AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **65(3)** (2016), Pages 200–211

Local gap colorings from edge labelings

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Abstract

We study a local version of gap vertex-distinguishing edge coloring. From an edge labeling $f\colon E(G) \to \{1,\dots,k\}$ of a graph G, an induced vertex coloring c is obtained by coloring the vertices with the greatest difference between incident edge labels. The local gap chromatic number $\chi^e_\Delta(G)$ is

* Partially funded by NSF GK-12 Transforming Experiences Grant DGE-0742434.
† Partially funded by Simons Foundation grant #276726.

the minimum k for which there exists an edge coloring such that $c(u) \neq c(v)$ for all edges uv. We prove that $\chi(G) \leq \chi^e_\Delta(G) \leq \chi(G) + 1$ for all graphs G, where $\chi(G)$ denotes the chromatic number of G. Further, we provide graph classes attaining both bounds.

1 Introduction

Unless otherwise stated, a graph G is simple, finite, and undirected with no isolated vertex. Standard graph theory notation ([13]) is used throughout.

Derived graph colorings, typically obtained from graph labelings, have been widely studied. In 1988, Chartrand et al. [2] introduced the *irregularity strength* of a graph G, which is the smallest positive integer k such that each edge can have a label from $[k] := \{1, \dots, k\}$ so that the sum of labels of edges incident to each vertex is distinct. This topic was further studied in [1, 3, 8], among others. In 2008, Gyori et al. [4] introduced a variation that seeks the smallest positive integer k such that each edge can have a label from $[k]$ so that the sets of the weights on edges incident to vertices are distinct.

In addition to the global constraints described above, local constraints have also been studied. The most active problem in this area, the 1-2-3 Conjecture, was proposed in 2004 by Karoński, Łuczak, and Thomason [6]. For a graph G, let $\chi^e_\Sigma(G)$ be the smallest positive integer k such that G has a labeling $\ell\colon E(G) \to [k]$ such that, for every edge $uv \in E(G)$, $\sum_{e \ni u} \ell(e) \neq \sum_{e \ni v} \ell(e)$.

1-2-3 Conjecture (Karoński, Łuczak, Thomason [6]). *If G has no component isomorphic to K_2, then $\chi^e_\Sigma(G) \leq 3$.*

Under the same assumptions, Kalkowski, Karoński, Pfender [5] showed that $\chi^e_\Sigma(G) \leq 5$. For a survey of work on the 1-2-3 Conjecture and derived colorings, we direct the reader to [11].

In this paper, we are interested in a particular derived vertex coloring called *gap vertex-distinguishing edge coloring*. Here, the derived coloring g_ℓ of a vertex is

$$g_\ell(v) = \begin{cases} \ell(e)_{e \ni v} & \text{if } d(v) = 1 \\ \max\limits_{e \ni v} \ell(e) - \min\limits_{e \ni v} \ell(e) & \text{otherwise,} \end{cases}$$

where $\ell : E(G) \to [k]$. An edge coloring $\ell\colon E(G) \to [k]$ of a graph G is called *gap vertex distinguishing* when all vertices have distinct colors. The minimum k such that a gap vertex-distinguishing edge coloring exists is called the *gap chromatic number* of G and denoted $\operatorname{gap}(G)$. Introduced by Tahraoui, Duchêne, and Kheddouci in [12], they conjectured that $\operatorname{gap}(G) \leq n(G) + 1$. In [9], Scheidweiler and Triesch prove this conjecture for connected graphs, but disprove it in general by finding a class of graphs with $\operatorname{gap}(G) = n(G) + 2$.

Tahraoui et al. [12] introduce a similar derived coloring that distinguished adjacent vertices. The *gap-adjacent-chromatic number* of a graph G, $\operatorname{gap}_{\mathrm{ad}}(G)$, is the minimum k for which a labeling $\ell : E(G) \to [k]$ exists so that g_ℓ induces a proper vertex coloring. Scheidweiler and Triesch [10] prove the following:

Theorem 1 ([10]). *If* $\chi(G) \in \{2,3\}$, *then* $\mathrm{gap}_{\mathrm{ad}}(G) \le \chi(G)+1$.

Theorem 2 ([10]). *If G is a graph without isolated edges, then*

$$\chi(G) - 1 \le \mathrm{gap}_{\mathrm{ad}}(G) \le \chi(G) + 5.$$

In this paper, we improve Theorem 2 by giving sharp bounds.

Theorem 3. *If G is a graph without isolated edges, then* $\chi(G) \le \mathrm{gap}_{\mathrm{ad}}(G) \le \chi(G)+1$ *unless G is a star, in which case* $\mathrm{gap}_{\mathrm{ad}}(G) = 1 = \chi(G) - 1$.

In order to define $\mathrm{gap}(G)$ for graphs with more than one leaf, the special treatment of leaves in the definition of g_ℓ is necessary. In the local version, leaves are permitted to have the same color. As such, it is natural to consider the following simpler definition for the gap color of vertices

$$c_\ell(v) = \max_{e \ni v} \ell(e) - \min_{e \ni v} \ell(e),$$

where $\ell : E(G) \to [k]$ and leaves do not receive special treatment. An edge coloring $\ell\colon E(G) \to [k]$ of a graph G is a *local gap k-coloring* when adjacent vertices have distinct colors under c_ℓ. The minimum k for which a local gap k-coloring exists is called the *local gap chromatic number* of G and denoted by $\chi^e_\Delta(G)$. In this paper, we prove the following:

Theorem 4. *If G has no isolated edges, then* $\chi^e_\Delta(G) \in \{\chi(G), \chi(G)+1\}$.

Despite the difference between c_ℓ and g_ℓ, we are able to use Theorem 4 to prove Theorem 3. All bounds in Theorem 3 and 4 are sharp, as we will discuss in the next section.

2 Sharpness Examples for Theorems 3 and 4

Assigning edge labels from $[k]$ allows for k vertex colors under c_ℓ, namely $0, \ldots, k-1$. Therefore, $\chi^e_\Delta(G) \ge \chi(G)$. Equality is achieved by many graphs, some of which we discuss in Section 5. A similar argument implies $\mathrm{gap}_{\mathrm{ad}}(G) \ge \chi(G) - 1$. However, not all graphs have $\chi^e_\Delta(G) = \chi(G)$. For example, consider a complete graph with a pendant edge added to each vertex. It is easy to see that $\chi^e_\Delta(G) = \chi(G) + 1$ and $\mathrm{gap}_{\mathrm{ad}}(G) = \chi(G)$.

For an example with $\chi^e_\Delta(G) = \mathrm{gap}_{\mathrm{ad}}(G) = \chi(G)+1$, consider the following graph. Let s and r be positive integers such that $s \ge r+1 \ge 3$, and let K^r_s be the complete r-partite graph with s vertices in each partite set $X_1, \ldots, X_r$. For all i and all $u, v \in X_i$, add a new vertex that is adjacent to u and v and call this graph G. Since G has no leaves, $\chi^e_\Delta(G) = \mathrm{gap}_{\mathrm{ad}}(G)$. We claim that $\chi^e_\Delta(G) = r+1 = \chi(G)+1$.

To see this, suppose that $\chi^e_\Delta(G) = r$. Let c_ℓ be a local gap r-coloring of G. Since $K^r_s \subseteq G$ and there exists exactly one partition of K^r_s into r independent sets, there is some partite set of K^r_s, say X_1, such that $c_\ell(v) = 0$ for all $v \in X_1$. For every $v \in X_1$, all edges incident to v have the same label. This partitions X_1 into at most

r classes depending on the label of the incident edges. Since $s \geq r+1$, there are two vertices $u, v \in X_1$ that have the same label on all incident edges. Thus, the vertex x outside K_s^r adjacent to u and v has $c_\ell(x) = 0$, a contradiction. Therefore, $\chi_\Delta^e(G) \geq r+1 = \chi(G)+1$; equality follows from Theorem 3.

One can generalize this construction by taking a large enough blow-up of any graph G and joining new vertices to every t-tuple from the independent sets of the blow-up corresponding to vertices of G.

3 An Upper Bound for $\chi_\Delta^e(G)$

We turn our attention to proving an upper bound for $\chi_\Delta^e(G)$. For this purpose, we define the following sets based on distance to any $X \subseteq V(G)$ (see Figure 1):

$$\begin{aligned} V_i(X) &= \{x \in V(G)\colon\ d(x, X) = i\}, \\ U_i(X) &= \{x \in V_i(X)\colon\ N(x) \subseteq V_{i-1}(X)\}, \\ E_i(X) &= \{xy \in E(G)\colon\ x \in V_{i-1}(X),\ y \in V_i(X)\}, \\ F_i(X) &= \{xy \in E_i(G)\colon\ y \in U_i(X)\}, \end{aligned}$$

where $i \in \mathbb{N}$, $V_0 = U_0 = X$. For $i \geq 2$, let

$$\begin{aligned} V'_{i-1}(X) &= \{x \in V_{i-1}(X)\colon\ \exists xy \in F_i(X)\} \\ F'_{i-1}(X) &= \{wx \in E_{i-1}(G)\colon\ x \in V'_{i-1}(X)\}. \end{aligned}$$

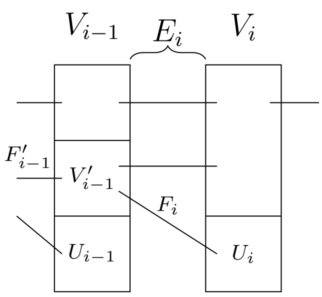


Figure 1: Sets defined for proofs.

For brevity we write v when $X = \{v\}$ and say that v has *gap color m* when $c_\ell(v) = m$. To prove Theorem 4, we cover three cases based on chromatic number in the following lemmas.

Lemma 1. *Let G be a connected bipartite graph not isomorphic to K_2. Then $\chi_\Delta^e(G) \leq 3$.*

Proof. Let v be a vertex in G that is not adjacent to a leaf. Define an edge labeling ℓ as follows for edge $e \in E_i(v)$ (see Figure 2):

$$\ell(e) = \begin{cases} 3 & \text{if } i \equiv 1 \mod 4, \text{ or both } i \equiv 0 \mod 4 \text{ and } e \in F_i(v), \\ 1 & \text{if } i \equiv 2 \mod 4, \text{ or both } i \equiv 3 \mod 4 \text{ and } e \in F'_i(v), \\ 2 & \text{otherwise.} \end{cases}$$

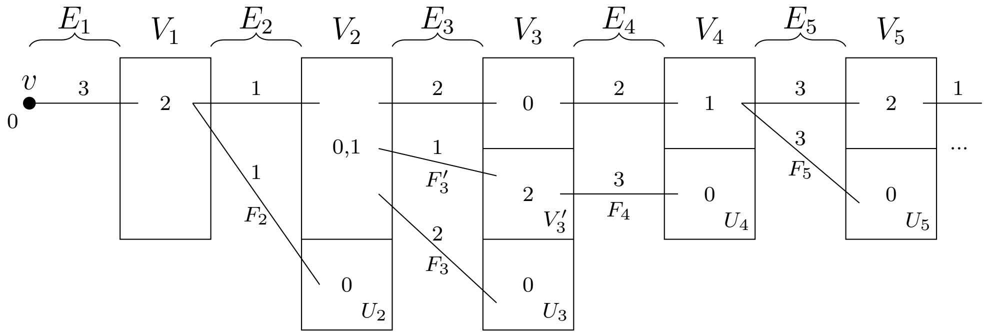


Figure 2: An edge labeling for connected bipartite graphs including derived gap colors.

Notice that each $V_i(v)$ is an independent set since G is bipartite. When i is even, the gap color of vertices in $V_i(v)$ is either 0 or 1. When i is odd, the gap color of vertices in $V_i(v)$ is either 0 or 2. Toward ℓ being a local gap 3-coloring, we show vertices with gap color 0 are independent. Since v is not adjacent to a leaf, $c_\ell(x) = 2$ for all $x \in V_1(v)$. Notice that every vertex in some $U_i(v)$ has gap color 0, and all of their neighbors have nonzero gap color. When $i \equiv 2 \mod 4$, every neighbor of a vertex in V_i with gap color 0 has gap color 2. Vertices in $V_i'(v)$ when $i \equiv 3 \mod 4$ have gap color 2 and are adjacent to vertices whose gap color is either 0 or 1. Thus, ℓ is a local gap 3-coloring. □

Lemma 2. *Let G be a connected tripartite graph. Then $\chi^e_\Delta(G) \le 4$.*

Proof. Let C_1, C_2, C_3 be the color classes of a proper coloring of $V(G)$ in which every vertex in C_i has a neighbor in C_j for $1 \le j < i \le 3$. Let $Y_1 = \{v \in C_1 \colon N(v) \subseteq C_2\}$, $Y_2 = C_1 \setminus Y_1$, $X_1 = \{v \in C_2 \colon N(v) \subseteq C_1\}$, and $X_2 = C_2 \setminus X_1$. We refine Y_1 and X_1 by defining $Y_{11} = \{v \in Y_1 \colon N(v) \subseteq X_1\}$, $Y_{12} = Y_1 \setminus Y_{11}$, $X_{11} = \{v \in X_1 \colon N(v) \subseteq Y_1\}$, and $X_{12} = X_1 \setminus X_{11}$. Define a partial edge labeling ℓ as follows (see Figure 3):

$$\ell(uv) = \begin{cases} 4 & \text{if } u \in C_3 \text{ and } v \in C_1, \\ 3 & \text{if } u \in C_3 \text{ and } v \in C_2, \\ 1 & \text{if } u \in X_2 \text{ and } v \in Y_2. \end{cases}$$

Notice that vertices in C_3, X_2, and Y_2 have gap colors 1, 2, and 3 respectively. Also notice that the remaining edges all live in a bipartite graph. We attempt to mimic Lemma 1 without changing the existing gap colors in C_3, X_2, and Y_2. We examine two completions of this coloring dependent upon Y_{12}.

First, assume that $Y_{12} = \emptyset$. Since G is connected, $X_{12} \neq \emptyset$. For all leaves $u \in X_{12}$ incident to some v, define $\ell(uv) = 1$. Complete the partial edge labeling for all

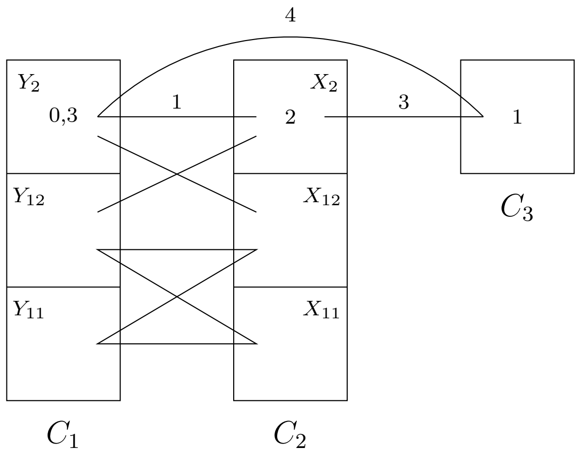


Figure 3: A partial edge labeling for connected tripartite graphs including derived gap colors.

remaining edges $uv \in E_i(Y_2)$ with $u \in C_2$ and $v \in C_1$ as follows (see Figure 4):

$$\ell(uv) = \begin{cases} 4 & \text{if } i \equiv 1 \mod 4, \\ 2 & \text{if } i \equiv 2 \mod 4, \\ 3 & \text{if } i \equiv 3 \mod 4, \\ 1 & \text{if } i \equiv 0 \mod 4. \end{cases}$$

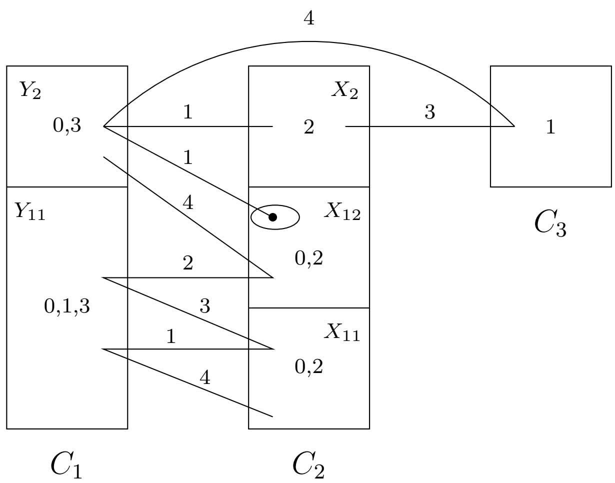


Figure 4: Completing the partial tripartite labeling from Figure 3 when $Y_{12} = \emptyset$.

The gap color of vertices in X_1 is 0 or 2, and the gap color of vertices in $Y_1 = Y_{11}$ is 0, 1, or 3. Hence, the vertices in C_2 have gap color 0 or 2, and vertices in C_1 have gap color 0, 1, or 3. Again, we must show the vertices with gap color 0 are independent. Notice that if a vertex in $(X_1 \cup Y_1) \cap V_i(Y_2)$ has a neighbor in $(X_1 \cup Y_1) \cap V_{i+1}(Y_2)$ then it has nonzero gap color. Let $x \in (X \cup Y) \cap V_i(Y_2)$ with $c_\ell(x) = 0$. Then, every neighbor of x is in $V_{i-1}(Y_2)$ and (the neighbor in $V_{i-1}(Y_2)$) has a neighbor in $V_{i-2}(Y_2)$, which is for $i = 2$ equal to Y_2 and for $i = 1$ equal to C_3. Hence, every neighbor of x has nonzero gap color. Therefore, ℓ is a local gap 4-coloring.

Otherwise, $Y_{12} \neq \emptyset$. Complete the edge labeling for all remaining edges $uv \in E_i(X_2)$ with $u \in C_2$ and $v \in C_1$ as follows (see Figure 5):

$$\ell(uv) = \begin{cases} 1 & \text{if } i \equiv 1 \mod 4,\ u \in X_{12}, \text{ and } v \in Y_2, \\ 3 & \text{if } i \equiv 1 \mod 4, \text{ and } u \in C_2 \setminus X_{12} \text{ or } v \in Y_1, \\ 2 & \text{if } i \equiv 2 \mod 4, \\ 4 & \text{if } i \equiv 3 \mod 4, \\ 1 & \text{if } i \equiv 0 \mod 4. \end{cases}$$

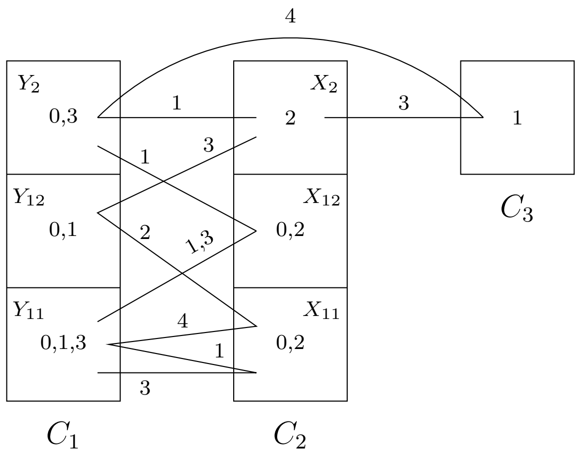


Figure 5: Completing the partial tripartite labeling from Figure 3 when $Y_{12} \neq \emptyset$.

The gap color of vertices in X_1 is either 0 or 2, and the gap color of vertices in Y_1 are 0, 1, or 3. As before, the neighbors of any vertex with gap color 0 in $X_1 \cup Y_1$ have nonzero gap color. Thus, ℓ is a local gap 4-coloring. □

Lemma 3. *Let G be a connected graph with $\chi(G) \geq 4$. Then $\chi^e_\Delta(G) \leq \chi(G) + 1$.*

Proof. Let $\chi = \chi(G)$, and let $C_1, \ldots, C_\chi$ be the color classes of a proper coloring on $V(G)$ in which every vertex in C_i has a neighbor in C_j for $1 \leq j < i \leq \chi$. Let X_{11}, X_{12}, X_2, Y_{11}, Y_{12}, and Y_2 be defined as in the proof of Lemma 2. Define a partial edge labeling for $uv \in E(G)$ as follows (see Figure 6).

$$\ell(uv) = \begin{cases} \chi - i + 2 & \text{if } u \in C_i \text{ and } v \in C_{i-1} \text{ for some } i = 4, \ldots, \chi \\ 1 & \text{if } u \in C_2 \text{ and } v \in C_i \text{ for some } i = 4, \ldots, \chi \\ 2 & \text{if } u \in C_1 \text{ and } v \in C_i \text{ for some } i = 4, \ldots, \chi \\ 2 & \text{if } u \in C_i \text{ and } v \in C_j \text{ for } 4 \leq i < j - 1 \leq \chi - 1 \\ \chi & \text{if } u \in C_3 \text{ and } v \in C_2 \\ 2 & \text{if } u \in C_3 \text{ and } v \in C_1 \\ \chi + 1 & \text{if } v \in Y_2 \end{cases}$$

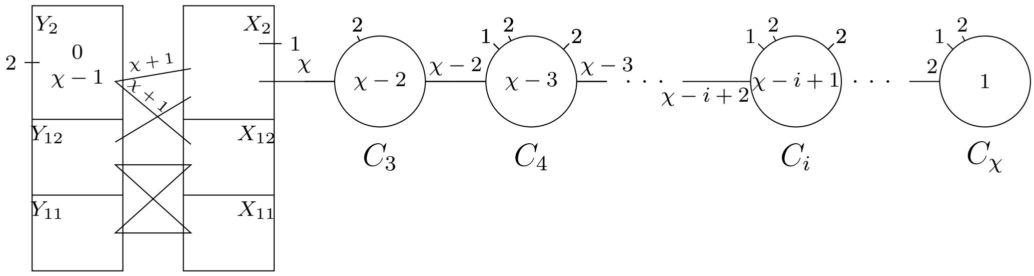


Figure 6: A partial edge labeling of a χ partite graph with $\chi \geq 4$.

Notice that for $i = 3, \ldots, \chi$ vertices in C_i have gap color $\chi - i + 1$, vertices in Y_2 with a neighbor in C_2 have gap color $\chi - 1$, and vertices in Y_2 with no neighbor in C_2 have gap color 0. As in Lemma 2, every remaining edge lives in a bipartite graph and we attempt to mimic Lemma 1 without changing the existing gap colors. We further refine X_2 and Y_{12} by defining $Y_{121} = \{v \in Y_{12} \colon N(v) \subseteq X_2\}$, $X_{21} = \{v \in X_2 \colon N(v) \subseteq C_3 \cup (Y_{12} \backslash Y_{121})\}$, $Y_{122} = \{v \in Y_{12} \backslash Y_{121} \colon N(v) \cap X_{21} \neq \emptyset, N(v) \cap X_{12} \neq \emptyset\}$, $X_{22} = X_2 \setminus X_{21}$, and $Y_{123} = Y_{12} \setminus (Y_{121} \cup Y_{122})$. Label edges $uv \in E(G)$ as follows:

$$\ell(uv) = \begin{cases} \chi + 1 & \text{if } u \in Y_{121}, \text{ or } u \in Y_{122} \text{ and } v \in X_1, \\ \chi - 1 & \text{if } u \in Y_{122} \text{ and } v \in X_{21}. \end{cases}$$

Let X' be the set of vertices in C_2 that have an incident edge already labeled. Complete the partial edge labeling for all remaining edges $uv \in E_i(X')$ with $u \in C_2$ and $v \in C_1$ as follows (see Figure 7):

$$\ell(uv) = \begin{cases} \chi & \text{if } i \equiv 1 \mod 4, \\ \chi - 2 & \text{if } i \equiv 2 \mod 4, \\ \chi - 3 & \text{if } i \equiv 3 \mod 4, \\ \chi - 1 & \text{if } i \equiv 0 \mod 4. \end{cases}$$

Every edge is labeled under this labelling even if $Y_{12} = \emptyset$ since G is connected. Notice that vertices in X_2 adjacent to $Y_{121} \cup Y_2$ have gap color 1 or χ, and the vertices with gap color 1 in X_2 have neighbors strictly in $C_1 \cup C_3$. Since vertices in X_2 with gap color 0 do not have neighbors in Y_{121}, vertices with gap color 0 in $X_1 \cup Y_1$ are independent. Notice that vertices in Y_{122} may have an incident edge labeled χ from X_{22} but do not have an incident edge labeled $\chi - 2$. Thus, vertices in Y_{122} have gap color 2. Vertices with gap color 0 in X_{12} have neighborhoods strictly in $Y_2 \cup Y_{122}$ and thus are independent from vertices with gap color 0. The remaining parity of colors guarantees ℓ is a local gap $(\chi + 1)$-coloring. □

Proof of Theorem 4. Apply Lemmas 1, 2, and 3 to appropriate components of G. □

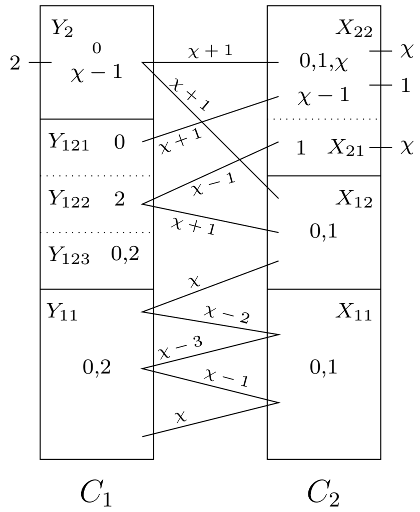


Figure 7: Completing the partial edge labeling from Figure 6.

4 Improved Bounds for $\mathrm{gap}_{\mathrm{ad}}(G)$

Recall that for a graph G, g_ℓ is the gap coloring associated with $\mathrm{gap}_{\mathrm{ad}}(G)$. In this section, we improve the bounds on the gap-adjacent-chromatic number of graphs. We begin by showing $\mathrm{gap}_{\mathrm{ad}}(G) \geq \chi(G)$ when G is not a star, and then provide a sharp upper bound for all graphs.

Lemma 4. *Let G be a connected graph. Then* $\mathrm{gap}_{\mathrm{ad}}(G) = \chi(G) - 1$ *if and only if G is a star.*

Proof. If G is a star, then $\ell \colon E(G) \to \{1\}$ gives $\mathrm{gap}_{\mathrm{ad}}(G) = \chi(G) - 1$. Assume G is not a star. Let L be the leaves of G and $\ell \colon E(G) \to [k]$. Since a proper coloring of $G - L$ can be extended to G without using an additional color, $\chi(G - L) = \chi(G)$. From the definition, $g_\ell(v) \in \{0, \ldots, k-1\}$ for all $v \in V(G - L)$. Thus, in order to properly color $G - L$, $k \geq \chi(G - L) = \chi(G)$. □

Lemma 5. *For any connected graph G not isomorphic to K_2,* $\mathrm{gap}_{\mathrm{ad}}(G) \leq \chi(G) + 1$.

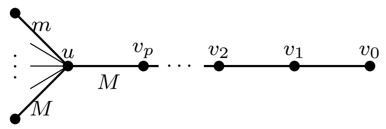
Proof. Theorem 1 gives the desired result when $\chi(G) \in \{2, 3\}$. Thus, we may assume $\chi(G) \geq 4$ and, in particular, that G is not a path. We proceed by induction on the number of leaves. If G has no leaves, then $\mathrm{gap}_{\mathrm{ad}}(G) = \chi^e_\Delta(G)$ since the two colorings associated with these parameters differ only on leaves. Lemma 3 completes the base case.

Assume $\mathrm{gap}_{\mathrm{ad}}(H) \leq \chi(H) + 1$ for all graphs H with k leaves and let G be a graph with $k + 1$ leaves. Let $P = v_0 v_1 \cdots v_p u$ be a minimum length path in G with $d(v_0) = 1$ and $d(u) \geq 3$. Let $G' = G - \{v_0, \ldots, v_p\}$. By the induction hypothesis, $\mathrm{gap}_{\mathrm{ad}}(G') \leq \chi(G') + 1 = \chi(G) + 1$. Let $\ell : E(G') \to [\chi(G) + 1]$ be a labeling such that g_ℓ induces a proper vertex coloring of G'. Let m, M be the minimum and maximum labels, respectively, incident to u in G'. We extend ℓ to $E(G)$.

Define $\ell(uv_p) = M$. Notice that $g_\ell(u)$ does not change. If $p = 0$, then

$$g_\ell(v_0) = M > M - m = g_\ell(u)$$

and we have extended ℓ from G' to G. Thus, we may assume $p \geq 1$. It is straightforward to iteratively label the remaining edges $v_p v_{p-1}, \ldots, v_1 v_0$ with values from $\{1, 2, 3\}$ so that g_ℓ induces a proper vertex coloring of G. To see this, notice that for each $i = 1, \ldots, p-1$, the 3 available labels for $\ell(v_{p-i+1}v_{p-i})$ give at least 2 possible values for $g_\ell(v_{p-i+1})$ when $v_{p-i+2}v_{p-i+1}$ is already labeled and $v_{p+1} = u$. Figure 8 illustrates the four cases for labeling $v_0 v_1$ when $p \geq 3$. For brevity, we omit the two remaining cases of $p = 1, 2$. □



$g_\ell(v_2)$	$\ell(v_1v_2)$	$g_\ell(v_1)$	$\ell(v_0v_1)$	$g_\ell(v_0)$
0	1	1	2	2
0	2	1	3	3
0	3	1	2	2
$\neq 0$	x	0	x	x

Figure 8: Extending ℓ from G' to G when $p \geq 3$. In the table, $x \in \{1, 2, 3\}$.

5 Further Pursuits

We determine $\chi^e_\Delta(G)$ exactly for cliques, cycles, and trees in the following propositions. It is easy to see that $\chi^e_\Delta(K_3) = 4$.

Proposition 1. *Let $n \geq 4$. Then $\chi^e_\Delta(K_n) = n$.*

Proof. Let $v_1 \cdots v_{n-1}$ be a cycle on all but one vertex, v_0, of K_n. Define $\ell(v_{n-1}v_1) = 1$, $\ell(v_i v_{i+1}) = i + 2$ for $i = 2, \ldots, n-2$, and $\ell(e) = 2$ for all remaining edges e. Note that $c_\ell(v_i) = i$ for $i = 0, \ldots, n-1$. □

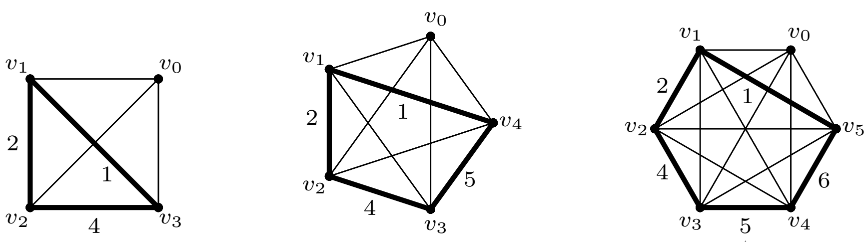


Figure 9: Partial labelings witnessing $c_\ell(K_n) = n$ for $n = 4, 5, 6$. Remaining edges are labeled 2.

Proposition 2. *Let C_n be a cycle on $n \geq 4$ vertices. Then*

$$\chi^e_\Delta(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \mod 4, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. A local gap 2-coloring must alternate gap colors between 0 and 1 along vertices. This is possible precisely when $n \equiv 0 \mod 4$. For $n \equiv 2 \mod 4$, Theorem 4 then implies $\chi^e_\Delta(C_n) = 3$. For odd $n \geq 5$, let $C_n = v_1 \dots v_n$. If $n \equiv 1 \mod 4$, define $\ell(v_n v_1) = 3$ and, for $i \in \{1, \dots, n-1\}$,

$$\ell(v_i v_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1, 2 \mod 4, \\ 2 & \text{if } i \equiv 0, 3 \mod 4. \end{cases}$$

If $n \equiv 3 \mod 4$, define $\ell(v_{n-1} v_n) = \ell(v_n v_1) = 3$ and, for $i \in \{1, n-2\}$,

$$\ell(v_i v_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1, 2 \mod 4, \\ 2 & \text{if } i \equiv 0, 3 \mod 4. \end{cases}$$

This completes the proof. □

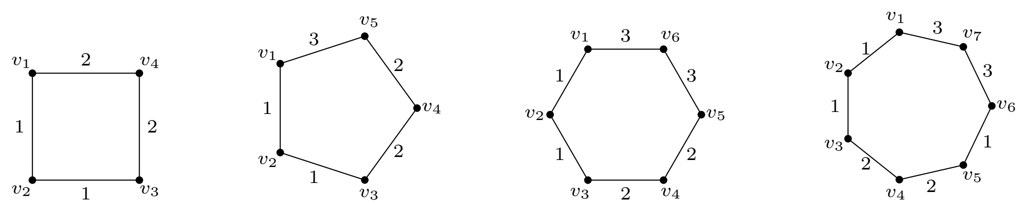


Figure 10: Labelings witnessing $c_\ell(C_n)$ for $n = 4, 5, 6, 7$.

Proposition 3. *Let T be a tree on n vertices, $n \geq 3$. Then $\chi^e_\Delta(T) = 2$ if all the leaves of T are in the same partite set of a bipartition of T; otherwise $\chi^e_\Delta(T) = 3$.*

Proof. Lemma 1 implies that $\chi^e_\Delta(T) \leq 3$. Since all leaves have gap color 0, a local gap 2-coloring is not possible if leaves appear in both partite sets of T.

Now, let all leaves be in the same partite set of T. Let v be a leaf of T. For each $e \in E_i(v)$, define $\ell(e) = 1$ if $i \equiv 0, 1 \mod 4$, and $\ell(e) = 2$ otherwise. Since T has no cycles, ℓ is a local gap 2-coloring of T. □

Since determining $\chi(G)$ is APX-hard in general [7] and $\chi^e_\Delta(G)$ is within an additive constant of $\chi(G)$, determining $\chi^e_\Delta(G)$ is APX-hard as well. However, it would be interesting to investigate when $\chi^e_\Delta(G)$ can be determined in polynomial time if $\chi(G)$ is given as part of the input.

References

[1] T. Bohman and D. Kravitz, On the irregularity strength of trees, *J. Graph Theory* 45(4) (2004), 241–254.

[2] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz and F. Saba, Irregular networks, *Congr. Numer.* 64 (1988), 197–210.

[3] A. Frieze, R. J. Gould, M. Karoński and F. Pfender, On graph irregularity strength, *J. Graph Theory* 41(2) (2002), 120–137.

[4] E. Győri, M. Horňák, C. Palmer and M. Woźniak, General neighbour-distinguishing index of a graph, *Discrete Math.* 308(5-6) (2008), 827–831.

[5] M. Kalkowski, M. Karoński and F. Pfender, Vertex-coloring edge-weightings: towards the 1-2-3-conjecture, *J. Combin. Theory Ser. B* 100(3) (2010), 347–349.

[6] M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours, *J. Combin. Theory Ser. B* 91(1) (2004), 151–157.

[7] C. Lund and M. Yannakakis, On the hardness of approximating minimization problems, *J. Assoc. Comput. Mach.* 41(5) (1994), 960–981.

[8] T. Nierhoff, A tight bound on the irregularity strength of graphs, *SIAM J. Discrete Math.* 13(3) (2000), 313–323.

[9] R. Scheidweiler and E. Triesch, New estimates for the gap chromatic number, *Discrete Math.* 328 (2014), 42–53.

[10] R. Scheidweiler and E. Triesch, Gap-neighbour-distinguishing colourings, *J. Combin. Math. Combin. Comput.* 94 (2015), 205–214.

[11] B. Seamone, The 1-2-3 conjecture and related problems: a survey, arXiv: 1211.5122.

[12] M. A. Tahraoui, E. Duchêne and H. Kheddouci, Gap vertex-distinguishing edge colorings of graphs, *Discrete Math.* 312(20) (2012), 3011–3025.

[13] D. B. West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

(Received 12 Nov 2014; revised 5 May 2016)