AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **67(2)** (2017), Pages 378–393

Kite systems of order 8; embedding of kite systems into bowtie systems

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Dedicated to the memory of Dan Archdeacon

Abstract

This article consist of two parts. In the first part, we enumerate the kite systems of order 8; in the second part, we consider embedding kite systems into bowtie systems.

1 Introduction

A *kite* is a connected 4-vertex graph consisting of a triangle and an edge incident with one of its vertices. A *bowtie* is a 5-vertex connected graph consisting of two triangles with a common vertex.

A kite as in Fig. 1 is denoted by $(a, b, c; d)$ (or $(b, a, c; d)$). Here $\{a, b, c\}$ is its *triangle*, and the edge $\{c, d\}$ is its *tail*. A bowtie as in Fig. 1(b) with its two triangles $\{a, b, c\}$ and $\{c, d, e\}$ is denoted by (a, b, c, d, e).

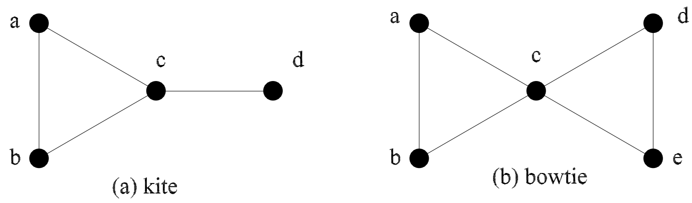

Fig. 1

(Later on, we may abbreviate the kite $(a, b, c; d)$ simply as *abcd*, and the bowtie (a, b, c, d, e) simply as *abcde*.) A decomposition of the complete graph K_n into edge-disjoint kites will be called a *kite system of order* n. A decomposition of the complete graph K_n into edge-disjoint bowties will be called a *bowtie system of order* n.

It is well-known that a necessary and sufficient condition for the existence of a kite system of order n is $n \equiv 0$ or 1 (mod 8), and for the existence of a bowtie system of order n is $n \equiv 1$ or 9 (mod 12).

Firstly, we want to enumerate, up to an isomorphism, all kite systems of order $n = 8$. Any such system contains 7 kites. The graph induced by the 7 tails in a kite system has to be a graph with 8 vertices and 7 edges, with all vertex degrees odd (the *tail graph*). There are just 3 possibilities for degree sequences of such graphs: (1) 71111111, (2) 53111111, and (3) 33311111. For (1) and (2), the corresponding graphs are unique, up to isomorphism, while for (3) there are two such graphs, one containing a triangle, and the other not (cf. [3, 4]). The diagrams of these 4 graphs are shown in Fig. 2.

Accordingly, we will call kite systems of type I, II, III, or IV, depending on the type of the tail graph.

The tail graph of type I is the 7-star, i.e. the complete bipartite graph $K_{1,7}$. It is not hard to show that up to an isomorphism there is a unique kite system of order 8 of type I (for the sake of brevity, brackets, commas etc. are omitted):

$$0137, 1247, 2357, 3467, 4507, 5617, 6027.$$

For the remaining types II, III, and IV, we will employ an isomorphism invariant which distinguishes the systems completely. This invariant uses graphs which are unions of two distinct kites in a system. There are 16 possible such graphs; they are presented in Fig. 3.

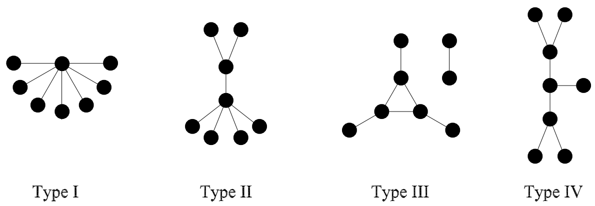

Fig. 2

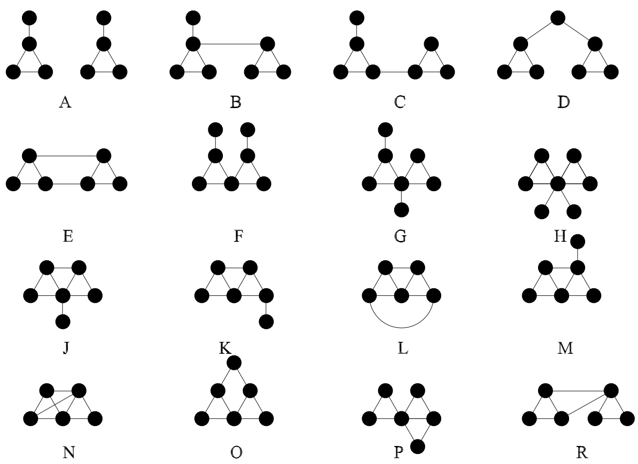

Fig. 3

However, a graph of type A cannot be present in a kite system of order 8. Indeed, suppose there are two vertex-disjoint kites present in a system of order 8, say, one with vertices $\{a, b, c, d\}$ and the other with vertices $\{e, f, g, h\}$. Any kite must use at least one edge *not* from the complete bipartite graph with partite sets $\{a, b, c, d\}$, $\{e, f, g, h\}$ but this means that there are only at most 4 edges available for the remaining 5 kites of the kite systems which is the desired contradiction.

All of the remaining 15 graphs (types B to R) may and actually do occur. We can assign now to any kite system of order eight a 7×7 matrix (with empty diagonal) whose rows and columns are labelled by the 7 kites $G_i, i = 1, 2, \ldots, 7$ of the system, and whose off-diagonal entry (i, j) is the type of the union of G_i and G_j. This is the *system matrix.* The system matrix is symmetric, and when listing it, we list only its above-diagonal elements, starting with the 6 elements of the first row, followed by the 5 elements of the second row, etc.

On the other hand, the *system vector* corresponding to a kite system is the 15-dimensional vector (the *system vector*) counting the number of occurrences of graphs of type $B, C, D, E, F, G, H, J, K, L, M, N, O, P, R$ (in this order) in the upper triangular part of the system matrix.

2 Systems of type II

There are up to an isomorphism exactly 17 kite systems of order 8 and type II. We list the systems together with their vectors and matrices. The edges of the tail graph are 06, 17, 26, 37, 47, 57, 67.

II.1 0137, 1247, 3526, 3467, 0457, 1560, 0271
100114111032330 $OGPOKNGPPFNMGMGOHBJME$
II.2 1306, 1247, 2537, 3467, 0457, 1562, 0271
100113112041240 $FGMGMJOPPKNPPKMOHBGME$
II.3 1306, 1247, 3526, 4637, 0457, 1567, 0271
100014030031341 $FOJGMJGPPONJGMGOPBPMR$
II.4 1307, 1426, 2537, 3647, 0457, 1567, 0271
100013031040251 $OGKGMJGJFMJPPOMPPBPMR$
II.5 0137, 1247, 3526, 3467, 4506, 1657, 0271
100006020031431 $OGPGONGPGONMOJGMPBJGR$
II.6 1306, 1426, 2537, 3467, 0547, 1657, 0271
100004022040251 $OGMGKJGMGKJPOPMPPBPMR$
II.7 1306, 1247, 2537, 3462, 0457, 1567, 0271
010022130041331 $FGMGMJOJPONJPOMFHCPMR$
II.8 0137, 1247, 3526, 3460, 0457, 1567, 0271
010005111032421 $OGJOONGGPONMGMGKHCPMR$
II.9 1306, 2417, 2356, 4637, 0547, 1567, 0726
000112031010381 $GFJGMPOOPPJPPPJPPEOKR$
II.10 1306, 2417, 2537, 3647, 0457, 1567, 0726
000103022010381 $GGKGMPOPOPJPPOJPPEPKR$
II.11 0317, 1247, 2537, 3462, 0457, 1567, 2706
000020142000462 $PPFOPJOJPOKJPOKFHRPJR$
II.12 0317, 1247, 2537, 3460, 0457, 1567, 0726
000002124000462 $PPKOPKOGPOJGPOJKHRPKR$
II.13 4567, 2357, 1347, 0736, 1726, 0516, 2406
000013031020362 $FGJJOPPPPGMOPGMOJRKRP$
II.14 2357, 5647, 0437, 2417, 0276, 1362, 1506
000005012040252 $PGOMKGPPRGKOMGGMJGRPM$

II.15 2357, 1247, 1637, 0517, 0276, 4562, 3406

000023030040252 $OPPMJFOPMJGPRGJMFGRPM$

II.16 2357, 0547, 0137, 2417, 0276, 1562, 3460

000040140040242 $PPOMJFOPMFJPMFJMJFRRH$

II.17 2357, 0547, 0137, 2417, 0276, 3462, 1560

000004104040242 $PPOMKGOPMGKPMGKMKGRRH$

It is clear from the above that the 17 kite systems of order 8 and type II. are distinguished completely by their system vectors, and therefore, of course, by their system matrices.

3 Systems of type III

There are 56 nonisomorphic kite systems of order 8 of type III. Next we list the solutions together with their system vector and system matrix. The edges of the tail graph are 02, 04, 07, 15, 26, 27, 37.

III.1 0137, 2415, 3520, 3640, 4572, 5607, 1762.

210004051140120 $GJJBPKJGLCJOMMBGMGMPJ$

III.2 0137, 1420, 2351, 3640, 4572, 5607, 1762

200113052040030 $KJJBPKJPMBMFJGEGMGMPJ$

III.3 0137, 1420, 3526, 3640, 4572, 5607, 6715

2001133140020 $KGJBPJHPMBGKMKEGMFMJL$

III.4 0137, 1427, 3526, 3640, 4751, 5602, 1673

100224212132100 $OFMEHNHGLBMKGNEGMKGJF$

III.5 1302, 2415, 3527, 3640, 4573, 0562, 1670

110122151041110 $JMMCPMJGKEGFNMBJJOFHJ$

III.6 0315, 1426, 3520, 3640, 4573, 5607, 1672

110122232141100 $JLKEJGHGFCNOKMBJMFMHM$

III.7 0315, 1427, 3526, 3640, 4573, 5602, 1670

110112234041100 $GKKEJJHGMBMKKNCJMOFHM$

III.8 0137, 1427, 3526, 3640, 4751, 5602, 1670

110106121132200 $OGJEGNHGLBMKGNCJMOGJM$

III.9 0137, 2415, 3526, 3640, 4572, 5602, 1670

110106121132200 $GGJBGNJGLEGKMNCGMOOHM$

III.10 0315, 1420, 2537, 3640, 4572, 5607, 1762

110105041041120 $JJKCJGGPMBMGNOEGMGMPJ$

III.11 0137, 1420, 3526, 3640, 4572, 0651, 1670

110103224121210 $KGJBKNHPMEOKMJCGKOGHL$

III.12 0137, 1426, 2351, 3640, 4572, 5602, 1670

110022161031201 $FJJBGNJJMRKFJJCGMOOHM$

III.13 0137, 1427, 3520, 3640, 4570, 0562, 6715

110021171011311 $OJJRFJHGMBJOONCPJFKJJ$

III.14 1307, 1427, 3520, 3640, 4573, 0651, 1762

110004143031201 $OMMRJKJPFRMFJJEJJFFHP$

III.15 0137, 1427, 3520, 3640, 4570, 0651, 1762

110003153011311 $OJJRKKHGMCNOOJBPKGJGJ$

III.16 0137, 1427, 3520, 3640, 4751, 0562, 1670

101114150112300 $OJJEFNHGLBMOGNDGJOGJJ$

III.17 0137, 1426, 3520, 3640, 4572, 0651, 1670

101112243121200 $FJJBKNHJMEKOMJDGKOGHL$

III.18 0315, 1427, 3520, 3640, 4573, 5607, 1762

101105132141100 $GLKEJGHGMDNOKMBJMGMGJ$

III.19 1302, 2415, 2537, 3640, 4572, 0562, 1670

101104142121210 $JJMDPMKGLEGGNKBGJOOHJ$

III.20 0315, 1427, 3526, 3640, 4570, 5602, 1673

100222224041010 $GKKEJJHGMBMKFNEPMKMHF$

III.21 1307, 1427, 2537, 3640, 4751, 5602, 1762

100205113121130 $OPMEHKPGLBNGJKEGMGGKP$

III.22 0315, 1427, 2537, 3640, 4570, 5602, 1762

100206022041030 $GJKEJGPGMBNGMKEPMGMGP$

III.23 1307, 1426, 3520, 3640, 4573, 0651, 1672

100131243031101 $FMMRJMHJFENOKJBJKFGHK$

III.24 1307, 1427, 3526, 3640, 4573, 5602, 6715

100131243031101 $OFMRHJHGMBJKKNEJMFFJK$

III.25 0137, 1426, 3520, 3640, 4570, 0651, 1672

100130253011211 $FJJRKMHJFENOOJBPKFJHK$

III.26 0137, 1427, 2351, 3640, 4570, 5602, 1762

100115041031121 $OJJRGKJGMBNFGJEPMGMGP$

III.27 0137, 1426, 3527, 3640, 4751, 5602, 1670

100123142031111 $FPJEGNHJKRKFJMBGMOGJM$

III.28 0137, 1427, 3526, 3640, 4570, 5602, 6715

100123142031111 $OGJRGJHGMBJKFNEPMFMJK$

III.29 1307, 1427, 2351, 3640, 4573, 5602, 1762

100123142031111 $OKMRHKJGMBNFJJEJMGFGP$
III.30 0315, 1420, 3527, 3640, 4573, 5607, 1762
100123142031111 $JKKEJGHPFBMFNORJMGMGJ$
III.31 0137, 1420, 3527, 3640, 4751, 0562, 1670
100122152011221 $KPJEFMHPKROFJMBGJOGJJ$
III.32 1302, 2415, 3527, 3640, 4570, 0562, 1673
100113134040021 $JMMBPKJGKEGFMMRPJKKHG$
III.33 1302, 2415, 2537, 3640, 4570, 0562, 1672
100112144020131 $JJMBPOKGKEJGMKRPJFKHP$
III.34 0137, 1426, 3520, 3640, 4572, 5607, 6715
120021160150110 $FJJBPJHJMCJOMMCGMFMJL$
III.35 1302, 2415, 3527, 3640, 4573, 5607, 1762
020013151041201 $JMMCHOJGKCJFNORJMGMGJ$
III.36 1302, 2415, 2537, 3640, 4572, 5607, 1762
011104141131210 $JJMDHOKGLCJGNOEGMGMPJ$
III.37 0137, 1426, 2351, 3640, 4570, 5602, 1672
000142150031112 $FJJRGMJJFRNFGJEPMFMHO$
III.38 0137, 1426, 3527, 3640, 4570, 5602, 6715
000141151040022 $FPJRGJHJFRJFMMEPMFMJK$
III.39 1307, 1426, 2351, 3640, 4573, 5602, 1672
000150251031102 $FKMRHMJJFRNFJJEJMFFHO$
III.40 1307, 1426, 3527, 3640, 4751, 5602, 1673
000132233031102 $FOMEHNHJKRKFJMRGMKGJF$
III.41 0137, 1420, 2351, 3640, 4570, 0562, 1672
000131152020132 $KJJRFMJPORMFGJEPJFKHP$
III.42 0137, 2415, 3526, 3640, 4570, 5602, 1672
000124132031112 $GGJRGMJGKEJKFNRPMFMHO$
III.43 1307, 1426, 2537, 3640, 4751, 5602, 1672
000124132031112 $FPMEHMGJKRNGJKRGMFGJO$
III.44 1307, 1420, 3527, 3640, 4573, 0651, 1762
000124132031112 $MOMRJKHPFEMFNGRJKGGGJ$
III.45 0137, 1420, 3526, 3640, 4570, 0651, 1672
000121235020122 $KGJRKMHPOEMKFJRPKFJHK$
III.46 0137, 2415, 3527, 3640, 4570, 5602, 1762
000115032040032 $GPJRGKJGKEJFMMRPMGMGP$
III.47 0137, 1420, 3527, 3640, 4570, 0651, 1762

000113134020132 $KPJRKKHPOEMFMGRPKGJGJ$

III.48 1307, 1420, 3526, 3640, 4573, 0651, 1672

000131234040012 $MFMRJMHPFEMKKJRJKFGHK$

III.49 4567, 2357, 0516, 0263, 3412, 0371, 2740

110111235050010 $BFHJMPMKJMMEJHKKMKCKG$

III.50 4567, 2357, 3416, 0263, 0512, 0371, 2740

100121235040011 $BFHJMPMKJMMKEHJKRKGKF$

III.51 4567, 2537, 1436, 2715, 0574, 2403, 0162

020131240151100 $CFHJJMMOLMHNJJXCEFMFGM$

III.52 4567, 2537, 1436, 1274, 2403, 0162, 0751

020011244050101 $CGJJJKMHKMMMKFJMOCRKH$

III.53 4567, 2537, 3416, 0215, 5704, 0362, 2471

030002233160100 $CGHJKMJMMMMLKJCOCMKGH$

III.54 4567, 2537, 3416, 4705, 0124, 1572, 0362

011114140151100 $CGJJMMOLMHMEGJNFGJMGD$

III.55 4567, 1237, 0536, 4725, 0143, 0261, 1570

011103224160000 $CGGJJLMMHKMKKEHMKGMDM$

III.56 4567, 1237, 0536, 0243, 0162, 2751, 1470

011112233160000 $CGHMMMJMJKMLFEJMKDKGH$

Unlike systems of type II., some of systems of type III. are not distinguished by their vectors alone: III.23 and III.24 have the same vectors, as have systems III.27, III.28, III.29, III.30, and also III.42, III.43, III.44. However, they are all distinguished by their matrices: there are no permutations of rows and columns of the matrices that send one system matrix to another.

4 Systems of type IV

There are 73 nonisomorphic kite systems of order 8 of type IV. Their list together with their vectors and system matrices follows. The tail graph is the same in each case, and its edges are 02, 16, 26, 27, 37, 47, 56.

IV.1 0316, 1427, 3520, 3462, 0547, 0673, 1756

200113133140020 $GKMFLPHMPBKMKEGGJBMJJ$

IV.2 1302, 1427, 2356, 3462, 0547, 0673, 5716

200105042031120 $MKOGJGGMPBJNGEPGJBMKJ$

IV.3 1302, 1426, 2356, 4637, 0547, 0761, 1572

200004053030051 $MKGGJOPKGRMJGBJPJBKMJ$

IV.4 0316, 1420, 3527, 3462, 0456, 0674, 1573
110122232141100 $JFMOKJHMKEFMGBNMJCLGH$
IV.5 0316, 1420, 3526, 4637, 0456, 0674, 1572
110112223131210 $JOJOKGHFKEMJPCMKNBLGH$
IV.6 1302, 1427, 2537, 3462, 0547, 0761, 1756
110112152021220 $MJOGJFPMPEKJOCJGHBKJN$
IV.7 1302, 2416, 3527, 3462, 0547, 0765, 1573
110106121132200 $JMOGGKGNGBGMOENGHCLMJ$
IV.8 1302, 1426, 2537, 3461, 0547, 0765, 1572
11010612113220 $MJKGGOGNGBMGOENGHCLMJ$
IV.9 1302, 1427, 2537, 3462, 0547, 0765, 5716
110104132121220 $MJOGGGPMPCJJOEKGHBLKN$
IV.10 1302, 1427, 2356, 3462, 0547, 0761, 1573
110104133041110 $MKOGJKGMPEMNGBJGHCKMJ$
IV.11 0316, 1420, 3526, 3647, 0456, 0673, 1572
110102234121210 $JOKOLJHGKCMKPEMJNBKGH$
IV.12 0316, 1420, 3526, 4637, 0547, 0765, 1572
110031160140111 $JOJFMGHFJCMJFRMPJBLMJ$
IV.13 1302, 2416, 3527, 3462, 0547, 0673, 1756
110014060031211 $JMOGJFGNGCPMORJGJBMJJ$
IV.14 1302, 2416, 2537, 3462, 0547, 0765, 1572
101114150112300 $JJOGGOFNGBJJOENGHDLMJ$
IV.15 0316, 1420, 2537, 3462, 0456, 0674, 1572
101105132141100 $JGMOKGGMKEMJGBNMJDLGH$
IV.16 1302, 1427, 3526, 3465, 0547, 0761, 1573
100231233041010 $MMFGJKHFPEMNFBKJHEKMJ$
IV.17 1302, 1426, 3527, 3461, 0547, 0765, 1573
100224212132100 $MMKGGKHNGBFFOENGHELMJ$
IV.18 1302, 1427, 3526, 4637, 0547, 0761, 1756
100222142021130 $MMGGJFHOPEKJFBPPJEKJN$
IV.19 13023, 1426, 3527, 4637, 0547, 0765, 5716
100205113121130 $MMGGGGHKGBPPOEKPJELKN$
IV.20 0316, 1427, 2537, 3426, 0547, 6702, 1765
100130520200141 $GGMFJPPMPRKJOEJGPBJJK$
IV.21 1302, 1427, 3526, 3465, 0547, 0673, 5716
100132151031111 $MMFGJGHFPBJNFEOJJRMKJ$

IV.22 1302, 2416, 2537, 3462, 0547, 0672, 1756
100123060011231 $JJOGPFFNGEPJORJGPBMJJ$
IV.23 1302, 2416, 2537, 3465, 0547, 0762, 1572
100122151021221 $JJFGPOFMGRJJOENJHBKMP$
IV.24 1302, 1427, 3526, 3461, 0547, 0673, 1756
100122143040021 $MMKGJFHKPBKMFEPGJRMJJ$
IV.25 0316, 1420, 2537, 3647, 0456, 0762, 1572
100105042031121 $JGKOMGGGKRMPJENJJBMGP$
IV.26 0316, 1426, 2537, 3647, 0456, 6702, 1572
100104052011231 $PGKOJGGJORMPJENJKBJGP$
IV.27 0316, 1420, 2356, 4637, 0547, 0762, 1572
100023061030122 $JOJFMGGFJRMJGRJPJBKMP$
IV.28 1302, 2416, 3526, 4637, 0547, 0765, 1572
100015041130221 $JMGGGOPKGBJJFRMPJBLMJ$
IV.29 1302, 2416, 2356, 4637, 0547, 0762, 1572
100004053010232 $JKGGPOOKGRJJGRJPJBKMP$
IV.30 0316, 1420, 3526, 3465, 0547, 0673, 1572
020131240151100 $JOMFLGHFJCMNFEMJJCMMH$
IV.31 1302, 2416, 3526, 3465, 0547, 0673, 1572
020122150051110 $JMFGJOPMGCJNFEMJJCMMH$
IV.32 0316, 1427, 3520, 3462, 0547, 0765, 1573
020112223160010 $GKMFMJHMPCMMKEKGHCLMJ$
IV.33 1302, 1426, 2356, 3461, 0547, 0673, 1572
020104132051110 $MKKGJOPNGCMMGEJGJCMMH$
IV.34 1302, 1426, 2537, 3465, 0547, 0761, 1572
020013151041200 $MJFGJOGMGRMJOCNJHCKMJ$
IV.35 03136, 1420, 3527, 3462, 0456, 0673, 1574
020012233150101 $JFMOLGHMKCKMGRMMJCKJH$
IV.36 0316, 1420, 2537, 3462, 0547, 0765, 1572
011114140151100 $JGMFMGGMJCMJOENGHDLMJ$
IV.37 1302, 2416, 2356, 3462, 0547, 0673, 1572
011104151022300 $JKOGJOONGCJNGEJGJDMMH$
IV.38 1302, 2416, 2356, 3462, 0547, 0672, 1573
010205122022220 $JKOGPKONGEGNGEJGPCMMH$
IV.39 0316, 1420, 2537, 3465, 0547, 0762, 1572
010123141041111 $JGMFMGGFJRMJOENJHCKMP$

IV.40 1302, 1426, 3527, 3465, 0547, 0761, 1573
010122233041101 $MMFGJKHMGRFKOCNJHEKMJ$
IV.41 0316, 1427, 2356, 3462, 0547, 6702, 1573
010115040041121 $GOMFJJGMPRMNGEJGPCJMG$
IV.42 0316, 1247, 3527, 3462, 0456, 6702, 1573
010115040041121 $GFMOJJPJGEMMGRNMPCJGG$
IV.43 1302, 1427, 3526, 4637, 0547, 0765, 5716
010114131121221 $MMGGGGHOPCJJFROPJELKN$
IV.44 1302, 1427, 3526, 3461, 0547, 0765, 1573
010113214150011 $MMKGGKHKPCMMFRKGHELMJ$
IV.45 1302, 1426, 3527, 4637, 0547, 0761, 1756
010113142021221 $MMGGJFHKGROPOCJPJEKJN$
IV.46 1302, 1426, 3527, 3461, 0547, 0673, 1756
010023151031202 $MMKGJFHNGCOFORJGJRMJJ$
IV.47 1302, 1426, 3527, 3465, 0547, 0672, 5716
010013143040112 $MMFGJGHMGCPKORKJJRMKJ$
IV.48 0316, 1420, 2537, 3462, 0456, 0672, 1574
010003143140112 0316, 1420, 2537, 3462, 0456, 0672, 1574
IV.49 0316, 1247, 2537, 3462, 0456, 6702, 1572
001205050022220 $GGMOJGOJGENJJENMPDJGP$
IV.50 1302, 2416, 3526, 3465, 0547, 0672, 1573
000224122041021 $JMFGPKPMGEGNFRKJGEMMH$
IV.51 1302, 2416, 3526, 4637, 0547, 0672, 1756
000223041030051 $JMGGPFPKGEPJFRPPMEMJJ$
IV.52 1302, 1426, 2356, 3461, 0547, 0672, 1573
000215113041021 $MKKGPKPNGRFMGEJGGEMMH$
IV.53 0316, 1420, 3526, 3465, 0547, 0672, 1573
000141242031102 $JOMFKJHFJRFNFRKJGEMMH$
IV.54 0316, 1420, 3526, 4637, 0547, 0672, 1756
000140151030132 $JOMFKPHFJRFJFRPPMEMJJ$
IV.55 0316, 1427, 3526, 3465, 0547, 6702, 1573
000133141031112 $GOMFJJHFPRMNFRKJGEJMG$
IV.56 0316, 1427, 3520, 3465, 0547, 0762, 1573
000122225040012 $GKMFMJHFPRMKKRKJHEKMG$
IV.57 1302, 1427, 2537, 3465, 0547, 0762, 5716
000122143020132 $MJFGPGPFPRJJOEKJHRKKM$

IV.58 0316, 1427, 3526, 4637, 0547, 6702, 1756
000121153000242 $GOJFJPHOPRKJFRPPKEJJK$
IV.59 1302, 2416, 3527, 3465, 0547, 0762, 1573
000115123031112 $JMFGPKGMGRGKORNJHEKMG$
IV.60 1302, 2416, 3527, 4637, 0547, 0762, 1756
000114042020142 $JMGGPFGKGRPPORJPJEKJM$
IV.61 1302, 1427, 2537, 3461, 0547, 0762, 1756
000113134020132 $MKJGPFPKPRKGOEJGHRKJM$
IV.62 0316, 1426, 3520, 4637, 0547, 0672, 1756
000113134020132 $PKJFKPHKGROGKRGPMEMJJ$
IV.63 1302, 1427, 2356, 3461, 0547, 0762, 1573
000105115040022 $MKKGPKGKPRMMGRJGHEKMG$
IV.64 1302, 1426, 2356, 4637, 0547, 0672, 5716
000105033020142 $MKGGPGPKGRPJGEPPMEMKJ$
IV.65 1302, 1427, 2356, 4637, 0547, 0762, 5716
000105033020142 $MKGGPGGOPRJJGRPPJEKKM$
IV.66 0316, 1420, 3527, 3647, 0456, 0762, 1573
000024142021203 $JFKOMJHGKRFOGRNJJRMGG$
IV.67 0316, 1426, 3527, 3647, 0456, 6702, 1573
000023152001313 $PFKOJJHJORFOGRNJKRJGG$
IV.68 0316, 1420, 3527, 4637, 0456, 0762, 1574
000023133040113 $JFMOMGHFKRKPGRMKJRMJG$
IV.69 0316, 1420, 3526, 3647, 0456, 0672, 1573
000022235010213 $JOKOKJHGKRFKPRFJMRKGH$
IV.70 0316, 1420, 3526, 4637, 0456, 0672, 1574
000021245010213 $JOJOKGHFKRKJPRFKMRKJH$
IV.71 1302, 1426, 2537, 3461, 0547, 0672, 1756
000016041021213 $MJKGPFGNGROGORJGGRMJJ$
IV.72 1302, 1426, 2537, 3465, 0547, 0672, 5716
000015042030123 $MJFGPGGMGRPJORKJGRMKJ$
IV.73 0316, 1426, 3527, 4637, 0456, 6702, 1574
000013134020223 $PFJOJGHKORKPGRMKKRJJG$

The systems of type IV. are distinguished by their system vectors except for the three pairs IV.7 , IV.8, and IV.40, IV.41, and IV.61, IV.62. However, each of these are distinguished by their system matrices.

Thus there are altogether $1 + 17 + 56 + 73 = 147$ nonisomorphic kite systems of order 8.

5 Embedding kite systems into bowtie systems

We will prove an analogue of the following recent important result in [2].

Theorem 5.1 *Any partial Steiner triple system of order v can be embedded in a Steiner triple system of order w provided $w \equiv 1$ or $3 \pmod 6$ and $w \geq 2v + 1$.*

We also need an analogue of the well-known Doyen-Wilson Theorem for embeddings of Steiner triple systems (cf., e.g., [9]).

Doyen-Wilson Theorem *Every Steiner triple system of order v can be embedded into a Steiner triple system of order w provided $w \geq 2v + 1$.*

In the literature, one finds Doyen-Wilson-type theorems about embedding systems other than STS, e.g. for Steiner systems $S(2, 4, v)$ [8], for kite systems [6, 7], for bull systems [5]. And although we have not been able to find in the literature explicitly an analogue of Doyen-Wilson Theorem for bowtie systems, it is actually a simple consequence of a following result in [1].

Theorem 5.2 *[1] If B is a pairwise balanced design with $M \leq 2m$, where M is the maximum block size and m is the minimum block size of B, then the block intersection graph $G(B)$ of B has a hamilton cycle.*

Theorem 5.3 *Any partial bowtie system of order v can be embedded in a bowtie system of order w provided $w \equiv 1$ or $9 \pmod{12}$, and $w \geq 2v + 1$.*

Proof. Any partial bowtie system $(V, \mathcal{B})$ of order v is embedded in an STS(w) $(W, \mathcal{C})$ (see Theorem 1). Requiring $w \equiv 1, 9 \pmod{12}$ ensures that the number of triples is even, and then $(W \setminus V, \mathcal{C} \setminus \mathcal{B})$ is a pairwise balanced design with block sizes 2 and 3, whose block intersection graph has, by Theorem 2, a hamilton cycle and thus a 1-factor. Consequently, the set of blocks $\mathcal{C} \setminus \mathcal{B}$ can be partitioned into bowties. □

Theorem 5.4 *A bowtie system of order v can be embedded in a bowtie system of order w if and only if $v, w \equiv 1$ or $9 \pmod{12}$, and $w \geq 2v + 1$.*

Proof. The proof is analogous to the previous one by using the Doyen-Wilson Theorem. □

A kite system $(V, \mathcal{B})$ is said to be embedded in a bowtie system $(W, \mathcal{C})$ if $V \subset W$ and for each kite $H \in \mathcal{B}$ there exists a bowtie $G \in \mathcal{C}$ such that H is contained in G as a subgraph.

Theorem 5.5 *Let a kite system of order v be embedded in a bowtie system of order w. If $v \equiv 0 \pmod 8$ then $w \geq 1/2(3v+18)$; if $v \equiv 1 \pmod 8$ then $w \geq 1/2(3v+15)$.*

Proof. Let a kite system $(V, \mathcal{B})$ of order v be embedded in a bowtie system $(W, \mathcal{C})$ of order w. An edge $\{x, y\}$ is called of type $\{V, W \setminus V\}$ if $x \in V$ and $y \in W \setminus V$. In order to complete a kite of $\mathcal{B}$ in a bowtie $\mathcal{C}$, two edges of type $\{V, W \setminus V\}$ are necessary. Therefore we have $\frac{|V||W \setminus V|}{2} \geq |\mathcal{B}|$, hence $\frac{v(w-v)}{2} \geq \frac{v(v-1)}{8}$, and so $w \geq \frac{(5v-1)}{4}$. The minimum value of $k \in \mathbb{Z}$ such that $w = \frac{5v-1}{4} + \sigma(v) = 12k+1$ or $w = \frac{5v-1}{4} + \sigma(v) = 12k+9$ is $k = h$ if $v = 8h$ or $v = 8h+1$. In particular if $v = 8h$ and $w \equiv 1 \pmod{12}$, $\sigma(v) = \frac{v+5}{4}$ and $w \geq \frac{5v-1}{4} + \sigma(v) = \frac{3v+2}{2}$. If $v = 8h$ and $w \equiv 9 \pmod{12}$, $\sigma(v) = \frac{v+37}{4}$ and $w \geq \frac{5v-1}{4} + \sigma(v) = \frac{3v+18}{2}$. If $v = 8h+1$ and $w \equiv 1 \pmod{12}$, $\sigma(v) = \frac{v-1}{4}$ and $w \geq \frac{5v-1}{4} + \sigma(v) = \frac{3v-1}{2}$. If $v = 8h+1$ and $w \equiv 9 \pmod{12}$, $\sigma(v) = \frac{v-1}{4} + 8$ and $w \geq \frac{5v-1}{4} + \sigma(v) = \frac{3v+15}{2}$. □

For example, the smallest order of a bowtie system into which a kite system of order 8 (or 9) can be embedded is 21. What follows are examples of such a smallest embedding.

Example 1. Embedding of a kite system of order 8 into a bowtie system of order 21.

Take as the kite system of order 8 the system $(V, \mathcal{B})$ of type IV (denoted by IV.1 in the preceding section) where $V = \{0, 1, \ldots 7\}$ and $\mathcal{B} = \{0316, 1427, 3520, 3462, 0547, 0673, 1756\}$. Let $W = \{8, 9, \ldots, 20\}$. By extending each kite in $\mathcal{B}$ to a bowtie we get the following 7 bowties: $(0,3,1,6,8), (1,4,2,7,8), (3,5,2,0,10), (3,4,6,2,9), (0,5,4,7,9), (0,6,7,3,10), (1,7,5,6,10)$.

Next we adjoin to this collection the following 28 more bowties on the set $V \cup W$:

$(11,12,7,13,20), (14,19,7,15,18), (7,16,17,2,14), (11,20,2,12,19),$
$(13,18,2,15,16), (8,20,4,10,19), (11,18,4,12,17), (13,16,4,14,15),$
$(9,20,1,10,18), (11,19,1,12,16), (13,14,1,15,17), (8,19,0,9,18),$
$(11,17,0,12,20), (13,15,0,14,16), (8,17,5,9,16), (11,15,5,12,18),$
$(13,19,5,14,20), (11,16,6,12,13), (14,18,6,15,19), (6,17,20,3,15),$
$(8,14,3,9,12), (11,13,3,16,19), (3,17,18,8,16), (9,13,8,10,11),$
$(8,12,15,9,10), (11,14,9,17,19), (12,14,10,13,17), (10,16,20,18,19).$

The result is a bowtie system of order 21 on $V \cup W$ containing the embedded kite system $(V, \mathcal{B})$ of order 8.

Example 2. Embedding of a kite system of order 9 into a bowtie system of order 21.

Here we take the cyclic kite system $(V, \mathcal{B})$ of order 9 where $V = Z_9$, $\mathcal{B} = \{1304 \bmod 9\}$. Let $W = \{9, 10, \ldots, 20\}$. Extend each kite in $\mathcal{B}$ to a bowtie by adding to it, for $i = 0, 1, \ldots, 8$, the two edges $\{i, i+9\}$ and $\{i+4 \bmod 9, i+9\}$. Adjoin to this collection of bowties the 26 more bowties on the set $V \cup W$:

$(10,20,0,11,12), (13,15,0,16,17), (0,19,18,1,16), (9,11,1,12,20),$
$(13,14,1,17,19), (9,10,2,12,13), (14,20,2,15,19), (2,17,18,3,20),$
$(9,13,3,10,11), (14,15,3,16,19), (10,12,4,11,16), (14,17,4,15,18),$
$(4,20,19,5,13), (9,15,5,11,17), (12,18,5,16,20), (9,20,6,10,18),$

$(12, 19, 6, 13, 17), (6, 16, 14, 7, 10), (9, 19, 7, 11, 15), (13, 18, 7, 17, 20),$
$(9, 16, 8, 10, 19), (11, 18, 8, 12, 14), (8, 20, 15, 10, 17), (12, 17, 9, 14, 18),$
$(10, 13, 16, 12, 15), (13, 20, 11, 14, 19).$

The resulting collection of bowties constitutes a bowtie system of order 21 on $V \cup W$ containing the embedded kite system $(V, \mathcal{B})$ of order 9.

The above two embeddings were found with the help of a computer.

The results above relate to an embedding of a specific kite system in a bowtie system. What can be said about embeddings of *any* kite system into a bowtie system?

Here we are able to make use of a recent important result [2].

Theorem 5.6 *[2] Any partial Steiner triple system of order v can be embedded in a Steiner triple system of order w provided $w \equiv 1$ or $3 \pmod 6$ and $w \geq 2v + 1$.*

This theorem enables us with little effort to prove the following:

Theorem 5.7 *Let $(V, \mathcal{B})$ be a kite system of order v whose tail graph has chromatic index c. The $(V, \mathcal{B})$ can be embedded in a bowtie system of order w for all $w \equiv 1, 9 \pmod{12}$, $w \geq 2v + 2c + 1$.*

Proof. Given a kite system $(V, \mathcal{B})$ of order v whose tail graph has chromatic index c, we can make it into a partial bowtie system $(V', \mathcal{B}')$ by adding suitable edges (in order to create triangles) and at most c new points. This partial bowtie system can be embedded in a Steiner triple system $(W, \mathcal{C})$ of order $w \geq 2v + 2c + 1$. Requiring $w \equiv 1, 9 \pmod{12}$ ensures that the number of triples is even. To complete the proof, we observe that the added triples of $(W, \mathcal{C})$ induce a PBD $(W \setminus V, \mathcal{C} \setminus \mathcal{B}')$ and then invoke Theorem 1 in a manner similar to that in the proof of Theorem 1. □

As a result of the preceding theorem, the possibility of embedding a given kite system of order v into a bowtie system of order w remains in doubt only for a small finite interval of values. In particular, Theorem 4 provides a complete solution for $v = 8, 9$.

Theorem 5.8 *The kite systems of order 8 and 9 from Examples 1, 2 can be embedded in a bowtie system of order w for all $w \equiv 1, 9 \pmod{12}$, $w \geq 21$.*

Proof. The tail graphs of bowtie systems in Examples 1 and 2 have chromatic index equal to 3. □

6 Conclusion

We have verified, with the aid of a computer, that *any* kite system of order 8 can be embedded into a bowtie system of order 21, and thus, by Theorem 2, into a bowtie system of any order $w \geq 45$. Whether a similar statement can be made in general about any kite system remains an open problem.

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(Received 26 Feb 2016)