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# A note on the independence number in bipartite graphs

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**Abstract**

The independence number of a graph $G$, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of vertices in $G$. The transversal number of $G$ is the minimum cardinality of a set of vertices that covers all the edges of $G$. If $G$ is a bipartite graph of order $n$, then it is easy to see that $\frac{n}{2} \leq \alpha(G) \leq n-1$. If $G$ has no edges, then $\alpha(G) = n = n(G)$. Volkmann [*Australas. J. Combin.* 41 (2008), 219–222] presented a constructive characterization of bipartite graphs $G$ of order $n$ for which $\alpha(G) = \lceil \frac{n}{2} \rceil$. In this paper we characterize all bipartite graphs $G$ of order $n$ with $\alpha(G) = k$, for each $\lceil \frac{n}{2} \rceil \leq k \leq n-1$. We also give a characterization on the Nordhaus-Gaddum type inequalities on the transversal number of trees.

## 1 Introduction

In this paper we study independence number and transversal number in bipartite graphs. For notation and also terminology not given here, we refer to [7]. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We denote by $n(G)$ and $m(G)$, or just $n, m$ if $G$ is specified, the order and size of $G$, respectively. For a vertex $v \in V$, let $N_G(v) = \{u | uv \in E(G)\}$ denote the *open neighborhood* of $v$. The *degree* of a vertex $v$, $\deg_G(v)$, or just $\deg(v)$, denotes the number of neighbors of $v$ in $G$. We refer $\Delta(G)$ and $\delta(G)$ as the *maximum degree* and the *minimum degree* of the vertices of $G$, respectively. A *leaf* in a graph is a vertex of degree one, and a *support vertex* is one that is adjacent to a leaf. We say that a support vertex is *strong* if it is adjacent to at least two leaves. An edge of $G$ is called a *pendant* edge if at least one of its vertices is a leaf of $G$. The *distance* between two vertices of a graph is the number of edges in a shortest path connecting them. The *eccentricity* of a vertex is the greatest distance between it and any other vertex. The *diameter* of a graph $G$, denoted by $\operatorname{diam}(G)$, is the maximum eccentricity among all vertices of $G$. For a subset $S$ of $V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. A *clique* is a subset of vertices such that its induced subgraph is complete. The *clique number*, $\omega(G)$, of a graph G is the number of vertices in a maximum clique in $G$. In this paper we denote by $P_n$ the path on $n$ vertices. A *star* $S_n$ is the complete bipartite graph $K_{1,n}$. The vertex with degree $n$ in the star $S_n$ is called *central vertex*. A *double star* is a tree with precisely two vertices that are not leaves, called the central vertices of the double star. A double-star with central vertices of degrees $m$ and $n$ is denoted by $S_{n,m}$. Note that the *corona* of a graph $G$, denoted by $\text{cor}(G)$, is a graph obtained from $G$ by adding a leaf for every vertex of $G$. If $T$ is a rooted tree, then for any vertex $v$ we denote by $T_v$ the subtree rooted at $v$.

A set $S$ of vertices in a graph $G$ is an *independent set* if no pair of vertices of $S$ are adjacent. The *independence number* of $G$, denote by $\alpha(G)$, is the maximum cardinality of an independent set in $G$. An independent set of cardinality $\alpha(G)$ is called an $\alpha(G)$-set. A *matching* (or *independent edge set*) in a graph is a set of edges without common vertices. The *matching number* of $G$, denote by $\alpha'(G)$, is the maximum cardinality of a matching in $G$. A vertex *covers* an edge if it is incident with the edge. A *transversal* in $G$ is a set of vertices that covers all the edges of $G$. We remark that a transversal is also called a *vertex-cover* in the literature. The *transversal number* of $G$, denoted by $\tau(G)$, is the minimum cardinality of a transversal in $G$. A transversal of cardinality $\tau(G)$ is called a $\tau(G)$-set. The independence number is one of the most fundamental and well-studied graph parameters (see, for example, [1, 2, 3, 4, 6, 7, 8, 10]). The following is well-known.

**Theorem 1.1** (Gallai [5]). *For any graph $G$ of order $n$, we have $\alpha(G) + \tau(G) = n$.*

According to the above relation, it is enough to discuss about only one of the independence number and transversal number. If $G$ is a graph with connected components $G_1, \ldots, G_k$, then it is obvious that $\tau(G) = \sum_{i=1}^{k} \tau(G_i)$. Therefore, in this paper we will consider connected graphs.

If $G$ is a bipartite graph with partite sets $V_1$ and $V_2$, then $V_1$ and $V_2$ are independent sets and also transversals. Thus the following holds for every bipartite graph $G$.

$$1 \leq \tau(G) \leq \frac{n}{2} \leq \alpha(G) \leq n - 1 \tag{1}$$

As mentioned above, $\alpha(G) = n = n(G)$ is possible, for example for $n = 1$. Volkmann in [11] characterized bipartite graphs $G$ of order $n$ with $\alpha(G) = \lceil \frac{n}{2} \rceil$. In this paper, we will characterize bipartite graphs $G$ of order $n$ with $\alpha(G) = k$, for each $\lceil \frac{n}{2} \rceil \leq k \leq n - 1$. We also give a characterization on the Nordhause-Gaddum type inequalities on the transversal number of trees. We make use of the following results for the next.

**Theorem 1.2** (König [9]). *If $G$ is a bipartite graph, then $\tau(G) = \alpha'(G)$.*

**Observation 1.3** (Volkmann [11]). *If $G$ is a connected graph with a maximum matching $M$, then $G$ contains a spanning tree with the maximum matching $M$.*

## 2 Main Results

We begin with the following straightforward observation.

**Observation 2.1.** *For the star $S_n$, the double star $S_{n,m}$ and the path $P_n$, we have $\tau(S_n) = 1$, $\tau(S_{n,m}) = 2$ and $\tau(P_n) = \lfloor \frac{n}{2} \rfloor$.*

**Proposition 2.2.** *For every integers $n$ and $k$ with $1 \leq k \leq \frac{n}{2}$, there exists a bipartite graph $G$ of order $n$ with $\tau(G) = k$.*

*Proof.* Let $n$ and $k$ be integers with $1 \leq k \leq \frac{n}{2}$. We construct a bipartite graph $G_{k,n}$ of order $n$ with transversal number $k$. Let $G$ be a bipartite graph of order $k$ with vertex set $V = \{v_1, \ldots, v_k\}$. We construct a graph $G_{k,n}$ from cor($G$) by adding $n - 2k$ new vertices $u_1, u_2, \ldots, u_{n-2k}$ together with new edges $v_i u_i$, $1 \leq i \leq n - 2k$, where the indices of vertices in $V$ are taken in modulo $k$ when $n - 2k > k$. It can be checked that $G_{k,n}$ is a bipartite graph of order $n$ with $\tau(G_{k,n}) = k$. □

We next wish to characterize bipartite graphs $G$ with $\tau(G) = k$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. For this purpose we first consider trees. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we define a family $\mathcal{T}_k$ of trees as follows. Let $\mathcal{T}_k$, be the collection of trees $T$ of order $n$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k$, of trees as follows. If $n$ is even, then $T_1 = P_2$ and otherwise $T_1 = P_3$, and let $v_1$ be the central vertex of $T_1$ (Note that each of vertices of $P_2$ is a central vertex of $P_2$). If $k \geq 2$ then $T_{i+1}$ can be obtained recursively from $T_i$ by the following operation.

- **Operation $\mathcal{O}$:** Assume that $v$ is an arbitrary vertex of $T_i$. Then $T_{i+1}$ is obtained from $T_i$ by adding a path $P_2$ with vertex set $\{v_{i+1}, w_{i+1}\}$ and joining $v$ to $v_{i+1}$.

Finally, add $n_i \geq 0$ leaves to $v_i$ for $i = 1, 2, ..., k$ in the tree $T_k$ such that $\sum_{i=1}^{k} n_i = n - 2k$ if $n$ is even and $\sum_{i=1}^{k} n_i = n - 2k - 1$ if $n$ is odd. We call $v_1, v_2, ..., v_k$ the *special vertices* of $T_k$.

We are now ready to establish the following result.

**Theorem 2.3.** *Let $T$ be a tree of order $n$. Then $\tau(T) = k$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, if and only if $T \in \mathcal{T}_k$.*

*Proof.* ($\Longleftarrow$) Let $T \in \mathcal{T}_k$. By definition of the family $\mathcal{T}_k$, $T$ is obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees, by adding some leaves to special vertices of $T_k$. If $k = 1$, then $T$ is a star. By Observation 2.1, $\tau(T) = 1$. Thus assume that $k \geq 2$, and so $T_{i+1}$ is obtained from $T_i$ according to Operation $\mathcal{O}$, for $i = 1, 2, ..., k - 1$, by adding a path $P_2 = v_{i+1} w_{i+1}$ and joining $v_{i+1}$ to a vertex of $T_i$. We prove that $\tau(T_{i+1}) = \tau(T_i) + 1$ for $i = 1, 2, ..., k - 1$. Let $S$ be a $\tau(T_i)$-set. Clearly $S \cup \{v_{i+1}\}$ is a transversal for $T_{i+1}$, and so $\tau(T_{i+1}) \leq \tau(T_i) + 1$. Since $V(T_i) \cap \{v_{i+1}, w_{i+1}\} = \emptyset$, no $\tau(T_i)$-set covers the edge $v_{i+1} w_{i+1}$ in $T_{i+1}$. Thus $\tau(T_{i+1}) \geq \tau(T_i) + 1$. Therefore, $\tau(T_{i+1}) = \tau(T_i) + 1$. Hence, $\tau(T_k) = k$, since $\tau(T_1) = 1$. It is easy to see that

$\{v_1, \ldots, v_k\}$ is a transversal for $T_k$. Since $T$ is obtained from $T_k$ by adding $n_i \geq 0$ leaves to $v_i$ for $i = 1, 2, ..., k$, $\{v_1, \ldots, v_k\}$ is also a transversal for $T$, and so $\tau(T) \leq k$. But $T_k$ is an induced subgraph of $T$, and thus $\tau(T) \geq \tau(T_k) = k$. Therefore, $\tau(T) = k$.

($\Longrightarrow$) We prove by an induction on $n$ to show that any tree $T$ of order $n$ with $\tau(T) = k$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, belongs to $\mathcal{T}_k$. It is obvious that $n \geq 2$. If diam$(T) = 1$, then $T = P_2 \in \mathcal{T}_1$. Now assume that diam$(T) = 2$. Thus $T$ is a star. By Observation 2.1, $\tau(T) = 1$. If $n$ is even then $T$ is obtained from a path $P_2$ by adding $n - 2$ leaves to a vertex of $P_2$, and thus $T \in \mathcal{T}_1$. If $n$ is odd then $T$ is obtained from a path $P_3$ by adding $n - 3$ leaves to the central vertex of $P_3$, and thus $T \in \mathcal{T}_1$. Assume that diam$(T) = 3$. Then $T$ is a double star. Let $abcd$ be a path of length three in $T$. If $n$ is even, then $T$ is obtained from the path $ab$ by adding a path $cd$, and then adding $\deg_T(b) - 2$ leaves to $b$, and $\deg_T(c) - 2$ leaves to $c$, and thus $T \in \mathcal{T}_2$. Thus assume that $n$ is odd. Then clearly we may assume, without loss of generality, that $\deg(b) \geq 3$. Let $b_1 \neq a$ be a leaf adjacent to $b$. Then $T$ is obtained from the path $abb_1$ by adding a path $cd$, and then adding $\deg_T(b) - 3$ leaves to $b$, and $\deg_T(c) - 2$ leaves to $c$, and thus $T \in \mathcal{T}_2$. These are sufficient for the base step of the induction. Now assume that diam$(T) \geq 4$. Assume that the result holds for every tree $T'$ of order $n' < n$. Assume that $T$ has some strong support vertices. We remove all leaves except one from each strong support vertex to obtain a tree $T'$ with no strong support vertex. Clearly $\tau(T') \leq \tau(T)$. Let $S$ be a $\tau(T')$-set. We can assume that $S$ contains every support vertex to cover each pendant edge. Then $S$ is also a transversal for $T$, and so $\tau(T) \leq \tau(T')$. Thus $\tau(T') = \tau(T) = k$. By the induction hypothesis, $T' \in \mathcal{T}_k$. Hence, $T'$ is obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees according to the Operation $\mathcal{O}$ and adding some leaves to the special vertices of $T_k$. Let $v_1, \ldots, v_k$ be the special vertices of $T_k$. It is easy to see that the support vertices of $T_k$ are a subset of $\{v_1, \ldots, v_k\}$. Since $T'$ is obtained from $T_k$ by adding leaves to the special vertices of $T_k$, and $T$ is obtained from $T'$ by adding leaves to some support vertices of $T'$, we obtain that $T \in \mathcal{T}_k$.

Thus assume for the next that $T$ has no strong support vertex. We now root $T$ at a leaf $x_0$ of a diametrical path $x_0x_1 \ldots x_d$, where $d =$diam$(T)$. Let $T' = T - T_{x_{d-1}}$, and let $S$ be a $\tau(T')$-set. Then $S \cup \{x_{d-1}\}$ is a transversal for $T$, and so $\tau(T) \leq \tau(T') + 1$. Since $V(T') \cap \{x_{d-1}, x_d\} = \emptyset$, no $\tau(T')$-set in $T$ covers the edge $x_{d-1}x_d$. Hence, $\tau(T) \geq \tau(T') + 1$. Thus, $\tau(T') = \tau(T) - 1 = k - 1$. By the induction hypothesis, $T' \in \mathcal{T}_{k-1}$. Then $T$ is obtained by adding the path $P_2 : x_{d-1}x_d$ and joining $x_{d-2}$ to $x_{d-1}$ according to Operation $\mathcal{O}$. Hence $T \in \mathcal{T}_k$. □

Now we present our main result. As an immediate consequence of Theorem 2.3, we have the following characterization of bipartite graphs of order $n$ with transversal number $k$, $1 \leq k \leq \frac{n}{2}$.

**Theorem 2.4.** *Let $G$ be a bipartite graph of order $n$. Then $\tau(G) = k$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, if and only if $G$ has a spanning tree $T \in \mathcal{T}_k$, and no spanning tree of $G$ belongs to $\mathcal{T}_{k'}$ for each $k' > k$.*

*Proof.* Let $\tau(G) = k$, where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Since $G$ is a bipartite graph, by Theorem 1.2, $G$ has a maximum matching $M$ of cardinality $k$. Hence, by Observation 1.3, $G$ contains a spanning tree $T$ with the maximum matching $M$. Then $\tau(T) = k$. Therefore, by Theorem 2.3, $T \in \mathcal{T}_k$. Suppose that $G$ has a spanning tree $T' \in \mathcal{T}_{k'}$ where $k' > k$. Then, by Theorem 2.3, $\tau(T') = k'$. But $\tau(G) \geq \tau(T') = k' > k$, a contradiction. Conversely, assume that $G$ has a spanning tree $T \in \mathcal{T}_k$ and no spanning tree of $G$ belongs to $\mathcal{T}_{k'}$ for each $k' > k$. By Theorem 2.3, $\tau(T) = k$. Thus $\tau(G) \geq \tau(T) = k$. Let $\tau(G) = k' > k$. By the first part of the theorem, $G$ has a spanning tree $T' \in \mathcal{T}_{k'}$, a contradiction. Therefore, $\tau(G) = k$. □

Theorem 2.4 is equivalent to a characterization of all bipartite graphs $G$ of order $n$ with $\alpha(G) = k$, for each $\lceil \frac{n}{2} \rceil \leq k \leq n - 1$ and also, all bipartite graphs $G$ of order $n$ with $\alpha'(G) = k$, for each $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

We end the paper with a characterization on the Nordhaus-Gaddum type inequalities on the transversal number of trees. If $G$ is a bipartite graph of order $n$, then $\omega(G) = 2$, and so by Theorem 1.1, we have

$$\tau(\overline{G}) = n - \alpha(\overline{G}) = n - \omega(G) = n - 2. \tag{2}$$

Therefore, by (1) and (2), we obtain the following bounds that are sharp by Observation 2.1.

**Observation 2.5.** *If $G$ is a bipartite graph of order $n$, then $n - 1 \leq \tau(G) + \tau(\overline{G}) \leq \frac{3}{2}n - 2$, and these bounds are sharp.*

As a consequence of Theorem 2.3, we have the following characterization.

**Corollary 2.6.** *Let $T$ be a tree of order $n$. Then $\tau(T) + \tau(\overline{T}) = k$ for $n - 1 \leq k \leq \frac{3}{2}n - 2$, if and only if $T \in \mathcal{T}_{k-n+2}$.*

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