

Skolem labellings of generalised Dutch windmills

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Abstract

In this paper, we consider Skolem (vertex) labellings and present (hooked) Skolem labellings for generalised Dutch windmills whenever such labellings exist. Specifically, we show that generalised Dutch windmills with more than two cycles cannot be Skolem labelled and that those composed of two cycles of lengths m and n , $n \geq m$, cannot be Skolem labelled if and only if $n - m \equiv 3, 5 \pmod{8}$ and m is odd. Showing that a Skolem labelling does not exist is, in general, a complex problem and we present a novel technique for doing so.

1 Introduction

A graph labelling is an assignment of labels, like integers or colours, to the vertices or edges of a graph under certain conditions. Graph labelling is a well studied subject in graph theory with many applications, including testing the reliability of networks; a multitude of papers have been written since its introduction more than fifty years ago. We study a vertex labelling problem called Skolem labelling that may be used in testing distance reliability in networks. In 1957, Skolem-type sequences were

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introduced to be sequences (s_1, \dots, s_m) of the integers in $\mathcal{D} = \{1, 2, \dots, n\}$ such that two occurrences of integer i are i positions apart [12]. A *Skolem sequence* of order n is a Skolem-type sequence of length $2n$ such that each integer appears exactly twice, and can be interpreted as a partition of the set $\{1, 2, \dots, 2n\}$ into a collection of disjoint ordered pairs $\{(a_i, b_i)\}_{i=1}^n$ such that $a_i < b_i$ and $b_i - a_i = i$ [12]. Positions in a Skolem-type sequence not occupied by integers contain null elements or hooks. A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod{4}$ and a *hooked Skolem sequence* of order n , a sequence with a hook at the penultimate position, exists if and only if $n \equiv 2, 3 \pmod{4}$ [8, 12]. Skolem-type sequences have many applications in design theory; they are used to construct triple systems, starters, and balanced ternary designs. They are also applied in finding factorizations of complete graphs and answering set partitioning problems.

In 1991, Mendelsohn and Shalaby extended the concept of Skolem labelling to graphs [7]. Similar to Skolem sequences, a graph on $2n$ vertices can be (weakly) Skolem labelled if the vertices can be assigned labels from the set $\mathcal{D} = \{1, \dots, n\}$ such that exactly two vertices at distance i are labelled i , for each integer $i \in \mathcal{D}$. If some of the vertices are not labelled, or are labelled by 0 or *, then the labelling is called *hooked Skolem* (and the graph is said to have been hook-Skolem labelled). A Skolem sequence of order n is equivalent to a Skolem labelling of a path with $2n$ vertices, and a graph with an odd number of vertices must have a vertex labelled by *. It is noted that Skolem labellings of graphs can be used to test communication networks for node, link and distance reliability, and have applications in areas such as the modeling and analysis of networks, as well as in designing monitoring systems such as radio antenna arrays [3, 7, 9]. Skolem labelling of various classes of graphs such as k -windmills, ladder graphs, and Cartesian products of paths have been studied [1, 2, 5, 6] and, in [9], the pseudo-Skolem sequences are introduced and are used to Skolem label a variety of classes of graphs.

In this paper, Skolem labellings of graphs with arbitrarily large chordless cycles are considered. With the exception of grids, this is the first time such graphs have been studied. In fact, until now, only tree-like and grid-like graphs have been Skolem labelled [1, 2, 5, 6]. Here, specifically, we study the Skolem labelling of *generalised Dutch windmills*, which are comprised of a number of cycles called *vanes* sharing a vertex called the *pivot*. Other types of labellings of these graphs, like graceful, harmonious, cordial, prime and Felicitous labellings, have been extensively studied [4]. But Skolem labellings of such graphs, up until now, have been conspicuously missing from the literature. As with windmills, networks with a central hub may be modeled using these graphs. A generalised Dutch windmill with more than two vanes has no Skolem labelling. Let C_m and C_n be the vanes of a generalised Dutch windmill $D_{m,n}$, $n \geq m$, with two vanes. We make use of Skolem-type sequences as well as Skolem labelled generalised 3-windmills to (hook-) Skolem label $D_{m,n}$, when $n - m \equiv 0, 1, 2, 4, 6, 7 \pmod{8}$ and when $n - m \equiv 3, 5 \pmod{8}$ and m is even. We also benefit from the Skolem parity of a sub-tree of $D_{m,n}$ to show that $D_{m,n}$ cannot be Skolem labelled when $n - m \equiv 3, 5 \pmod{8}$ and m is odd. Showing that a Skolem labelling does not exist is, in general, a complex problem and in Sections 3.1.4 and

3.1.5 we offer this new technique for doing so.

2 Existence results

In this section, we present the existence results that are used as tools in subsequent sections. A k -near Skolem sequence of order n is a Skolem-type sequence (s_1, \dots, s_m) with $\mathcal{D} = \{1, 2, \dots, n\} \setminus \{k\}$. For $k \leq n$, a k -near Rosa sequence of order n , denoted k -near \mathcal{R}_n , is a k -near Skolem-type sequence of order n with $m = 2n - 1$ and $s_n = 0$.

Theorem 2.1 (Shalaby [11]) *A k -near Rosa sequence of order n exists if and only if $n \equiv 0,3 \pmod{4}$ and k is even, or $n \equiv 1,2 \pmod{4}$ and k is odd, with exceptions when $(n, k) = (3, 2), (4, 2)$.*

We will also use the concept of Skolem parity of a tree T with an even number of vertices [6]. If T is a tree and $u \in V(T)$, then the Skolem parity of T is defined to be $\mathcal{D}(u, V(T)) \equiv \sum_{v \in V(T)} d(u, v) \pmod{2}$, where $d(u, v)$ denotes the distance between vertices u and v .

Theorem 2.2 (Mendelsohn and Shalaby [6]) *The Skolem parity of a tree T with $2k$ vertices is independent of $u \in V(T)$.*

A k -windmill is a tree consisting of k paths of equal length, the vanes, which meet at a pivot. In a generalised k -windmill, denoted gk -windmill, the vanes may have different lengths.

Theorem 2.3 (Baker and Manzer [2]) *A $g3$ -windmill T has a Skolem labelling if and only if T satisfies the following Skolem parity conditions: either*

- (i) $n \equiv 0,3 \pmod{4}$ and the parity of T is even, or
- (ii) $n \equiv 1,2 \pmod{4}$ and the parity of T is odd.

Note that if a tree with an even number of vertices is Skolem labelled, then it satisfies one of the Skolem parity conditions (i) and (ii) above ([6]).

3 Skolem labellings of Dutch windmills

The Dutch windmill graph D_n^m has as its vanes m copies of C_n which meet at a pivot. A Dutch windmill, which has $m(n - 1) + 1$ vertices, can only be Skolem labelled if $m(n - 1) + 1$ is even. In addition, in order to use label i , there must be two vertices at distance i . The longest distance between two vertices of a Dutch windmill is n when n is even and $n - 1$ otherwise. Therefore, the number of vertices being labelled is at most $2n$; i.e. only the vertices of two vanes can be labelled. As well, D_n^2 can only be hook-Skolem labelled since $|V(D_n^2)|$ is odd. Here, we consider the class of generalised Dutch windmills in which the two vanes may have different lengths.

3.1 Generalised Dutch windmills

As above, we study generalised Dutch windmills in which the lengths of the vanes can be different. Following the argument above, such graphs with more than two vanes cannot be (hook-) Skolem labelled. As in the Introduction, let $D_{m,n}$ be a Dutch windmill with C_m and C_n as its vanes, with $n \geq m$. For convenience, we adopt a drawing of $D_{m,n}$ that has an axis of symmetry ℓ with the pivot p and the farthest vertex of an even cycle on the symmetry axis as shown in Figure 1.

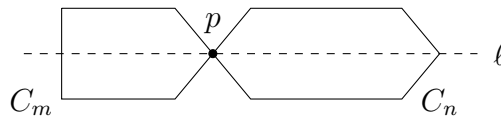


Figure 1: Generalised Dutch windmill $D_{m,n}$ where m is odd and n is even.

When Skolem labelling generalised Dutch windmills, the distance between two vertices of C_n is at most $\frac{n}{2}$ or $\frac{n-1}{2}$, depending on the parity of n . Therefore, if u is the maximum label of the Skolem labelling, then the labels $u, u-1, \dots, \frac{n}{2}+1$ or $\frac{n+1}{2}$, depending on the parity of n , are forced. We call these labels *permanent*. As well, if two vertices x and y of C_n are labelled such that the shortest path between them contains vertices with permanent labels, then we say they are *labelled around C_n* . In the figures, by a dashed arrow with ± 2 , we mean that we label the vertices in the direction of the arrow by increasing or decreasing 2 at a time. The main result of this paper is Theorem 3.1 below. Theorems 3.5 and 3.7 prove that when $n - m \equiv 3, 5 \pmod{8}$ and m is odd, $D_{m,n}$ does not have a Skolem labelling, and Theorems 3.2, 3.3, 3.4, 3.6, and 3.8 prove the existence of a labelling with the Skolem property for $D_{m,n}$ in all remaining cases.

Theorem 3.1 *The generalised Dutch windmill $D_{m,n}$, $n \geq m$, does not have a (hooked) Skolem labelling if and only if $n - m \equiv 3, 5 \pmod{8}$ and m is odd.*

In the following subsections, we find the necessary and sufficient conditions for $D_{m,n}$ to be Skolem labellable as stated in Theorem 3.1. To achieve this, we examine the graph based on the value of $n - m$ modulo 8.

3.1.1 $n - m \equiv 0, 2, 6 \pmod{8}$

In these cases, m and n have the same parity and the maximum label in a Skolem labelling is $u = \frac{n+m-2}{2}$. When $n - m \equiv 0 \pmod{8}$, u is odd when m is even and is even when m is odd. When $n - m \equiv 2, 6 \pmod{8}$, u is even when m is even and is odd when m is odd. We use Skolem sequences to Skolem label the graph $D_{m,n}$.

Theorem 3.2 *If $n - m \equiv 0, 2, 6 \pmod{8}$, then $D_{m,n}$ can be hook-Skolem labelled.*

Proof First we consider the case when $n - m \equiv 0, 6 \pmod{8}$. We assign a hook to a vertex of the m -cycle which is farthest from the pivot. Then we assign the labels in descending order until all the vertices of the m -cycle, except the pivot, are labelled. The number of vertices awaiting a label is $n - m + 2$; these vertices are all consecutive along C_n . Since $\frac{n-m+2}{2} \equiv 0, 1 \pmod{4}$ when $n - m \equiv 0, 6 \pmod{8}$, we can complete the labelling using a Skolem labelling of order $\frac{n-m+2}{2}$.

If $n - m \equiv 2 \pmod{8}$, then we assign a hook to a vertex of C_n which is farthest from the pivot, and then assign labels in descending order until all vertices of C_m except the pivot are labelled. The number of vertices awaiting labels is $n - m$. Since $\frac{n-m}{2} \equiv 1 \pmod{4}$, we can complete the labelling using a Skolem labelling of order $\frac{n-m}{2}$. \square

3.1.2 $n - m \equiv 1, 7 \pmod{8}$

When $n - m \equiv 1, 7 \pmod{8}$, then m and n have different parities and the maximum label in a Skolem labelling is $u = \frac{n+m-1}{2}$. If $n - m \equiv 1 \pmod{8}$, u is odd when m is odd and is even when m is even. If $n - m \equiv 7 \pmod{8}$, u and m have different parities. We make use of near Rosa sequences to Skolem label $D_{m,n}$.

Theorem 3.3 *If $n - m \equiv 1, 7 \pmod{8}$, then $D_{m,n}$ can be Skolem labelled.*

Proof First we partially label the graph as illustrated in Figures 2 and 3 depending on the parities of m and n . If there is a 1-near $\mathcal{R}_{\frac{n-m+3}{2}}$ then we can complete the labelling. By Theorem 2.1, a 1-near $\mathcal{R}_{\frac{n-m+3}{2}}$ exists if and only if $\frac{n-m+3}{2} \equiv 1, 2 \pmod{4}$, which implies that $n - m \equiv 1, 7 \pmod{8}$. It is easy to show that the number of unlabelled vertices is even and the pivot is in the middle of the vertices. \square

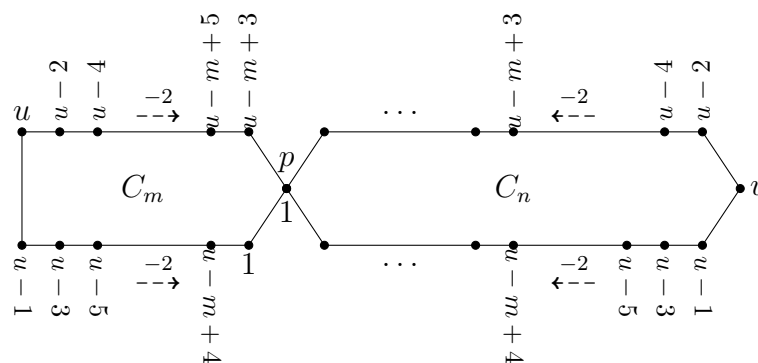


Figure 2: Partial labelling of $D_{m,n}$ when $n - m \equiv 1, 7 \pmod{8}$, m is odd and n is even.

Note that we may obtain a Skolem labelling of $D_{m,n}$ when $n - m \equiv 7 \pmod{8}$, m is odd and n is even from a hooked Skolem labelling of $D_{m+1,n}$, and conversely. This is illustrated in Example 3.1 below.

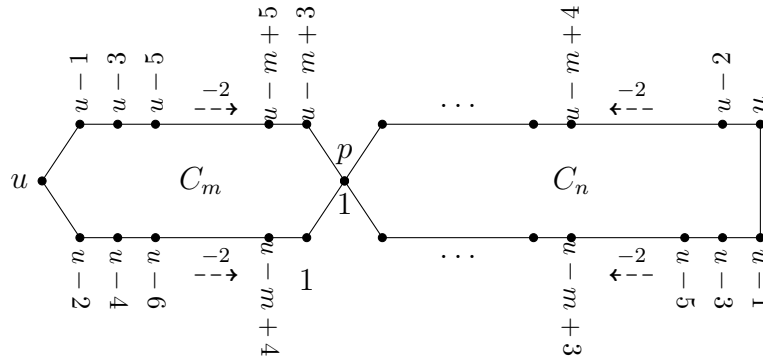


Figure 3: Partial labelling of $D_{m,n}$ when $n - m \equiv 1,7 \pmod{8}$, m is even and n is odd.

Example 3.1 Figure 4 illustrates how we Skolem label $D_{9,16}$ using a 1-near \mathcal{R}_5 and then use this labelling to hook-Skolem label $D_{10,16}$.

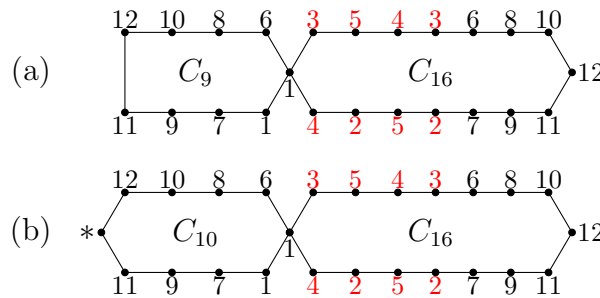


Figure 4: (a) Skolem labelled $D_{9,16}$ using a 1-near \mathcal{R}_5 , (b) hook-Skolem labelled $D_{10,16}$ using the labelling of $D_{9,16}$ in (a).

3.1.3 $n - m \equiv 4 \pmod{8}$

Theorem 3.4 *If $n - m \equiv 4 \pmod{8}$, then $D_{m,n}$ can be Skolem labelled.*

Proof When $n - m \equiv 4 \pmod{8}$, then the greatest label is $u = \frac{m+n-2}{2}$. We consider two cases based on the parities of m and n .

Case 1: both n and m are even. We first assign the hook to a vertex of C_n which is at distance $\frac{n}{2} - 1$ from the pivot, and assign the labels $u, u - 1, u - 2, \dots, \frac{n-m}{2} + 2$ as shown in Figure 5.

Removing all the labelled vertices leaves us with a $g3$ -windmill T with $|V(T)| = n - m + 2$. Note that by symmetry $D(p, V(T)) \equiv (d(z, p) + d(x, p) + d(y, p)) \pmod{2}$, where $D(p, V(T))$ is the Skolem parity of T , x, y are the two vertices of T farthest

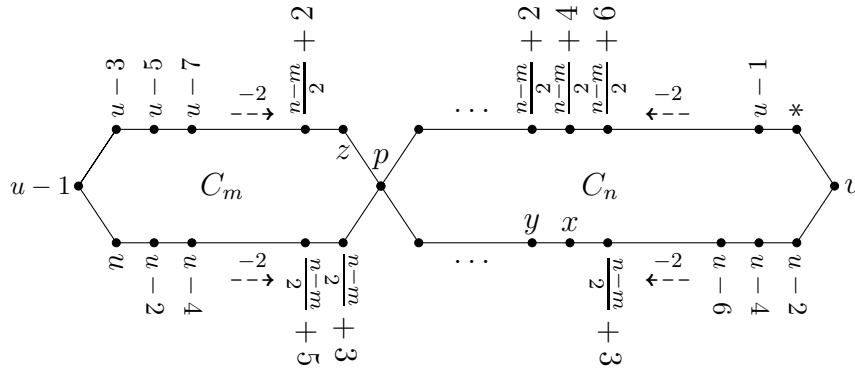


Figure 5: Partial labelling of $D_{m,n}$ when $n - m \equiv 4 \pmod{8}$ and n and m are both even.

from p , and z is the leaf of the vane of T which is of length one; see Figure 5. Therefore,

$$D(p, V(T)) \equiv (1 + d(x, p) + d(y, p)) \pmod{2}.$$

Since either $d(x, p)$ or $d(y, p)$ is odd, then the parity of T is even. Since $\frac{|V(T)|}{2} = \frac{n-m}{2} + 1 = 4k + 3$, for some integer k , we can complete the partial labelling by combining it with a Skolem labelling of T , by Theorem 2.3.

Case 2: both n and m are odd. This case is also divided into two subcases based on the parity of $\frac{m+1}{2}$:

(a) $\frac{m+1}{2} \equiv 0 \pmod{2}$: see Figure 6 for a Skolem labelling of $D_{m,n}$.

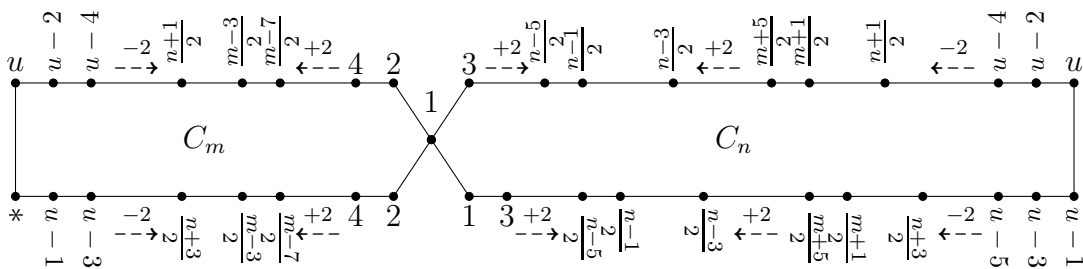


Figure 6: Skolem labelling of $D_{m,n}$ when $n - m \equiv 4 \pmod{8}$, m and n are odd, and $\frac{m+1}{2} \equiv 0 \pmod{2}$.

(b) $\frac{m+1}{2} \equiv 1 \pmod{2}$: see Figure 7 for a Skolem labelling of $D_{m,n}$.

□

The only remaining cases are when $n - m \equiv 3, 5 \pmod{8}$. In what follows, we show that when $m \in 2\mathbb{Z} + 1$ and $n \in 2\mathbb{Z}$, then $D_{m,n}$ does not have a Skolem labelling. Otherwise, we present a Skolem labelling of the graph.

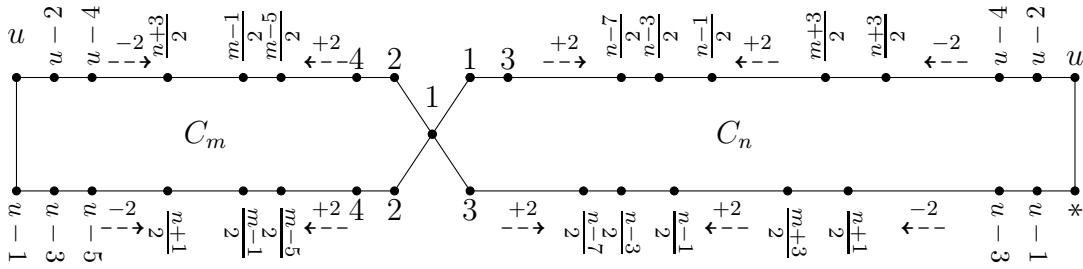


Figure 7: Skolem labelling of $D_{m,n}$ when $n - m \equiv 4 \pmod{8}$, m and n are odd, and $\frac{m+1}{2} \equiv 1 \pmod{2}$.

3.1.4 $n - m \equiv 3 \pmod{8}$

First we show that when m is odd and n is even, then $D_{m,n}$ does not have a Skolem labelling.

Theorem 3.5 *If $n - m \equiv 3 \pmod{8}$, $m \in 2\mathbb{Z} + 1$, and $n \in 2\mathbb{Z}$, then $D_{m,n}$ cannot be Skolem labelled.*

Proof To the contrary, assume that $D_{m,n}$ has a Skolem labelling \mathcal{L} obtained by placing the $\frac{m-1}{2}$ permanent labels $\frac{n}{2}+1, \frac{n}{2}+2, \dots, \frac{m+n-1}{2}$ on C_m and C_n , and assigning the rest of the labels $1, 2, \dots, \frac{n}{2}$. By removing the vertices with labels in $\{\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, \frac{m+n-1}{2}\}$, we obtain a $g4$ -windmill T (this is always the case when m and n have opposite parity). It is not difficult to see that the labelling of the vertices of T is not a Skolem labelling of T ; this is because T does not satisfy the Skolem parity conditions. If some vertices x and y are labelled around C_n by $\frac{n}{2} - i$, for some positive integer i , then their distance in T is $\frac{n}{2} + i$. In other words, if a_j and b_j are the vertices of T labelled j , for $1 \leq j \leq \frac{n}{2}$, and $d(a_j, b_j)$ is their distance in T , then

$$d(a_{\frac{n}{2}-i}, b_{\frac{n}{2}-i}) = \begin{cases} \frac{n}{2} + i, & \text{if } a_{\frac{n}{2}-i} \text{ and } b_{\frac{n}{2}-i} \text{ are labelled around } C_n \\ \frac{n}{2} - i, & \text{otherwise.} \end{cases} \quad (*)$$

As before, let $D(p, V(T))$ denote the Skolem parity of T . By Lemmas 2.1 and 2.2 of [6],

$$\begin{aligned} D(p, V(T)) &\equiv \left(D(p, V(T) \setminus \{a_{\frac{n}{2}}, b_{\frac{n}{2}}\}) + d(a_{\frac{n}{2}}, b_{\frac{n}{2}}) \right) \pmod{2} \\ &\equiv \left(D(p, V(T) \setminus \{a_{\frac{n}{2}}, b_{\frac{n}{2}}\}) + \frac{n}{2} \right) \pmod{2} \\ &\equiv \left(D(p, V(T) \setminus \{a_{\frac{n}{2}}, b_{\frac{n}{2}}, a_{\frac{n}{2}-1}, b_{\frac{n}{2}-1}\}) + d(a_{\frac{n}{2}-1}, b_{\frac{n}{2}-1}) + \frac{n}{2} \right) \pmod{2}. \end{aligned}$$

Repeating, we get $\frac{n}{2}$ terms as shown in (1) below, including those associated with vertices with labels $\frac{n}{2} - i$ going around C_n and at distance $\frac{n}{2} + i$, for $1 \leq i \leq \frac{n}{2} - 1$:

$$D(p, V(T)) \equiv \left(1 + \dots + d(a_{\frac{n}{2}-3}, b_{\frac{n}{2}-3}) + d(a_{\frac{n}{2}-2}, b_{\frac{n}{2}-2}) + d(a_{\frac{n}{2}-1}, b_{\frac{n}{2}-1}) + \frac{n}{2}\right) \pmod{2}. \tag{1}$$

Now note that since $\frac{n}{2} - i$ and $\frac{n}{2} + i$ have the same parity then, in (1), we can replace each $d(a_{\frac{n}{2}-i}, b_{\frac{n}{2}-i}) = \frac{n}{2} + i$, given by (*) above, with $\frac{n}{2} - i$ without changing the parity of T . Therefore,

$$D(p, V(T)) \equiv 1 + 2 + \dots + \left(\frac{n}{2} - 1\right) + \frac{n}{2} = \frac{\frac{n}{2}(\frac{n}{2} + 1)}{2} \pmod{2}. \tag{2}$$

Now we show that the parity of $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ obtained in (2) above is different from the actual parity of T (once calculated) to get our desired contradiction. We consider the following four cases to calculate the parity of T based on the structure of T in each case, i.e. the symmetry and the length of each vane.

1. $n \equiv 0 \pmod{8}$: If $n = 8k$ for some integer k , then $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ is even. Now let us calculate the actual parity of T in this case. Since $n - m \equiv 3 \pmod{8}$, then $m \equiv 5 \pmod{8}$ and the number of permanent labels is $\frac{m-1}{2}$ which is even. By the structure of T (Figure 8), the parity of T is equivalent to that of $\frac{n-m-1}{2}$ which is odd (contradiction). To see this, note that by symmetry

$$D(p, V(T)) \equiv d(x, p) \pmod{2},$$

where x is the vertex of T farthest from p . Therefore, $D(p, V(T)) \equiv \frac{n-m-1}{2} \pmod{2}$. Given that $\frac{n-m-1}{2} \equiv 1 \pmod{2}$, we conclude that $D(p, V(T)) \equiv 1 \pmod{2}$ which means the parity of T is odd.

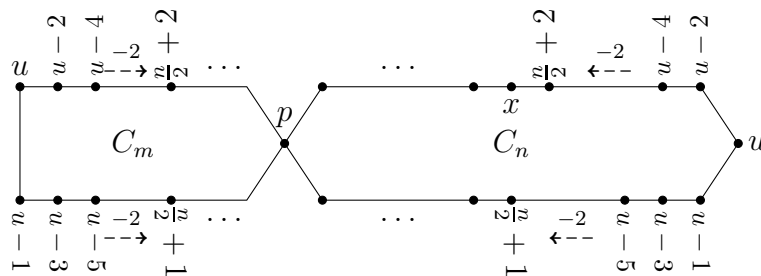


Figure 8: Partial labelling of $D_{m,n}$ when $n - m \equiv 3 \pmod{8}$, n is even and m is odd.

We omit the details in cases 2–4 below as they are similar to the argument in case 1 above.

2. $n \equiv 2 \pmod{8}$: If $n = 8k + 2$ for some integer k , then $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ is odd. Given that $n - m \equiv 3 \pmod{8}$, then $m \equiv 7 \pmod{8}$ and the number of permanent labels, $\frac{m-1}{2}$, is odd. By the structure of T , the parity of T is equivalent to that of $\frac{m-m-1}{2}$ which is even (contradiction).

3. $n \equiv 4 \pmod{8}$: If $n = 8k + 4$ for some integer k , then $\frac{n(\frac{n}{2}+1)}{2}$ is odd. Since $n - m \equiv 3 \pmod{8}$, then $m \equiv 1 \pmod{8}$ and the number of permanent labels, $\frac{m-1}{2}$, is even. By the structure of T , the parity of T is equivalent to that of $\frac{n-m-1}{2}$ which is even (contradiction).
4. $n \equiv 6 \pmod{8}$: If $n = 8k + 6$ for some integer k , then $\frac{n(\frac{n}{2}+1)}{2}$ is even. Given that $n - m \equiv 3 \pmod{8}$, then $m \equiv 3 \pmod{8}$ and the number of permanent labels, $\frac{m-1}{2}$, is odd. By the structure of T , the parity of T is equivalent to that of $\frac{m-m-1}{2}$ which is odd (contradiction).

Therefore, $D_{m,n}$ does not have such a Skolem labelling. □

We now give a Skolem labelling of $D_{m,n}$, when m is even and n is odd.

Theorem 3.6 *If $n - m \equiv 3 \pmod{8}$, $m \in 2\mathbb{Z}$, and $n \in 2\mathbb{Z} + 1$, then $D_{m,n}$ can be Skolem labelled.*

Proof We consider two sub-families of such graphs based on the parity of $\frac{n+1}{2}$. If $\frac{n+1}{2} \equiv 0 \pmod{2}$, we use the labelling in Figure 9 and if $\frac{n+1}{2} \equiv 1 \pmod{2}$, we use the labelling in Figure 10.

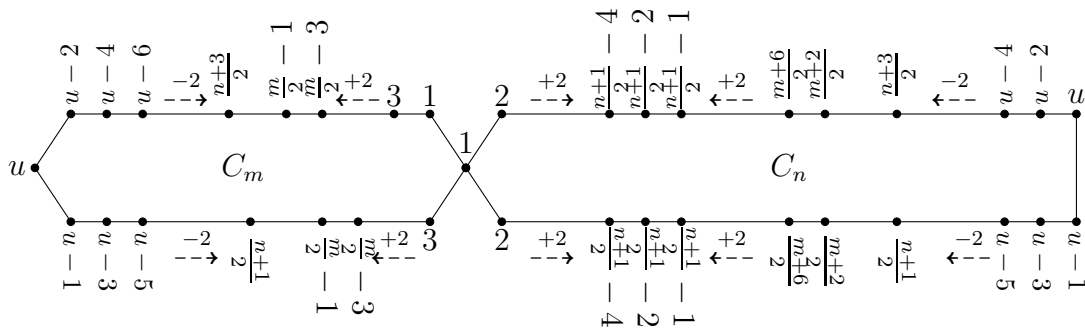


Figure 9: Skolem labelling of $D_{m,n}$ when $n - m \equiv 3 \pmod{8}$ and $\frac{n+1}{2} \equiv 0 \pmod{2}$.

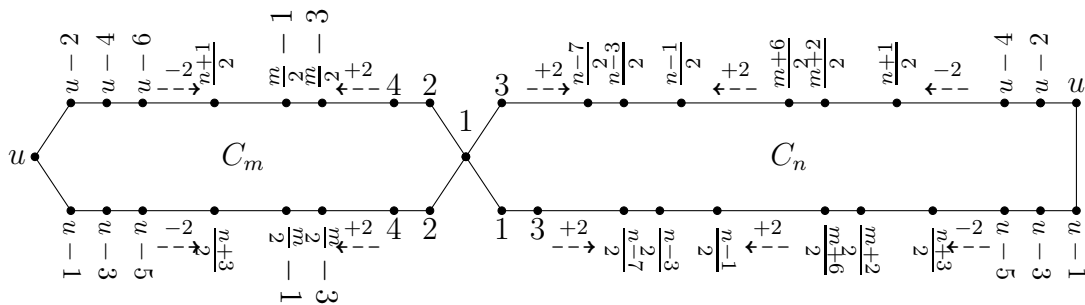


Figure 10: Skolem labelling of $D_{m,n}$ when $n - m \equiv 3 \pmod{8}$ and $\frac{n+1}{2} \equiv 1 \pmod{2}$.

□

3.1.5 $n - m \equiv 5 \pmod{8}$

Again, we divide the case $n - m \equiv 5 \pmod{8}$ into two subcases based on the parities of n and m . First we show that when m is odd and n is even, $D_{m,n}$ does not have a Skolem labelling.

Theorem 3.7 *If $n - m \equiv 5 \pmod{8}$, $m \in 2\mathbb{Z} + 1$, and $n \in 2\mathbb{Z}$, then $D_{m,n}$ cannot be Skolem labelled.*

Proof The argument follows as per the proof of Theorem 3.5. If a Skolem labelling of the graph exists, then the Skolem parity of its subtree T arising from the labelling is

$$D(p, V(T)) \equiv \frac{\frac{n}{2}(\frac{n}{2} + 1)}{2} \pmod{2}.$$

We show that the parity of $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ is different from the actual parity of T .

1. $n \equiv 0 \pmod{8}$: If $n = 8k$ for some integer k , then $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ is even. Since $n - m \equiv 5 \pmod{8}$, then $m \equiv 3 \pmod{8}$ and the number of permanent labels, $\frac{m-1}{2}$, is odd. By the structure of T , the parity of T is equivalent to that of $\frac{m - \frac{m-1}{2}}{2}$ which is odd (contradiction).
2. $n \equiv 2 \pmod{8}$: If $n = 8k + 2$ for some integer k , then $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ is odd. Given that $n - m \equiv 5 \pmod{8}$, then $m \equiv 5 \pmod{8}$ and the number of permanent labels, $\frac{m-1}{2}$, is even. By the structure of T , the parity of T is equivalent to that of $\frac{n - \frac{m-1}{2}}{2}$ which is even (contradiction).
3. $n \equiv 4 \pmod{8}$: If $n = 8k + 4$ for some integer k , then $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ is odd. Since $n - m \equiv 5 \pmod{8}$, then $m \equiv 7 \pmod{8}$ and the number of permanent labels, $\frac{m-1}{2}$, is odd. By the structure of T , the parity of T is equivalent to that of $\frac{m - \frac{m-1}{2}}{2}$ which is even (contradiction).
4. $n \equiv 6 \pmod{8}$: If $n = 8k + 6$ for some integer k , then $\frac{\frac{n}{2}(\frac{n}{2} + 1)}{2}$ is even. Given that $n - m \equiv 5 \pmod{8}$, then $m \equiv 1 \pmod{8}$ and the number of permanent labels, $\frac{m-1}{2}$, even. By the structure of T , the parity of T is equivalent to that of $\frac{n - \frac{m-1}{2}}{2}$ which is odd (contradiction).

Therefore, $D_{m,n}$ does not have such a Skolem labelling. □

Now we consider the case when m is even and n is odd.

Theorem 3.8 *If $n - m \equiv 5 \pmod{8}$, $m \in 2\mathbb{Z}$, and $n \in 2\mathbb{Z} + 1$, then $D_{m,n}$ can be Skolem labelled.*

Proof If $\frac{m}{2} \equiv 1 \pmod{2}$, then we use the Skolem labelling of Figure 11; otherwise $\frac{m}{2} \equiv 0 \pmod{2}$ and we use the Skolem labelling of Figure 12.

□

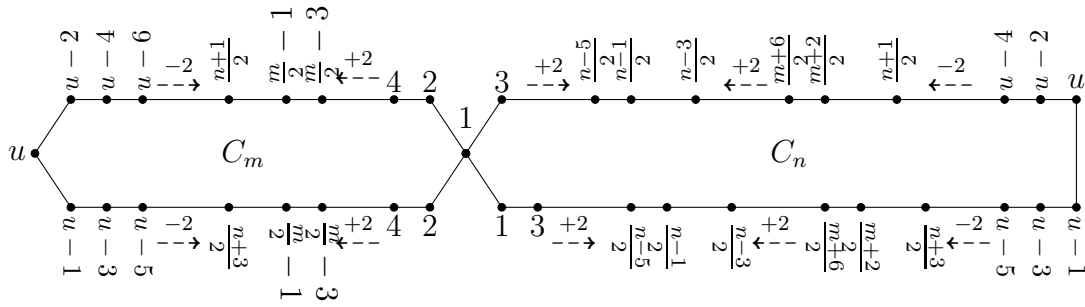


Figure 11: Skolem labelling of $D_{m,n}$ when $n - m \equiv 5 \pmod{8}$ and $\frac{m}{2} \equiv 1 \pmod{2}$.

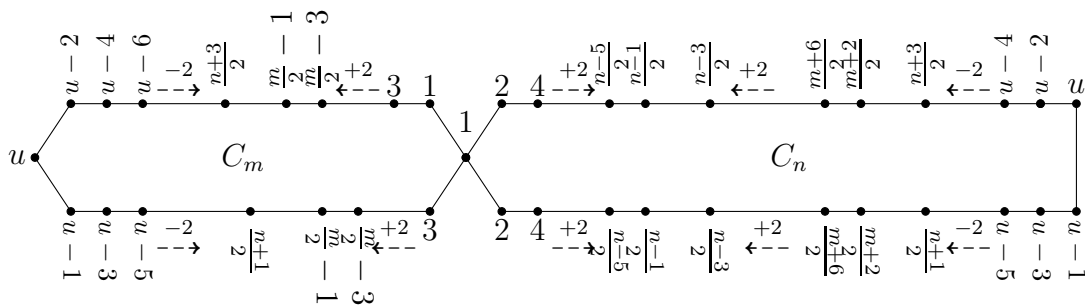


Figure 12: Skolem labelling of $D_{m,n}$ when $n - m \equiv 5 \pmod{8}$ and $\frac{m}{2} \equiv 0 \pmod{2}$.

References

- [1] C. Baker, P. Kergin and A. Bonato, Skolem arrays and Skolem labellings of ladder graphs, *Ars Combin.* **63** (2002), 97–107.
- [2] C. Baker and J.D.A. Manzer, Skolem-labeling of generalised three-vane windmills, *Australas. J. Combin.* **41** (2008), 175–204.
- [3] F. Biraud, E. J. Blum and J. C. Ribes, On optimum synthetic linear arrays with application to radioastronomy, *IEEE Trans. Ant. Prop.* **22** (1) (1974), 108–109.
- [4] J. A. Gallian, A dynamic survey of graph labeling, *Elect. J. Combin.* **16**, article DS6, 2013.
- [5] A. J. Graham, D. A. Pike and N. Shalaby, Skolem labelled trees and $P_s \square P_t$ Cartesian products, *Australas. J. Combin.* **38** (2007), 101–115.
- [6] E. Mendelsohn and N. Shalaby, On Skolem labelling of windmills, *Ars Combin.* **53** (1999), 161–172.
- [7] E. Mendelsohn and N. Shalaby, Skolem labelled graphs, *Discrete Math.* **97** (1991), 301–317.
- [8] E. S. O’Keefe, Verification of a conjecture of Th. Skolem, *Math. Scand.* **9** (1961), 80–82.

- [9] D. A. Pike, A. Sanaei and N. Shalaby, Pseudo-Skolem sequences and graph Skolem labelling, *Math. Scand.* (to appear).
- [10] A. Rosa, A note on cyclic Steiner triple systems, *Mat.-Fyz.Časopis Slov. Akad. Vied* **16** (1966), 285–290.
- [11] N. Shalaby, The existence of near-Rosa and hooked near-Rosa sequences, *Discrete Math.*, **261** (2003), 435–450.
- [12] Th. Skolem, On certain distribution of integers in pairs with given differences, *Math. Scand.* **6** (1957), 273–280.

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