 2-$\{v; (v-1)/2, (v-1)/2; (v-3)/2\}$ supplementary difference sets are called complementary difference sets and these difference sets were used to construct skew-Hadamard matrices.

Lemma 3.1. *The matrix* $X(R_1, R_2)$ *in* (12) *is skew-symmetric if and only if* $r_{1,1} = 1$ *and* $r_{1,i} = -r_{1,v+2-i}$ $(i = 2, 3, \ldots, v)$. *If* $X(R_1, R_2)$ *is skew-symmetric, then* $r = \frac{v-1}{2}$.

Proof. The elementary proof is omitted. □

By Lemma 2.2, we have the following:

Proposition 3.2. *If there exists a skew-symmetric* 2-$\{v; r, k; \lambda\}$ *supplementary difference set satisfying* (4)–(6). *Then there exists a matrix* M *satisfying* (1)–(3) *for* $(\alpha, \beta) = (4(r + k - \lambda), 2(v - 2(r + k - \lambda)))$.

Now we give a remark on the condition (5) for skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary difference sets.

Proposition 3.3. *Suppose that* $k \le \frac{v-1}{2}$. *If there exists a skew-symmetric* 2-$\{v; r, k; \lambda\}$ *supplementary difference set* (A, B), *then* $r + k - \lambda \ge 1$, *that is,* (A, B) *satisfies* (5).

Proof. By Lemma 3.1, $r = \frac{v-1}{2}$. Hence, it follows from (7) that

$$r + k - \lambda = \frac{(v + 2k)(v - 2k) - 1 + 4kv}{4(v - 1)}.$$

From the assumption, $r + k - \lambda > 0$. The result follows. □

For the case $k \in \{0, 1\}$, 2-$\{v; r, k; \lambda\}$ supplementary difference sets are characterized as follows. Although the following characterization is somewhat trivial, it was not explicitly stated in the literature. We give a proof for the sake of completeness.

Proposition 3.4. *The following statements are equivalent.*

(1) *There exists a skew-symmetric* 2-$\{4m - 1; 2m - 1, k; m - 1\}$ *supplementary difference set with* $k = 0$ *and* 1.

(2) *There exists a circulant Hadamard* 2-$(4m - 1, 2m - 1, m - 1)$ *design with incidence matrix* M *satisfying that* $M + M^T + I = J$.

Proof. Suppose that there exists a skew-symmetric 2-$\{4m - 1; 2m - 1, k; m - 1\}$ supplementary difference set (A, B) with $k \in \{0, 1\}$. Then A is a $(4m - 1, 2m - 1, m - 1)$-difference set. Let M be an incidence matrix of A. Then M is an incidence matrix of a circulant Hadamard 2-$(4m-1, 2m-1, m-1)$ design. Since A satisfies the condition that if $i \in A$ then $-i \notin A$, M satisfies the condition that $M + M^T + I = J$.

Suppose that there exists a circulant Hadamard 2-$(4m - 1, 2m - 1, m - 1)$ design with incidence matrix M satisfying that $M + M^T + I = J$. By reversing the above argument, a $(4m - 1, 2m - 1, m - 1)$-difference set A satisfying the condition that if $i \in A$ then $-i \notin A$ is constructed. Then $(A, \emptyset)$ and $(A, \{0\})$ are skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary difference sets with parameters $(v, r, k, \lambda) = (4m-1, 2m-1, 0, m - 1)$ and $(4m - 1, 2m - 1, 1, m - 1)$, respectively. □

Suppose that p is a prime with $p \equiv 3 \pmod 4$. Then it is well known that there exists a circulant Hadamard 2-$(p, \frac{p-1}{2}, \frac{p-3}{4})$ design with incidence matrix A satisfying that $A + A^T + I = J$ (see e.g., [7, Lemma 7.10]). This implies the existence of skew-symmetric supplementary difference sets with parameters 2-$\{p; \frac{p-1}{2}, 0; \frac{p-3}{4}\}$ and 2-$\{p; \frac{p-1}{2}, 1; \frac{p-3}{4}\}$.

4 Classification method

In this section, we describe how to classify skew-symmetric supplementary difference sets satisfying (4)–(6).

4.1 Equivalent supplementary difference sets

If (A, B) is a supplementary difference set, then the following pairs

(E0) $(\mathbb{Z}_v \backslash A, B)$ and $(A, \mathbb{Z}_v \backslash B)$,

(E1) (B, A),

(E2) $(\pm A + a, \pm B + b)$ for any $a, b \in \mathbb{Z}_v$,

(E3) (dA, dB) for any $d \in U(\mathbb{Z}_v)$

are also supplementary difference sets, where $U(\mathbb{Z}_v) = \{d \in \{1, 2, \ldots, v-1\} \mid \gcd(d, v) = 1\}$ and d is regarded as an integer for $\gcd(d, v) = 1$. These supplementary difference sets are called *equivalent* [10].

4.2 Classification method

Let (A, B) be a skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary difference set satisfying (4), (6). By Lemma 3.1, $r = \frac{v-1}{2}$. By (E0), we may assume without loss of generality that $k \le \frac{v-1}{2}$. We note that (A, B) satisfies (5) by Proposition 3.3 under this assumption. In addition, if $r = k$, then it follows from (7) that $2(v - 2(r + k - \lambda)) = -2$. Hence, we may assume without loss of generality that

$$k < \frac{v-1}{2} = r. \tag{13}$$

Since A corresponds to a skew-symmetric matrix, there exists $A' \subset \mathbb{Z}'_v$ such that $A = A' \cup \{v - j \mid j \in \mathbb{Z}'_v \setminus A'\}$, where $\mathbb{Z}'_v = \{1, 2, \ldots, (v-1)/2\} \subset \mathbb{Z}_v$. By (E2), $(A, B + b)$ is a skew-symmetric supplementary difference set for any $b \in \mathbb{Z}_v$. We classify skew-symmetric 2-$\{v; (v-1)/2, k; \lambda\}$ supplementary difference sets satisfying (4), (6) by the following steps.

(i) We calculate $\overline{\mathcal{A}} = \{A' \cup \{v - j \mid j \in \mathbb{Z}'_v \setminus A'\} \mid A' \subset \mathbb{Z}'_v\}$. Then we find $\mathcal{A} = \{A \in \overline{\mathcal{A}} \mid P_A(i) \le \lambda \text{ for all } i\}$.

(ii) We calculate $\overline{\mathcal{B}} = \{B \subset \mathbb{Z}_v \mid |B| = k, B \preceq B + b \text{ for any } b \in \mathbb{Z}_v\}$, where $\preceq$ is a natural lexicographic order on k-subsets of $\mathbb{Z}_v$. Then we find $\mathcal{B} = \{B \in \overline{\mathcal{B}} \mid P_B(i) \leq \lambda \text{ for all } i\}$.

(iii) We construct $\mathcal{AB} = \{(A, B) \in \mathcal{A} \times \mathcal{B} \mid P_A + P_B = (\lambda, \lambda, \dots, \lambda)\}$.

(iv) We classify $\mathcal{AB}$.

In Step (i) (resp. (ii)), we found all $((v-1)/2)$-subsets (resp. k-subsets) of $\mathbb{Z}_v$ by a computer program implemented in `C` language using functions from the GNU Scientific Library (GSL) software library, then we output A and $(\lambda, \lambda, \dots, \lambda) - P_A$ (resp. B and P_B) to a file. We sorted the above data by $(\lambda, \lambda, \dots, \lambda) - P_A$ (resp. P_B). We found a pair (A, B) with $(\lambda, \lambda, \dots, \lambda) - P_A = P_B$ in Step (iii). Two skew-symmetric 2-$\{v; (v-1)/2, k; \lambda\}$ supplementary difference sets (A, B) and (A', B') are equivalent if and only if (A', B') is an element of $\{(\pm dA + a, \pm dB + b) \mid d \in U(\mathbb{Z}_v), a, b \in \mathbb{Z}_v\}$. In Step (iv), for (A, B) and (A', B'), we determined whether there exist $d \in U(\mathbb{Z}_v)$ and $a, b \in \mathbb{Z}_v$ such that $(A', B') = (dA + a, dB + b), (dA + a, -dB + b), (-dA + a, dB + b)$ or $(-dA + a, -dB + b)$. This was done by using the program implemented in `C` language.

5 Classification of skew-symmetric supplementary difference sets

In this section, we give a classification of skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary differences sets satisfying (4)–(6) for $v \leq 51$. This is the main result of this paper. As described in Proposition 3.2, a skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary difference set satisfying (4)–(6), gives a matrix M satisfying (1)–(3) for $(\alpha, \beta) = (4(r + k - \lambda), 2(v - 2(r + k - \lambda)))$.

We call (v, r, k, λ) *feasible parameters* for supplementary difference sets if (v, r, k, λ) satisfies (4)–(6), (7) and (13) (see Proposition 3.3 for (5)). In Table 1, we list the feasible parameters (v, r, k, λ) for $v \leq 75$.

By an approach given in Section 4, our exhaustive computer search completed a classification of skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary difference sets satisfying (4)–(6) for the feasible parameters in Table 1 with $v \leq 51$. We used a computer with CPU Intel(R) Core(TM) i7 4790k, 4 Core.

Theorem 5.1. *Suppose that* $v \leq 51$. *If there exists a skew-symmetric* 2-$\{v; r, k; \lambda\}$ *supplementary difference sets satisfying* (4)–(6)*, then it is equivalent to one of the supplementary difference sets* (A, B) *with* $v \leq 51$ *in Table 3.*

For $v \geq 53$, due to the computational complexity, our exhaustive computer search completed a classification of skew-symmetric 2-$\{v; r, k; \lambda\}$ supplementary difference

Table 1: Parameters of skew-symmetric supplementary difference sets

(v, r, k, λ)	$N(v, r, k, \lambda)$	(v, r, k, λ)	$N(v, r, k, \lambda)$
(3, 1, 0, 0)	1	(43, 21, 15, 15)	0
(7, 3, 0, 1)	1	(45, 22, 11, 13)	0
(7, 3, 1, 1)	1	(47, 23, 0, 11)	1
(11, 5, 0, 2)	1	(47, 23, 1, 11)	1
(11, 5, 1, 2)	1	(49, 24, 9, 13)	0
(13, 6, 3, 3)	1	(51, 25, 0, 12)	0
(15, 7, 0, 3)	0	(51, 25, 1, 12)	0
(15, 7, 1, 3)	0	(53, 26, 14, 16)	?
(19, 9, 0, 4)	1	(55, 27, 0, 13)	0
(19, 9, 1, 4)	1	(55, 27, 1, 13)	0
(21, 10, 6, 6)	1	(57, 28, 21, 21)	?
(23, 11, 0, 5)	1	(59, 29, 0, 14)	1
(23, 11, 1, 5)	1	(59, 29, 1, 14)	1
(25, 12, 4, 6)	0	(61, 30, 6, 15)	0
(27, 13, 0, 6)	0	(61, 30, 10, 16)	0
(27, 13, 1, 6)	0	(61, 30, 15, 18)	?
(29, 14, 7, 8)	1	(63, 31, 0, 15)	0
(31, 15, 0, 7)	1	(63, 31, 1, 15)	0
(31, 15, 1, 7)	1	(67, 33, 0, 16)	1
(31, 15, 6, 8)	1	(67, 33, 1, 16)	1
(31, 15, 10, 10)	1	(67, 33, 12, 18)	?
(35, 17, 0, 8)	0	(67, 33, 22, 23)	?
(35, 17, 1, 8)	0	(69, 34, 18, 21)	?
(37, 18, 10, 11)	0	(71, 35, 0, 17)	1
(39, 19, 0, 9)	0	(71, 35, 1, 17)	1
(39, 19, 1, 9)	0	(71, 35, 15, 20)	?
(41, 20, 5, 10)	0	(71, 35, 21, 23)	?
(43, 21, 0, 10)	1	(73, 36, 28, 28)	?
(43, 21, 1, 10)	1	(75, 37, 0, 18)	0
(43, 21, 7, 11)	0	(75, 37, 1, 18)	0

sets satisfying (4)–(6) for the following feasible parameters:

$$\begin{aligned}(v,r,k,\lambda) = \ & (55,27,0,13),(55,27,1,13),(59,29,0,14),\\ & (59,29,1,14),(61,30,6,15),(61,30,10,16),\\ & (63,31,0,15),(63,31,1,15),(67,33,0,16),\\ & (67,33,1,16),(71,35,0,17),(71,35,1,17),\\ & (75,37,0,18),(75,37,1,18).\end{aligned} \quad (14)$$

The skew-symmetric 2-$\{v;r,k;\lambda\}$ supplementary difference sets (A,B) with parameters (14) are listed in Table 3. For the feasible parameters (v,r,k,λ) given in Table 1, the numbers $N(v,r,k,\lambda)$ of the inequivalent skew-symmetric 2-$\{v;r,k;\lambda\}$ supplementary difference sets are also listed in the table.

6 Classification of skew-symmetric circulant D-optimal designs meeting (8)

Skew-symmetric circulant D-optimal designs $X(R_1,R_2)$ in (10) are corresponding to a certain class of skew-symmetric supplementary difference sets satisfying (4)–(6). According to [10], we say that circulant D-optimal designs meeting (8) are *equivalent* if the supplementary difference sets constructed by Lemma 2.3 are equivalent. In this section, as a consequence of the previous section, we give a classification of skew-symmetric circulant D-optimal designs meeting (8) for orders up to 110.

Let D be a circulant D-optimal design $X(R_1,R_2)$ in (10) of order $n=2v$ meeting (8). Here we suppose that r and k are the numbers of -1's in the first rows of R_1 and R_2, respectively. If D is skew-symmetric, then $r=\frac{v-1}{2}$ by Lemma 3.1.

We call (n,r,k) *feasible parameters* for skew-symmetric circulant D-optimal designs if (n,r,k) satisfies $r=\frac{v-1}{2}$ and (11). In Table 2, we list the feasible parameters (n,r,k) for $n\le 200$.

Table 2: Parameters of skew-symmetric circulant D-optimal designs

(n,r,k)	$N(n,r,k)$
$(6,1,0)$	1
$(14,3,1)$	1
$(26,6,3)$	1
$(42,10,6)$	1
$(62,15,10)$	1
$(86,21,15)$	0
$(114,28,21)$	?
$(146,36,28)$	?
$(182,45,36)$	?

Let S_3, S_7, S_{13}, S_{21} and S_{31} be the skew-symmetric 2-$\{v;r,k;\lambda\}$ supplementary difference sets in Table 3 with $(v,r,k,\lambda)=(3,1,0,0)$, $(7,3,1,1)$, $(13,6,3,3)$,

$(21, 10, 6, 6)$ and $(31, 15, 10, 10)$, respectively. Let D_6, D_{14}, D_{26}, D_{42} and D_{62} be the skew-symmetric circulant D-optimal designs $X(R_1, R_2)$ in (10) of orders $6, 14, 26, 42$ and 62 meeting (8), constructed by Lemma 2.3 from S_3, S_7, S_{13}, S_{21} and S_{31}, respectively. From the classification in Theorem 5.1, we have the following:

Corollary 6.1. *Suppose that $n \leq 110$. If there exists a skew-symmetric circulant D-optimal design $X(R_1, R_2)$ in* (10) *of order n meeting* (8)*, then it is equivalent to one of D_6, D_{14}, D_{26}, D_{42} and D_{62}.*

The numbers $N(n, r, k)$ of the inequivalent skew-symmetric circulant D-optimal designs meeting (8) are also listed in Table 2 for the feasible parameters (n, r, k).

A classification of circulant D-optimal designs $X(R_1, R_2)$ in (10) meeting (8) was given in [10] for orders $n \leq 58$ and $n = 66$, and in [1] for orders $n = 62, 74$ (see [1] for the revised classification for order 26). Our computer search found that D_n $(n = 6, 14, 26)$ is equivalent to the circulant D-optimal design, which is constructed by Lemma 2.3 from the first supplementary difference set given in [10, Table 1], D_{42} is equivalent to the circulant D-optimal design, which is constructed by Lemma 2.3 from the 19th supplementary difference set given in [10, Table 1], and D_{62} is equivalent to the circulant D-optimal design, which is constructed by Lemma 2.3 from the 50th supplementary difference set given in [1, Appendix].

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Table 3: Skew-symmetric supplementary difference sets (A, B)

$(v,r,k,\lambda) = (3,1,0,0)$ $A = \{2\}$ $B = \emptyset$
$(v,r,k,\lambda) = (7,3,0,1), (7,3,1,1)$ $A = \{3,5,6\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (11,5,0,2), (11,5,1,2)$ $A = \{2,6,7,8,10\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (13,6,3,3)$ $A = \{4,7,8,10,11,12\}$ $B = \{0,2,8\}$
$(v,r,k,\lambda) = (19,9,0,4), (19,9,1,4)$ $A = \{2,3,8,10,12,13,14,15,18\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (21,10,6,6)$ $A = \{2,3,9,11,13,14,15,16,17,20\}$ $B = \{0,1,7,9,12,17\}$
$(v,r,k,\lambda) = (23,11,0,5), (23,11,1,5)$ $A = \{5,7,10,11,14,15,17,19,20,21,22\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (29,14,7,8)$ $A = \{4,5,6,8,9,10,12,13,15,18,22,26,27,28\}$ $B = \{0,1,11,13,15,18,21\}$
$(v,r,k,\lambda) = (31,15,0,7), (31,15,1,7)$ $A = \{3,6,11,12,13,15,17,21,22,23,24,26,27,29,30\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (31,15,6,8)$ $A = \{3,6,11,12,13,15,17,21,22,23,24,26,27,29,30\}$ $B = \{0,1,15,20,22,28\}$
$(v,r,k,\lambda) = (31,15,10,10)$ $A = \{4,6,7,12,16,17,18,20,21,22,23,26,28,29,30\}$ $B = \{0,1,4,5,8,11,16,18,20,29\}$
$(v,r,k,\lambda) = (43,21,0,10), (43,21,1,10)$ $A = \{2,3,5,7,8,12,18,19,20,22,26,27,28,29,30,32,33,34,37,39,42\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (47,23,0,11), (47,23,1,11)$ $A = \{5,10,11,13,15,19,20,22,23,26,29,30,31,33,35,38,39,40,41,43,44,45,46\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (59,29,0,14), (59,29,1,14)$ $A = \{2,6,8,10,11,13,14,18,23,24,30,31,32,33,34,37,38,39,40,42,43,44,47,50,52,54,55,56,58\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (67,33,0,16), (67,33,1,16)$ $A = \{2,3,5,7,8,11,12,13,18,20,27,28,30,31,32,34,38,41,42,43,44,45,46,48,50,51,52,53,57,58,61,63,66\}$ $B = \emptyset, \{0\}$
$(v,r,k,\lambda) = (71,35,0,17), (71,35,1,17)$ $A = \{7,11,13,14,17,21,22,23,26,28,31,33,34,35,39,41,42,44,46,47,51,52,53,55,56,59,61,62,63,65,66,67,68,69,70\}$ $B = \emptyset, \{0\}$

References

[1] M. Araya and M. Harada, Classification of circulant D-optimal designs of orders 62 and 74, *J. Statis. Appl.* **4** (2009), 301–312.

[2] D. Blatt and G. Szekeres, A skew Hadamard matrix of order 52, *Canad. J. Math.* **21** (1969), 1319–1322.

[3] Th. Chadjipantelis and S. Kounias, Supplementary difference sets and D-optimal designs for $n \equiv 2 \pmod 4$, *Discrete Math.* **57** (1985), 211–216.

[4] H. Ehlich, Determinantenabschätzungen für binäre Matrizen, *Math. Z.* **83** (1964), 123–132.

[5] R. J. Fletcher, C. Koukouvinos and J. Seberry, New skew-Hadamard matrices of order $4 \cdot 59$ and new D-optimal designs of order $2 \cdot 59$, *Discrete Math.* **286** (2004), 251–253.

[6] A. V. Geramita and J. Seberry, *Orthogonal Designs, Quadratic Forms and Hadamard Matrices*, Marcel Dekker Inc., New York, 1979.

[7] A. S. Hedayat, N. J. A. Sloane and J. Stufken, *Orthogonal Arrays*, Springer-Verlag, New York, 1999.

[8] S. J. Kirkland and B. L. Shader, Tournament matrices with extremal spectral properties, *Linear Algebra Appl.* **196** (1994), 1–17.

[9] C. Koukouvinos, S. Kounias and J. Seberry, Supplementary difference sets and optimal designs, *Discrete Math.* **88** (1991), 49–58.

[10] S. Kounias, C. Koukouvinos, N. Nikolaou and A. Kakos, The nonequivalent circulant D-optimal designs for $n \equiv 2 \bmod 4, n \leq 54, n = 66$, *J. Combin. Theory Ser. A* **65** (1994), 26–38.

[11] H. Nozaki and S. Suda, A characterization of skew Hadamard matrices and doubly regular tournaments, *Linear Algebra Appl.* **437** (2012), 1050–1056.

[12] H. Nozaki and S. Suda, Complex spherical codes with two inner products, *European J. Combin.* **51** (2016), 511–518.

[13] K. B. Reid and E. Brown, Doubly regular tournaments are equivalent to skew Hadamard matrices, *J. Combin. Theory Ser. A* **12** (1972), 332–338.

[14] P. Rowlinson, On 4-cycles and 5-cycles in regular tournaments, *Bull. London Math. Soc.* **18** (1986), 135–139.

[15] G. Szekeres, Tournaments and Hadamard matrices, *Enseignement Math. (2)* **15** (1969), 269–278.

[16] J. (Seberry) Wallis, On supplementary difference sets, *Aequat. Math.* **8** (1972), 242–257.

[17] J. (Seberry) Wallis, A note on supplementary difference sets, *Aequat. Math.* **10** (1974), 46–49.

[18] A.L. Whiteman, An infinite family of skew Hadamard matrices, *Pacific J. Math.* **38** (1971), 817–822.

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