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Degree associated reconstruction number of certain connected digraphs with unique end vertex

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Abstract

A vertex-deleted unlabeled subdigraph of a digraph D is a *card* of D. A *dacard* specifies the degree triple (a, b, c) of the deleted vertex along with the card, where a and b are respectively the indegree and outdegree of v and c is the number of symmetric pairs of arcs (each pair considered as unordered edge) incident with v. The *degree* (*triple*) *associated reconstruction number*, $drn(D)$, of a digraph D is the size of the smallest collection of dacards of D that uniquely determines D. A *P-digraph* is a connected digraph of order $p \geq 4$ with exactly two blocks; only one of them has just two vertices and the other block has a vertex of degree triple $(0, 0, p-2)$ other than the cutvertex. In this paper, we prove that the *drn* is at most 3 for all P-digraphs except one type and show that the *drn* of all connected digraphs D, with a unique end vertex in D and an end vertex in $\overline{D}$, is at most $\max\{3, k\}$ if the *drn* of the exceptional type of P-digraphs is at most k for some k.

1 Introduction

We shall mostly follow the graph theoretic terminology of [5]. A *digraph* D consists of a finite set $V(D)$ of vertices and a set $A(D)$ of ordered pairs of distinct vertices. Any such pair (u, v) is called an *arc* and will usually be denoted uv. If $uv \in A(D)$, then we say that u is *adjacent to* v and v is *adjacent from* u. We say that v is *adjacent with* u if u is adjacent to or from v. Two vertices u and v of a digraph D are *nonadjacent* if u is neither adjacent to nor adjacent from v. If uv and vu are both arcs, then they together are called *symmetric pair of arcs*. The ordered triple (a, b, c) where a, b and c are respectively the number of unpaired out-arcs, unpaired in-arcs and symmetric pair of arcs incident with v in D, is called the *degree triple* *of* v and is denoted by $\deg t(v)$; also v is called an (a, b, c)-vertex. A *card* $D - v$ of a digraph (graph) D is obtained from D by deleting a vertex v and all arcs (edges) incident with v. The deck of D is the collection of all its cards and it is denoted by $\mathscr{D}(D)$.

The well-known Reconstruction Conjecture (RC) of Kelly [8] and Ulam [16] has been open for more than 50 years. It asserts that every graph G with at least three vertices can be (uniquely) reconstructed from $\mathscr{D}(G)$. The conjecture has been proved for many special classes, and many properties of G may be deduced from $\mathscr{D}(G)$. Nevertheless, the full conjecture remains open. Surveys of results on the RC and related problems include [4, 9, 10]. Harary and Plantholt [7] defined the reconstruction number of a graph G, $rn(G)$, to be the minimum number of cards which can only belong to the deck of G and not to the deck of any other graph H, $H \ncong G$, these cards thus uniquely identifying G. Reconstruction numbers are known for various classes of graphs [2].

An extension of the Reconstruction Conjecture to digraphs is the *Digraph Reconstruction Conjecture* (DRC), proposed by Harary [6], which asserts that every digraph D with at least seven vertices can be (uniquely) reconstructed from $\mathscr{D}(D)$. The DRC was disproved by Stockmeyer [15] by exhibiting several infinite families of counter-examples. Ramachandran [11] then proposed a variation in the DRC and introduced the degree associated reconstruction and the corresponding reconstruction number [12, 13].

The *degree associated card* or *dacard* of a digraph (graph) is a pair (d, C) consisting of a card C and the degree triple (degree) d of the deleted vertex. The *dadeck* of a digraph is the multiset of all its dacards. A digraph is said to be *N-reconstructible* if it can be uniquely determined from its dadeck. The *new digraph reconstruction conjecture* (NDRC) asserts that all digraphs are N-reconstructible. The *degree* (*triple*) *associated reconstruction number* of a digraph D is the size of the smallest collection of dacards of D that uniquely determines D. We abbreviate the term to $drn(D)$. Articles [1] and [3] are recent papers on degree associated reconstruction number.

A *connected* (*disconnected,* 2-*connected, separable*, respectively) digraph D is a digraph whose underlying graph is connected (disconnected, 2-connected, separable, respectively). If uv is an arc in a digraph D, we say that u is a neighbour of v and vice versa. The number of neighbours of v in D is called the *degree* of v and is denoted by $d(v)$. A vertex of degree n is called an *n-vertex.* A k-vertex which is a neighbour of v is called a k-neighbour of v. A 1-*vertex* is called an *end vertex* and the unique neighbour of a 1-vertex is called its *base.*

Definition: A digraph D with p vertices is called a *P-digraph* if

(i) there exist only two blocks in D and exactly one of them has just two vertices (denote the end vertex by x and its base by$r$), and

(ii) there exists a vertex $u \neq r$ with $\deg t(u) = (0, 0, p - 2)$.

Throughout this paper, u, r *and* x *are used in the sense of the above definition.*

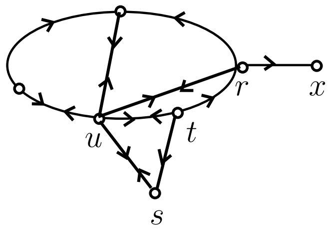

Figure 1. A P-digraph on 7 vertices

Notation: For a P-digraph D, let S denote the set of all 2-vertices of D, and T denote the set of neighbours of the 2-vertices of D other than u and r. We use the letters t and s to denote, respectively, a member of T and a 2-vertex neighbour of t.

Let $\mathbb{D}'$ denote the family of all P-digraphs with at least two 2-vertices such that each $t \in T$is adjacent with at most $p-3$ vertices, at least two $t \in T$ have unique 2-neighbour, $\deg t(r) \neq (0,0,p-3)$ and no $t \in T$ has degree triple $(0,0,p-3)$. The *complement* $\overline{D}$ of digraph D is defined as the digraph having the same vertex set as D and uw is an arc of $\overline{D}$ if and only if it is not an arc of D. A vertex of degree $(0,0,p-2)$ or $(1,0,p-2)$ or $(0,1,p-2)$ in D is called a *ce-vertex* since such a vertex becomes an end vertex in the complement $\overline{D}$.

For clarity, we classify all digraphs into four disjoint families as below:
$\mathbb{F}_1$: All disconnected digraphs.
$\mathbb{F}_2$: All separable digraphs without endvertices.
$\mathbb{F}_3$: All separable digraphs with endvertices.
$\mathbb{F}_4$: All 2-connected digraphs.

The NDRC is proved [11] for the family $\mathbb{F}_1 \cup \mathbb{F}_2$ and it remains open for the family $\mathbb{F}_3 \cup \mathbb{F}_4$. Ramachandran and Monikandan [14] proved that the NDRC is true for $\mathbb{F}_3$ if it is true for $\mathbb{F}_4$. For proving this result, they first proved that the NDRC is true for all P-digraphs if it true for $\mathbb{F}_4$ by using the well-known result that a digraph D is N-reconstructible if and only if $\overline{D}$ is N-reconstructible. It is clear from their definitions that, for each digraph $D \in \mathbb{F}_3$, the complement $\overline{D}$ is in $\mathbb{F}_1 \cup \mathbb{F}_2 \cup \mathbb{F}_4$, $\overline{D}$ is a P-digraph or the underlying graph of $\overline{D}$ (denoted by $U(\overline{D})$) is in the family of two types of graphs G and H defined in Figure 2. To prove $\mathbb{F}_3$ is N-reconstructible, they, in fact, proved that all P-digraphs are N-reconstructible if the family $\mathbb{F}_4$ is N-reconstructible and that all digraphs whose underlying graphs are in the family of two types of graphs G and H defined in Figure 2, are N-reconstructible.

In the problem of determining the drn of digraphs, it was proved that $drn(D) = drn(\overline{D})$; but the drn of the family $\mathbb{F}_1 \cup \mathbb{F}_2$ is not known. Also it is clear that, since the complement of most of the P-digraphs are again so, we cannot exclude P-digraphs in order to N-reconstruct $\mathbb{F}_3$ and hence to determine the drn of $\mathbb{F}_3$. We also observe that the drn of P-digraphs turns out to be of great use while shuttling between a digraph in $\mathbb{F}_3$ and its complement in order to determine its drn. Consequently, any result finding the drn of P-digraphs is of interest. In this paper, we prove that the drn is at most 3 for all P-digraphs except those in $\mathbb{D}'$ and show that the drn of all connected digraphs with exactly one end vertex and a ce-vertex is at most $\max\{3, k\}$ if $drn(D') \le k$ for some k for all $D' \in \mathbb{D}'$.

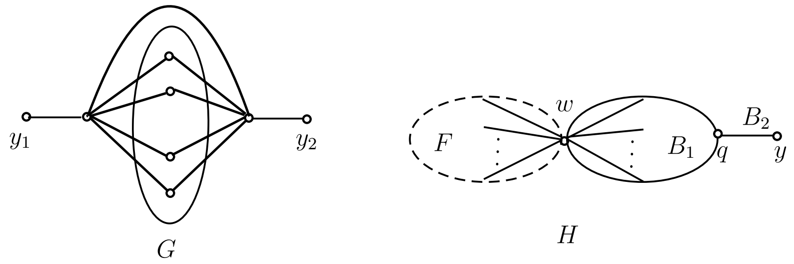

Figure 2. The underlying graph $U(\overline{D})$

(The graph G, shown in Figure 2, contains two endvertices y_1 and y_2 with distinct bases and each base is adjacent to all the other possible vertices; the graph H contains exactly two cutvertices w and q, and exactly one endvertex y. The vertex w is adjacent to all the vertices except y and the graph H is the union of three subgraphs B_1 (the non-endblock containing w and q), F (the union of end-blocks containing w) and the end-block B_2 consisting only two vertices.)

2 The drn of P-digraphs

It is known that if D is a P-digraph with $\deg t(r) = (0, 0, p-1)$ or its dacard $D - x$ of D is vertex-transitive, then $drn(D) = 1$. The drn of digraphs on at most four vertices is shown [12] to be 1, 2 or 3. In this paper, *we address the drn of only P-digraphs D of order at least 5 such that $\deg t(r) \neq (0, 0, p-1)$ and its dacard $D - x$ is not vertex-transitive and thus $drn(D) > 1$.*

For a P-digraph D, the following hold.

(i) $D - x$ is the only dacard without end vertices in the dadeck $\mathscr{D}(D)$.

(ii) $D - r$ is the only disconnected dacard in $\mathscr{D}(D)$.

(iii) Any connected dacard with as few arcs as possible in $\mathscr{D}(D)$ is isomorphic to $D - u$. (For, if $D - u_1$ is such a connected dacard in $\mathscr{D}(D)$, then $u_1 \neq r$ and $\deg t(u_1) = \deg t(u) = (0, 0, p-2)$, since $D - u$ is a connected dacard with minimum number of arcs in $\mathscr{D}(D)$. Hence u_1 and u have same neighbourhood in D and so $D - u_1 \cong D - u$.)

2.1 At most one 2-vertex

An *extension* of a dacard $((a, b, c), C)$ of a digraph D is a digraph obtained from the dacard by adding a new vertex v and joining it with r vertices by unpaired out-arcs, s vertices by unpaired in-arcs and t vertices by symmetric pair of arcs of the dacard.

Theorem 1. *If D is a P-digraph with no 2-vertices, then $drn(D) = 2$.*

Proof. We use the two dacards $(\deg t(x), D-x)$ and $((0,0,p-2), D-u)$. The dacard $D-x$ forces every its extension to have exactly one end vertex. Since $D-u$ has exactly one end vertex, D can be obtained uniquely from $D-u$ by annexing a vertex and joining it with all vertices other than the unique end vertex by means of symmetric pair of arcs. Thus the above two dacards uniquely determine the P-digraph D and hence $drn(D) = 2$. □

Theorem 2. *If none of the* 2*-vertices of a P-digraph* D *have degree triple* $\deg t(x) + (0,0,1)$ (*where '+' means vector addition*), *then* $drn(D) = 2$.

Proof. Consider the dacards $D-x$ and $D-u$. The dacard $D-x$ forces every extension to have exactly one end vertex, say x of degree triple $\deg t(x)$. In $D-u$, the end vertex x of degree triple $\deg t(x)$ can be distinguished from all other end vertices as no other end vertices of $D-u$ have the same degree triple as x does. Hence D can be obtained uniquely from $D-u$ by annexing a vertex and joining it with all vertices other than x by means of symmetric pair of arcs. □

Theorem 3. *If* D *is a P-digraph having a* 2*-vertex adjacent with* r, *then* $drn(D) = 2$.

Proof. If $d(r)$ were three, then the vertex r would have only one 2-neighbour, say s and the set $\{u, r, s\}$ would induce a block of D with u and r as cutvertices for $p > 4$, a contradiction. So assume $d(r) > 3$. Let s be a 2-vertex adjacent with r.

Case 1. All 2-vertices are adjacent with r.

Consider $D-x$ and $D-u$. The dacard $D-x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$. Hence in any extension of $D-u$, the newly added vertex v must be adjacent with all vertices other than an end vertex of degree triple $\deg t(x)$ by symmetric pair of arcs and the resulting digraph is isomorphic to D.

Case 2. At least one 2-vertex is nonadjacent with r.

Case 2.1. At least two 2-vertices are adjacent with r.

In this case, we use the dacards $D-r$ and $D-s$. The dacard $D-s$ forces every extension to have at most one end vertex and hence $D-r$ forces every extension to have exactly one end vertex whose base is of degree triple $\deg t(r)$ and all 2-vertices have a common neighbour. Hence all digraphs obtained from $D-s$, by adding a vertex v and joining it with the base by suitable arcs so that the degree triple of the base becomes $\deg t(r)$ and with the neighbour common to all 2-vertices by symmetric pair of arcs, are isomorphic and they are D.

Case 2.2. Exactly one 2-vertex is adjacent with r.

Case 2.2.1. There are at least two t's.

Here we use the dacards $D-r$ and $D-s$ and the proof is similar to Case 2.1.

Case 2.2.2 There is exactly one t.

Case 2.2.2.1. t is adjacent with at least two 2-vertices.

Consider $D - s$ and $D - u$. The dacard $D - s$ forces every extension to have exactly one end vertex or two adjacent 2-vertices. Since no extensions of $D - u$ have two adjacent 2-vertices, the only possibility is that every extension must have exactly one end vertex and its base must be adjacent with a 2-vertex and all 2-vertices must have a neighbour of degree triple $(0, 0, p-2)$. Now all digraphs obtained from $D-s$, by annexing a vertex v and joining it with a $(0, 0, p-3)$-vertex by symmetric pair of arcs and with the base by suitable arcs so that the degree triple of v in the resulting digraph remains $\deg t(s)$, are isomorphic and they are D.

Case 2.2.2.2. t is adjacent with exactly one 2-vertex.

Clearly D has exactly two 2-vertices. Consider $D - x$ and $D - u$. The dacard $D - x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$ with exactly one 2-vertex or exactly two nonadjacent 2-vertices with exactly one common neighbour. Hence, in any extension of $D - u$, the newly added vertex v must be adjacent with all vertices other than an end vertex of degree triple $\deg t(x)$ whose base has two 1-neighbours and the resulting digraph is isomorphic to D. □

Theorem 4. *If a P-digraph D has a vertext* $\in T$ *with* $\deg t(t) = (0, 0, p - 2)$, *then* $drn(D) = 2$.

Proof. Since any $(0, 0, p - 2)$-vertex other than r in D must be adjacent with all vertices other than the end vertex, every 2-vertex in D must be adjacent with the two $(0, 0, p - 2)$-vertices t and u and so it will not be adjacent with r.

Case 1. D has exactly one 2-vertex.

The dacards $D - x$ and $D - u$ are used in this case.

Case 1.1. $d(r) = 3$.

The dacard $D-x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$. Hence in any extension of $D - u$, the newly added vertex must be adjacent with all vertices other than an end vertex of degree triple $\deg t(x)$ by symmetric pair of arcs and from the resulting digraph it is clear that D has exactly one 2-vertex and the degree triple of the unique 2-vertex, say s can be determined. Now, D can be uniquely obtained from $D - x$ by annexing a vertex v and joining it with a 2-vertex which is not of degree triple $\deg t(s)$ (If both 2-vertices are of same degree triple, then v can be joined with any one of them).

Case 1.2. $d(r) > 3$.

The dacard $D - x$ forces every extension to have exactly one end vertex, say x and hence $D - u$ forcesevery extension to have exactly one 2-vertex and the base of the end vertex is not adjacent with the 2-vertex. Therefore $D - x$ forces every extension to have a unique 2-vertex whose neighbours are of degree triple $(0, 0, p-2)$. Hence D is obtained from $D - u$ by annexing a vertex v and joining it with all non end vertices and an end vertex whose base is of degree triple $(0, 0, p - 3)$ by means of symmetric pair of arcs.

Case 2. D has at least two 2-vertices.

Consider the dacards $D-r$ and $D-u$. The dacard $D-r$ forces every extension to have a base of degree triple $\deg t(r)$ that is not adjacent with any 2-vertex. Hence in any extension of $D-u$ the newly added vertex v must be joined with all vertices other than the end vertex whose base has exactly one 1-neighbour by means of symmetric pair of arcs and the resulting digraph is D. □

Theorem 5. *If* D *is a P-digraph having a vertex* $s \in D$ *with* $\deg t(s) = (0, 1, 1)$ *and a corresponding* t *with* $\deg t(t) = (1, 0, p-3)$ *or* $\deg t(s) = (1, 0, 1)$ *and a corresponding* t *with* $\deg t(t) = (0, 1, p-3)$, *then* $drn(D) = 2$.

Proof. *Case 1.* $\deg t(s) = (0, 1, 1)$ and $\deg t(t) = (1, 0, p-3)$.

Case 1.1 D has exactly one 2-vertex.

Consider here $D-x$ and $D-u$. The dacard $D-x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$ and hence $D-u$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$ whose base is not adjacent with any 2-vertex. Therefore $D-x$ forces every extension to have two vertices other than the base of degree triple $(0, 0, p-2)$ and $(1, 0, p-3)$. In $D-u$, let the two end vertices be x_1 and x_2. Then D can be obtained from $D-u$ by annexing a vertex v and joining it with all non end vertices and with either an end vertex which is not of degree triple $\deg t(x)$ (when $\deg t(x_1) \neq \deg t(x_2)$) or an end vertex whose base is a $(1, 0, p-4)$-vertex (when $\deg t(x_1) = \deg t(x_2)$) and the resulting digraph is D.

Case 1.2. D has at least two 2-vertices.

The proof is similar to Case 2 of Theorem 4.

Case 2. $\deg t(s) = (1, 0, 1)$ and $\deg t(t) = (0, 1, p-3)$.

The proof is similar to Case 1. □

Theorem 6. *If* D *is a P-digraph with exactly one* 2*-vertex, then* $drn(D) \leq 3$.

Proof. In light of Theorem 3, we can assume that the 2-vertex is nonadjacent with r.

Case 1. $\deg t(t) \neq \deg t(r)$.

The two dacards $D-r$ and $D-u$ are used for this case. The dacard $D-u$ forces every extension to be connected with at most one 1-vertex and hence $D-r$ forces every extension to have exactly one 1-vertex, say x, whose base is of degree triple $\deg t(r)$. In $D-u$, we can distinguish x from the other end vertex by their bases and hence D can be obtained uniquely from $D-u$. Hence $drn(D) = 2$.

Case 2. $\deg t(t) = \deg t(r)$.

Case 2.1 $d(r) = 3$.

Consider the three dacards $D-x$, $D-s$ and $D-u$. The dacard $D-x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$ and hence $D-u$ forcesevery extension to have exactly one 2-vertex adjacent with a 3-vertex and a $(0, 0, p-2)$-vertex. Hence in any extension of $D-s$, the newly added vertex must be adjacent with a $(0, 0, p-3)$-vertex by symmetric pair of arcs and with the unique 2-vertex by suitable arcs and the digraph thus obtained is D. Hence $drn(D) \leq 3$.

Case 2.2. $d(r) > 3$.

The two dacards $D-x$ and $D-u$ are used here. The dacard $D-x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$. Hence D can be obtained from $D-u$ by adding a vertex u and joining it with all vertices except an end vertex of degree triple $\deg t(x)$. In $D-u$, let x_1 and x_2 be the two end vertices and r_1 and r_2 be the two distinct bases. If $\deg t(x_1) \neq \deg t(x_2)$, then D can be uniquely determined from $D-u$. If $\deg t(x_1) = \deg t(x_2)$ and if there is an automorphism of $D-u$ taking r_1 to r_2, then this automorphism takes r_2 to r_1 since r_1 and r_2 are the only vertices of $D-u$ that occurs as bases of end vertices and hence D can be obtained uniquely from $D-u$ by joining u to all vertices except r_1 or r_2. *So, we assume that there is no automorphism on $D-u$ taking r_1 to r_2.*

In $D-x$, a vertex of degree triple $(0, 0, p-2)$, say u and the only 2-vertex, say s, are identifiable and hence $(D-x)-u$ is known from $D-x$. Hence the base of the unique end vertex, say t, is known in $(D-x)-u$. Obviously there exists an isomorphism from $(D-x)-u$ to an induced subgraph of $D-u$ and this isomorphism should map t to r_1 or r_2. Without loss of generality, let α be such an isomorphism taking t to r_1. If there exists another isomorphism β from $(D-x)-u$ to an induced subdigraph of $D-u$ taking t to r_2, then an obvious extension of $\beta\alpha^{-1}$ gives an automorphism of $D-u$ taking r_1 to r_2, contradicting our assumption. Hence all isomorphisms from $(D-x)-u$ to $D-u$ take t to r_1 so that r_1 in $D-u$ is the actual t of D. Hence D can be obtained uniquely and $drn(D) = 2$.

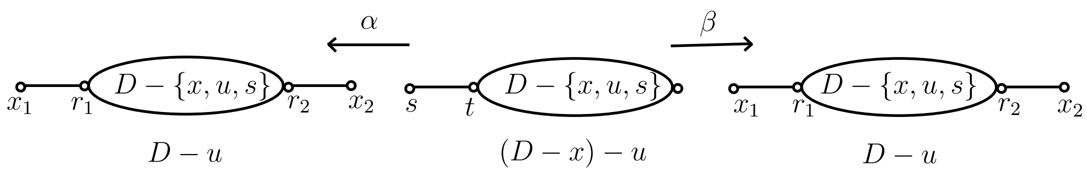

Figure 3. The isomorphisms α and β

□

2.2 At least two 2-vertices

Let us assume that D is a P-digraph satisfying none of the hypotheses of Theorems 2 to 5.

Theorem 7. *Let D be a P-digraph with at least two 2-vertices. If there is a $t \in T$ adjacent with $p-2$ vertices, then $drn(D) = 2$.*

Proof. Consider $D-x$ and $D-u$. The dacard $D-x$ forces every extension to have exactly one end vertex and all 2-vertices with same neighbourhood and hence D can be obtained from $D-u$, by adding a vertex v and joining it with all vertices except an end vertex. Since all 2-vertices of D have the same neighbourhood, in $D-u$, vertex v must be adjacent with all vertices by symmetric pair of arcs, other than the end vertex whose base has exactly one 1-neighbour and the resulting digraph is D. □

Theorem 8. *Let D be a P-digraph with at least two 2-vertices. If each vertex of T is adjacent with at most $p-3$ vertices and if there is a $t \in T$ with $\deg t(t) = (0, 0, p-3)$, then $drn(D) \leq 3$.*

Proof. Consider the dacards $D - x$, $D - u$, $D - t$ (obtained from D by removing a $(0, 0, p - 3)$-vertex) and $D - s$ (obtained from D by removing a 2-vertex adjacent with t).

Case 1. t is adjacent with at least two 2-vertices.

For this case, we use only $D - s$, $D - t$ and $D - u$. As in Case 2.2.2.1 of Theorem 3, using the two dacards $D - s$ and $D - u$, we can determine that D has exactly one end vertex and a $(0, 0, p-2)$-vertex adjacent to all 2-vertices. Also, $D-t$ forces every extension to have a $(0, 0, p - 3)$-vertex other than the base having a 2-neighbour. Hence D can be determined uniquely from $D - s$ by annexing a vertex and joining it with the unique $(0, 0, p - 3)$-vertex adjacent to all 2-vertices and to a $(0, 0, p - 4)$-vertex other than the base having a 2-neighbour. Hence $drn(D) \leq 3$.

Case 2. t is adjacent with exactly one 2-vertex.

Case 2.1. $d(r) = 3$.

The two dacards $D - x$ and $D - r$ are used here. The dacard $D - x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$ and maximum degree to be $p - 2$ or $p - 1$ and hence $D - r$ forces every extension to have the unique base of degree triple $\deg t(r)$ with no 2-neighbours and maximum degree $p-2$. Therefore every extension of $D-x$ has two vertices of degree triple $(0, 0, p-2)$ and $(0, 0, p-3)$. Hence D is obtained uniquely from $D - r$ by adding a vertex v and joining it with the unique isolated vertex by suitable arcs so that the degree triple of the end vertex is $\deg t(x)$ and with a $(0, 0, p-4)$-vertex and $(0, 0, p-3)$-vertex by symmetric pair of arcs.

Case 2.2. $d(r) > 3$.

Consider here $D - x$, $D - u$ and $D - s$. The dacard $D - x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$ and hence $D - u$ forces every extension to have two 2-vertices and the base with no 2-neighbours. Therefore every extension of $D - x$ has two 2-vertices havinga $(0, 0, p - 2)$-vertex as a common neighbour and the other neighbours are say t of degree triple $(0, 0, p - 3)$ and t'. Hence D is obtained from $D - s$ by annexing a vertex v and joining it with a $(0, 0, p - 4)$-vertex and $(0, 0, p - 3)$-vertex by means of symmetric pair of arcs and $drn(D) \leq 3$. □

Theorem 9. *Let D be a P-digraph with at least two 2-vertices. If each vertex of T is adjacent with at most $p - 3$ vertices, $\deg t(r) = (0, 0, p - 3)$ and no vertices of T has degree triple $(0, 0, p - 3)$, then $drn(D) = 2$.*

Proof. Case 1. There exists $t \in T$ adjacent with all 2-vertices.

This is similar to the proof of Theorem 7.

Case 2. No $t \in T$ is adjacent with all 2-vertices.

Consider the dacards $D-s$ and $D-u$. The dacard $D-s$ forces every extension to have two adjacent 2-vertices or exactly one end vertex whose base is either a $(p-2)$-vertex or $(0,0,p-3)$-vertex. Since no extensions of $D-u$ have two adjacent 2-vertices, D must have exactly one end vertex, say x. In $D-u$, x can be distinguished from other end vertices by their bases as exactly one base is a $(0,0,p-4)$-vertex and the degree triple of other bases is not $(0,0,p-4)$. Hence D can be obtained from $D-u$ by annexing a vertex v and joining it with all vertices other than x. □

Theorem 10. *Let D be a P-digraph with at least two* 2*-vertices. If each vertex of T is adjacent with at most $p-3$ vertices, at most one vertex of T has unique* 2*-neighbour,* $\deg t(r) \neq (0,0,p-3)$ *and no vertex of T has degree triple* $(0,0,p-3)$*, then $drn(D) \leq 3$.*

Proof. Case 1. No $t \in T$ has a unique 2-neighbour.

Then every $t \in T$ has at least two 2-neighbours. In this case, we use the three dacards $D-x$, $D-u$ and $D-r$. The dacards $D-x$ and $D-r$ force D to have exactly one end vertex of degree triple $\deg t(x)$ and the base with exactly one 1-neighbour. Hence in any extension of $D-u$, the newly added vertex v must be joined with all vertices other than an end vertex whose base has exactly one 1-neighbour. Therefore the resulting digraph obtained in this way is D and $drn(D) \leq 3$.

Case 2. Exactly one $t \in T$ has a unique 2-neighbour.

Case 2.1. $\deg(t) \neq \deg t(r)$.

Consider $D-u$ and $D-r$. The dacard $D-u$ forces every extension to be connected with at most one end vertex and hence $D-r$ forces every extension to have exactly one end vertex with the base of degree triple $\deg t(r)$ with no 2-neighbours. Hence in any extension of $D-u$, the newly added vertex v must be adjacent with all vertices other than an end vertex whose base is a $(\deg t(r)-(0,0,1))$-vertex (where '$-$' means vector subtraction) with exactly one 1-neighbour and the resulting digraph is D.

Case 2.2. $\deg t(t) = \deg t(r)$.

The case when $d(r) = 3$ is just similar to Case 2.1 of Theorem 6. So assume $d(r) > 3$. Consider $D-x$ and $D-u$. The dacard $D-x$ forces every extension to have exactly one end vertex of degree triple $\deg t(x)$ and the base with no 2-neighbour, since otherwise the resulting digraph has no $(0,0,p-2)$-vertex or the removal of any $(0,0,p-2)$-vertex would result in a dacard having the number of bases reduced by one when compared to $D-u$. Hence in $D-u$, among the bases, say r_1 and r_2 with exactly one 1-neighbours, say x_1 and x_2, one must be the actual t. Now proceeding as in Case 2.2 of Theorem 6, we have $drn(D) = 2$. □

3 Concluding remarks

Among all the dacards of a P-digraph, the dacards $D-u, D-x, D-r, D-t$ and $D-s$ are more easily identifiable than the others in the deck. This is why we could determine, in the above section, the drn of all P-digraphs except those in $\mathbb{D}'$, by using at most three of them in each case. In general, these dacards are not enough to determine the drn of all P-digraphs in $\mathbb{D}'$. It appears that "case by case" analysis with more dacards may lead to the solution of the following problem.

Problem 1. Prove that $drn(D') \leq k$ for some k for all $D' \in \mathbb{D}'$.

If the above problem is proved, then we can determine the drn of a more natural type of digraph in the family $\mathbb{F}_3$ as discussed in the next theorem.

Theorem 11. *The drn of all connected digraphs D with exactly one end vertex and a ce-vertex is at most* $\max\{3, k\}$ *if $drn(D') \leq k$ for some k for all $D' \in \mathbb{D}'$.*

Proof. From the hypothesis, we conclude that drn is at most $\max\{3, k\}$ for all P-digraphs.

Case 1. D has exactly one ce-vertex.

Let x and u be, respectively, the end vertex and the ce-vertex.

Case 1.1 u and x are nonadjacent.

Now u has degree triple $(0, 0, p-2)$. Let r be the base of x. If r is the only cutvertex of D, then D is a P-digraph and hence $drn(D) \leq \max\{3, k\}$. If D has one more cutvertex, then it must be u and hence D is the union of three subdigraphs, say B_{ur} (the non end block containing u and r), B_u (the union of end blocks containing u) and the end block B containing x (which has just two vertices) (Figure 4).

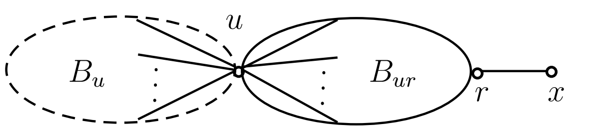

Figure 4. Underlying graph of D

Consider the dacards $D-x$ and $D-v$ (obtained from D by removing a vertex of B_u). In $D-v$, the vertices u, x and r are identifiable as the only $(0, 0, p-3)$-cutvertex (if there are two candidates for u, then there exists an automorphism of $D-v$ taking one of them to the other), the only end vertex nonadjacent with u and the base of x, respectively. Hence the nonendblock B_{ur} containing u and r is known with u and r labeled. The only cutvertex of the card $D-x$ is u. Suppose there is an isomorphism α from B_{ur} onto a block of $D-x$ such that $\alpha(u) = u$. Denote the extension obtained from $D-x$ by adding a new vertex and joining it only with $\alpha(r)$ by suitable arcs by D_α. If β is another such isomorphism and D_β is the corresponding extension, then

$D_\alpha \cong D_\beta$ under the mapping ψ where

$$\psi = \begin{cases} \beta\alpha^{-1} \text{ on the vertices of } \alpha(B_{ur}) \\ \alpha\beta^{-1} \text{ on the vertices of } \beta(B_{ur}) \\ \text{identity on all other vertices} \end{cases}$$

when $\alpha(B_{ur})$ and $\beta(B_{ur})$ are different blocks of $D - x$ and

$$\psi = \begin{cases} \beta\alpha^{-1} \text{ on the vertices of } \alpha(B_{ur}) \\ \text{identity on all other vertices} \end{cases}$$

when $\alpha(B_{ur})$ and $\beta(B_{ur})$ are one and the same block of $D - x$. Hence D is known up to isomorphism and $drn(D) = 2$.

Case 1.2. u and x are adjacent.

Case 1.2.1. $\deg t(x) = (1, 0, 0)$ or $(0, 1, 0)$.

If $\deg t(x) = (1, 0, 0)$, then u must have degree triple $(0, 1, p-2)$. Consider $D-u$ and $D - x$. The dacard $D - u$ shows that D has a $(1, 0, p - 2)$-vertex and hence D can be obtained (uniquely up to isomorphism) from $D-x$ by annexing a vertex and joining it with $(0, 0, p - 2)$-vertex by suitable arcs. Therefore $drn(D) = 2$.

The proof is similar for the case when $\deg t(x) = (0, 1, 0)$.

Case 1.2.2. $\deg t(x) = (0, 0, 1)$.

If $\deg t(x) = (0, 0, 1)$, then the degree triple of u is one of $(0, 0, p-2)$, $(1, 0, p-2)$ or $(0, 1, p - 2)$. In $\overline{D}$, u is the only end vertex and x is the only ce-vertex and they are nonadjacent. Hence $drn(D) \leq 3$ as in Case 2.2.1.

Case 2. D has at least two ce-vertices, say u, v.

At least one of u and v has degree triple $(0, 0, p - 2)$ because otherwise the set $\{\deg t(u), \deg t(v)\}$ would be a subset of $\{(0, 1, p-2), (1, 0, p-2)\}$ and so D would not contain an end vertex, a contradiction. Thus D is a P-digraph and hence $drn(D) \leq \max\{3, k\}$. □

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