

List strong edge coloring of some classes of graphs

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Abstract

A *strong edge coloring* of a graph is a proper edge coloring in which every color class is an induced matching. The *strong chromatic index* of a graph is the minimum number of colors needed to obtain a strong edge coloring. In an analogous way, we can define the list version of strong edge coloring and list version of strong chromatic index. In this paper we prove that if G is a graph with maximum degree at most four and maximum average degree less than 3, then the list strong chromatic index is at most $3\Delta + 1$, where Δ is the maximum degree of G . In addition, we prove that if G is a planar graph with maximum degree at least 4 and girth at least 7, then the list strong chromatic index is at most 3Δ .

1 Introduction

A *proper edge coloring* of a graph is an assignment of colors to the edges such that adjacent edges receive distinct colors. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors needed to obtain a proper edge coloring of G . We denote the minimum and maximum degrees of vertices in G by $\delta(G)$ and $\Delta(G)$, respectively. The well-known result on edge coloring is Vizing's theorem, which says that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

An edge coloring is a *strong edge coloring* if every color class is an induced matching. That is, an edge coloring is *strong* if for each edge uv , the color of uv is distinct from the colors of the edges (other than uv) incident with $N_G(u) \cup N_G(v)$, where

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$N_G(u)$ and $N_G(v)$ respectively denote the neighborhood of u and v in G . The *strong chromatic index* $\chi'_s(G)$ of a graph G is the minimum number of colors needed to obtain a strong edge coloring of G . The concept of strong edge coloring was introduced by Fouquet and Jolivet [8, 9].

For a graph with maximum degree at most two, we can easily obtain the following result.

Proposition 1. *If G is a graph with maximum degree one, then $\chi'_s(G) \leq 1$. If G is a graph with maximum degree two, then $\chi'_s(G) \leq 5$.*

In 1985, Erdős and Nešetřil constructed graphs with strong chromatic index $\frac{5}{4}\Delta(G)^2$ when $\Delta(G)$ is even, and $\frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1)$ when $\Delta(G)$ is odd. Inspired by their construction, they proposed the following strong edge coloring conjecture during a seminar in Prague.

Conjecture 1 (Erdős and Nešetřil [6]). *If G is a graph with maximum degree Δ , then*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

Andersen [1] and Horák et al. [13] independently confirmed the conjecture for $\Delta = 3$. Kostochka et al. [15] proved that the strong chromatic index of a subcubic planar multigraph without loops is at most 9. Some other classes of graphs have been investigated, such as degenerate graphs [4, 5, 18, 22, 23] and Halin graphs [3, 16, 17, 20, 21].

The *degree* of a vertex v in G , denoted by $\deg(v)$, is the number of incident edges of v in G . A vertex of degree k , at most k and at least k is called a k -vertex, k^- -vertex and k^+ -vertex, respectively. A k_t -vertex is a k -vertex adjacent to exactly t vertices of degree two. Two distinct edges $e_1 = uv$ and e_2 are *within distance two*, if e_2 is incident with at least one vertex in $N_G(u) \cup N_G(v)$. The *girth* of a graph G is the length of a shortest cycle in G ; if G has no cycle we define the girth of G to be ∞ .

The *maximum average degree* $\text{mad}(G)$ of a graph G is the largest average degree of its subgraphs, that is,

$$\text{mad}(G) = \max_{H \subseteq G} \left\{ \frac{2|E(H)|}{|V(H)|} \right\}.$$

Hocquard et al. [11, 12] studied the strong chromatic index of subcubic graphs in terms of maximum average degree. The following result was proved in [11] and it is the best result in the literature.

Theorem 1.1 (Hocquard et al. [11]). *If G is a subcubic graph, then*

- (a) $\chi'_s(G) \leq 6$ when $\text{mad}(G) < \frac{7}{3}$;
- (b) $\chi'_s(G) \leq 7$ when $\text{mad}(G) < \frac{5}{2}$;

- (c) $\chi'_s(G) \leq 8$ when $\text{mad}(G) < \frac{8}{3}$;
 (d) $\chi'_s(G) \leq 9$ when $\text{mad}(G) < \frac{20}{7}$.

In 1990, Faudree et al. [7] studied the strong edge coloring of planar graphs. They gave an upper bound on the strong chromatic index in the following theorem.

Theorem 1.2 (Faudree et al. [7]). *If G is a planar graph, then $\chi'_s(G) \leq 4\chi'(G) \leq 4\Delta(G) + 4$.*

By a similar argument, we can get the following result.

Theorem 1.3. *If G is a planar graph with girth at least 7, then $\chi'_s(G) \leq 3\chi'(G) \leq 3\Delta(G) + 3$.*

The famous Four Color Theorem is used in the proof of Theorem 1.2. Theorem 1.3 was proved by using Grötzsch's theorem [10], which says that the chromatic number of a triangle-free planar graph is at most 3. If someone wants to improve the upper bounds to $4\Delta(G)$ and $3\Delta(G)$ in Theorems 1.2 and 1.3 respectively by adding some additional conditions, then all are attributed to the $\Delta(G)$ -edge-coloring problem on planar graphs, and thus we do not address them here.

Hudák et al. [14] also proved the following two results.

Theorem 1.4 (Hudák et al. [14]). *If G is a planar graph with girth at least 6, then $\chi'_s(G) \leq 3\Delta(G) + 5$.*

Theorem 1.5 (Hudák et al. [14]). *If G is a planar graph with girth at least 7, then $\chi'_s(G) \leq 3\Delta(G)$.*

Bensmail et al. [2] and the authors (in the first version of the current paper, see <http://arxiv.org/abs/1402.5677v1>) independently improved the upper bound in Theorem 1.4 to $3\Delta(G) + 1$.

Theorem 1.6 (Bensmail et al. [2]). *If G is a planar graph with girth at least 6, then $\chi'_s(G) \leq 3\Delta(G) + 1$.*

Similar to all the other kinds of coloring parameters, we can define the list strong edge coloring and list strong chromatic index χ'_{list} . In fact, most of the results for strong edge coloring are also true for list strong edge coloring, since the proofs just indicate the numbers of forbidden colors and available colors for uncolored edges in each step.

Ma et al. [19] investigated the list strong edge coloring of subcubic graphs, and gave the following results.

Theorem 1.7 (Ma et al. [19]). *If G is a subcubic graph, then*

- (a) $\chi'_{\text{list}}(G) \leq 6$ when $\text{mad}(G) < \frac{15}{7}$;
 (b) $\chi'_{\text{list}}(G) \leq 7$ when $\text{mad}(G) < \frac{27}{11}$;

(c) $\chi'_{\text{slist}}(G) \leq 8$ when $\text{mad}(G) < \frac{13}{5}$;

(d) $\chi'_{\text{slist}}(G) \leq 9$ when $\text{mad}(G) < \frac{36}{13}$. □

Zhu et al. [24] improved the above result to the following.

Theorem 1.8 (Zhu et al. [24]). *If G is a subcubic graph, then*

(a) $\chi'_{\text{slist}}(G) \leq 7$ when $\text{mad}(G) < \frac{5}{2}$;

(b) $\chi'_{\text{slist}}(G) \leq 8$ when $\text{mad}(G) < \frac{8}{3}$;

(c) $\chi'_{\text{slist}}(G) \leq 9$ when $\text{mad}(G) < \frac{14}{5}$. □

In Section 2, we prove that if G is a graph with $\text{mad}(G) < 3$ and $\Delta(G) \leq 4$, then $\chi'_{\text{slist}}(G) \leq 3\Delta(G) + 1$. In Section 3, we prove that if G is a planar graph with maximum degree at least 4 and girth at least 7, then $\chi'_{\text{slist}}(G) \leq 3\Delta(G)$.

2 Graphs with maximum average degree less than 3 and maximum degree at most 4

In this section, we prove the following result with restriction on maximum average degree and maximum degree.

Theorem 2.1. *If G is a graph with $\text{mad}(G) < 3$ and $\Delta(G) \leq 4$, then $\chi'_{\text{slist}}(G) \leq 3\Delta(G) + 1$.*

Note that Andersen [1] and Horák et al. [13] independently proved that the strong chromatic index of a subcubic graph is at most 10. However, none of their proofs is true for the list strong chromatic index.

Corollary 1. *If G is a planar graph with girth at least 6 and $\Delta(G) \leq 4$, then $\chi'_{\text{slist}}(G) \leq 3\Delta(G) + 1$.*

Here, we do not prove Theorem 2.1 directly, but alternately prove the corresponding result for original strong chromatic index, because the following proof can be trivially extended to the list strong chromatic index and it is easy for writing.

Theorem 2.2. *If G is a graph with $\text{mad}(G) < 3$ and $\Delta(G) \leq 4$, then $\chi'_s(G) \leq 3\Delta(G) + 1$.*

Proof. Let G be a minimum counterexample to the theorem. The minimum means that every proper subgraph H of G admits a strong edge coloring with at most $3\Delta(H) + 1$ colors. By the minimality, the graph G is connected and $\delta(G) \geq 1$. By Proposition 1, the maximum degree is at least three.

Claim 1. *The minimum degree of G is at least two.*

Proof. Suppose that a 1-vertex v is adjacent to a vertex u . The graph $G - v$ has a strong edge coloring with at most $3\Delta(G) + 1$ colors. Note that the edge uv has at most $3\Delta(G)$ colored edges within distance two. Thus, we can assign an available color to uv , which is a contradiction. \square

Claim 2. *Every 2-vertex is adjacent to a 4-vertex.*

Proof. Suppose that a 2-vertex v is adjacent to two 3^- -vertices u and w . By the minimality of G , the graph $G - v$ has a strong edge coloring with at most $3\Delta(G) + 1$ colors. Both uv and wv have at most $3\Delta(G) - 1$ colored edges within distance two, thus both have at least two available colors, a contradiction. \square

Since $\text{mad}(G) < 3$, we have the following inequality:

$$\sum_{v \in V(G)} (\text{deg}(v) - 3) < 0,$$

which implies $\Delta(G) = 4$ by Claims 1 and 2.

Claim 3. *Let v_1 be a 2-vertex with $N_G(v_1) = \{v, w_1\}$. If v is adjacent to at least two 2-vertices, then w_1 is a 4-vertex.*

Proof. Suppose to the contrary that w_1 is a 3^- -vertex and v is adjacent to another 2-vertex v_2 . By the minimality of G , the graph $G - v_1$ admits a strong edge coloring with at most $3\Delta(G) + 1$ colors. The edge vv_1 has at most $2\Delta(G) + 4$ colored edges within distance two, and the edge v_1w_1 has at most $2\Delta(G) + 3$ colored edges within distance two. Hence, the edge vv_1 has at least one available color and v_1w_1 has at least two available colors. Thus, we can color vv_1 and v_1w_1 in this order, which is a contradiction. \square

Claim 4. *There is no 4_4 -vertex in G .*

Proof. Suppose that v is a 4_4 -vertex with $N_G(v) = \{v_1, v_2, v_3, v_4\}$. By the minimality of G , the graph $G - v$ admits a strong edge coloring with at most $3\Delta(G) + 1$ colors. Note that every edge vv_i has at most $\Delta(G) + 3$ colored edges within distance two, thus each edge has at least $2\Delta(G) - 2 \geq 4$ available colors, a contradiction. \square

Claim 5. *If a 2-vertex is adjacent to a 4_3 -vertex, then the other neighbor is a 4_1 -vertex.*

Proof. Let v be a 4_3 -vertex with three 2-neighbors v_1, v_2 and v_3 . Let w_1 be the other neighbor of v_1 . By contradiction and Claims 3 and 4, the vertex w_1 is a 4_2 - or 4_3 -vertex. By the minimality of G , the graph $G - v_1$ admits a strong edge coloring with at most $3\Delta(G) + 1$ colors. Now, we remove the color on edge vv_2 and denote the resulting coloring by σ . The edge v_1w_1 has at most $2\Delta(G) + 4$ colored edges within distance two, thus it has at least one available color, so we can assign a color to v_1w_1 . After v_1w_1 was colored, both vv_1 and vv_2 have at least two available colors, so we can assign colors to them, which is a contradiction. \square

Next, we use the discharging method to get a contradiction and complete the proof. We assign an initial charge $\deg(v) - 3$ to every vertex v , and then design appropriate discharging rules and redistribute charges among vertices, such that the final charge of every vertex is nonnegative, which derives a contradiction.

The Discharging Rules:

(R1) Every 4_1 -vertex sends 1 to the adjacent 2-vertex.

(R2) Every 4_2 -vertex sends $\frac{1}{2}$ to every adjacent 2-vertex.

- **Let v be a 4-vertex.** If v is a 4_0 -vertex, then the final charge is $4 - 3 = 1$. If v is a 4_1 -vertex, then the final charge is $4 - 3 - 1 = 0$. If v is a 4_2 -vertex, then the final charge is $4 - 3 - 2 \times \frac{1}{2} = 0$. If v is a 4_3 -vertex, then the final charge is $4 - 3 = 1$. By Claim 4, there is no 4_4 -vertex.
- The final charge of every 3-vertex is zero.
- **Let v be a 2-vertex.** If v is adjacent to a 4_1 -vertex, then the final charge is at least $2 - 3 + 1 = 0$.

By Claim 5, if v is adjacent to a 4_3 -vertex, then it is adjacent to a 4_1 -vertex, but this case has been treated above. So we may assume that v is not adjacent to any 4_1 - or 4_3 -vertex. By Claim 2, the 2-vertex v is adjacent to a 4-vertex. By Claim 3 and the excluded cases, the vertex v is adjacent to two 4_2 -vertices, and then the final charge is $2 - 3 + 2 \times \frac{1}{2} = 0$. \square

3 Planar graphs with girth at least 7

The following is the main result in this section.

Theorem 3.1. *Let G be a planar graph with maximum degree at most Δ , where $\Delta \geq 4$. If G has girth at least 7, then $\chi'_{\text{slis}}(G) \leq 3\Delta$.*

Similar to the previous section, we do not prove the above result directly, but alternatively prove the following corresponding result for original strong chromatic index. Note that the following result has been proved in [14] by using Grötzsch's theorem. It is well known that Grötzsch's theorem is not true for the list vertex coloring, so the proofs due to Hudák et al. [14] cannot be extended to the proof for Theorem 3.1.

Theorem 3.2. *Let G be a plane graph with maximum degree at most Δ , where $\Delta \geq 4$. If G has girth at least 7, then $\chi'_s(G) \leq 3\Delta$.*

Proof. Let G be a minimum counterexample to the theorem. The minimum means that every proper subgraph H of G has a strong edge coloring with at most 3Δ colors. By the minimality, the graph G is connected and $\delta(G) \geq 1$.

Claim 1. *Every 1-vertex v is adjacent to a 4^+ -vertex u ; if u is a 4-vertex, then it is adjacent to three Δ -vertices.*

Proof. Suppose that a 1-vertex v is adjacent to a vertex u . The graph $G - v$ admits a strong edge coloring with at most 3Δ colors. If uv has at most $3\Delta - 1$ colored edges within distance two, then we can extend the coloring to G , a contradiction. Hence, the edge uv has at least 3Δ colored edges within distance two, which implies the claim. \square

Claim 2. *Every 2-vertex is adjacent to a 4^+ -vertex.*

Proof. Suppose that a 2-vertex v is adjacent to two 3^- -vertices u and w . By the minimality of G , the graph $G - v$ admits a strong edge coloring with at most 3Δ colors. Both uv and wv have at most $2\Delta + 2 \leq 3\Delta - 2$ colored edges within distance two, thus each of them has at least two available colors, a contradiction. \square

Claim 3. *Every vertex is adjacent to at least one 3^+ -vertex.*

Proof. Suppose that v is a τ -vertex with $N_G(v) = \{v_1, v_2, \dots, v_\tau\}$ and it is not adjacent to 3^+ -vertices. By the minimality of G , the graph $G - v$ admits a strong edge coloring with at most 3Δ colors. Note that every edge vv_i has at most $\Delta + \tau - 1 \leq 2\Delta - 1$ colored edges within distance two, thus each has at least $\Delta + 1$ available colors, a contradiction. \square

Claim 4. *If a 2-vertex v is adjacent to a 4-vertex u and a 2-vertex w , then u is a 4_1 -vertex.*

Proof. Suppose that u is not a 4_1 -vertex. The graph $G - v$ admits a strong edge coloring with at most 3Δ colors. The edge uv has at most $(2\Delta + 2) + 1 = 2\Delta + 3$ colored edges within distance two, so we can assign a color to it. After the edge uv was colored, the edge wv has at most $\Delta + 4$ colored edges within distance two, so we also can assign an available color to it. \square

Claim 5. *If a 2-vertex v is adjacent to a 4-vertex u and a 3-vertex w , then u is not a 4_3 -vertex.*

Proof. Suppose to the contrary that u is a 4_3 -vertex. The graph $G - v$ admits a strong edge coloring with at most 3Δ colors. The edge wv has at most $2\Delta + 3$ colored edges within distance two, so we can assign a color to it. After wv was colored, the edge uv has at most $(\Delta + 4) + 3 = \Delta + 7 < 3\Delta$ colored edges within distance two, so we also can assign an available color to it. \square

Claim 6. *Let v_1 be a 2-vertex with $N_G(v_1) = \{v, w_1\}$. If v is a 3_2 -vertex, then w_1 is a 5^+ -vertex or 4_1 -vertex.*

Proof. By contradiction and Claim 3, suppose that w_1 is a 3^- -vertex or 4_2 -vertex or 4_3 -vertex. By the minimality of G , the graph $G - v_1$ admits a strong edge coloring with at most 3Δ colors. Let v be adjacent to another 2-vertex v_2 . Now, we remove the color on edge vv_2 and denote the resulting coloring by σ . The edge v_1w_1 has at most $1 + (2\Delta + 2) = 2\Delta + 3 \leq 3\Delta - 1$ colored edges within distance two, thus we can assign an available color to v_1w_1 . After the edge v_1w_1 was colored, the edge vv_1 has at most $(\Delta + 1) + 4 = \Delta + 5$ colored edges within distance two, and vv_2 has at most $(\Delta + 1) + \Delta = 2\Delta + 1$ colored edges within distance two. Thus, both vv_1 and vv_2 have at least three available colors, and we can extend σ to a strong edge coloring of G , which is a contradiction. \square

Claim 7. *Suppose that a k -vertex v is adjacent to exactly one 3^+ -vertex. If u is a neighbor of v and it is a 2^- -vertex, then it is a 2-vertex, and the other neighbor of u is a 4^+ -vertex.*

Proof. If u is a 1-vertex, then uv has at most $\Delta + 2(k - 2) \leq 3\Delta - 4$ edges (other than uv) within distance two, so we can first color $G - u$ and extend the coloring to G , a contradiction. So we may assume that v is adjacent to exactly $k - 1$ vertices of degree two.

Suppose that w is the other neighbor of u and it is a 3^- -vertex. The graph $G - u$ admits a strong edge coloring with at most 3Δ colors. The edge uw has at most $2\Delta + (k - 1) \leq 3\Delta - 1$ colored edges within distance two, so we can assign an available color to uw . After the edge uw was colored, the edge uv has at most $\Delta + 2(k - 2) + 3 \leq 3\Delta - 1$ colored edges within distance two, so we also can assign an available color to uv , and then we obtain a strong edge coloring of G , a contradiction. \square

Claim 8. *Let $k \geq 5$ be an integer. Suppose that a k -vertex v is adjacent to exactly two 3^+ -vertices. Then v is adjacent to at most $k - 5$ pendent vertices. Moreover, if v is adjacent to exactly $k - 5$ pendent vertices, then one of the adjacent 2-vertex is adjacent to a 4^+ - or 3_1 -vertex.*

Proof. Let ℓ be the number of adjacent 1-vertices. Suppose to the contrary that $\ell \geq k - 4 \geq 1$, and uv is a pendent edge incident with v . Note that uv has at most $2\Delta + (\ell - 1) + 2(k - 2 - \ell) = 2\Delta + 2k - \ell - 5 \leq 3\Delta$ edges (other than uv) within distance two, a contradiction.

Suppose that v is adjacent to exactly $k - 5$ pendent vertices, and three 2-vertices v_1, v_2, v_3 . Let $N_G(v_1) = \{v, w_1\}$, $N_G(v_2) = \{v, w_2\}$ and $N_G(v_3) = \{v, w_3\}$. Suppose that all the vertices w_1, w_2 and w_3 are 2- or 3_2 -vertices. By the minimality, $G - v_1$ has a strong edge coloring with at most 3Δ colors. Now, we remove the colors on edges v_2w_2 and v_3w_3 . The edge vv_1 has at most $2\Delta + (k - 3) + 2 \leq 3\Delta - 1$ colored edges within distance two, so we can assign an available color to vv_1 . After vv_1 was colored, we can assign available colors to v_1w_1, v_2w_2, v_3w_3 in this order. \square

From Euler’s formula, we have the following equality:

$$\sum_{v \in V(G)} \left(\frac{5}{2} \deg(v) - 7 \right) + \sum_{f \in F(G)} (\deg(f) - 7) = -14. \tag{1}$$

Similar to the previous section, we use the discharging method to get a contradiction. We first assign an initial charge $\frac{5}{2}\deg(v) - 7$ to every vertex v and $\deg(f) - 7$ to every face f . We design appropriate discharging rules and redistribute charges among vertices and faces, such that the final charge of every vertex and every face is nonnegative, which leads to a contradiction.

The Discharging Rules:

- (R1) Every 1-vertex receives 2 from its incident face.
 - (R2) Every 1-vertex receives $\frac{5}{2}$ from the adjacent vertex.
 - (R3) Every 4_1 -vertex sends 3 to the adjacent 2-vertex.
 - (R4) Every 4_2 -vertex sends $\frac{3}{2}$ to every adjacent 2-vertex.
 - (R5) Every 4_3 -vertex sends 1 to every adjacent 2-vertex.
 - (R6) If a 2-vertex v is adjacent to a 5^+ -vertex u and a 2-vertex, then v receives 2 from u .
 - (R7) If a 2-vertex v is adjacent to a 4_2 -vertex u and a 3_1 -vertex w , then v receives $\frac{1}{2}$ from w .
 - (R8) If a 2-vertex v is adjacent to a 5^+ -vertex u and a 3_1 -vertex w , then v receives $\frac{3}{2}$ from u and $\frac{1}{2}$ from w .
 - (R9) If a 2-vertex v is adjacent to a 5^+ -vertex u and a 3_2 -vertex w , then v receives 2 from u .
 - (R10) If a 2-vertex v is adjacent to a 5^+ -vertex u and a 4^+ -vertex w , then v receives 1 from u .
- If f is a face incident with t vertices of degree one, then the degree of f is at least $7 + 2t$, and then the final charge of f is at least $(7 + 2t) - 7 - 2t = 0$ by (R1).
 - If v is a 1-vertex, then it receives 2 from incident face and $\frac{5}{2}$ from the adjacent vertex, and then the final charge is at least $\frac{5}{2} - 7 + 2 + \frac{5}{2} = 0$ by (R1) and (R2).
 - **Let v be a 2-vertex.** By Claim 1, the vertex v is not adjacent to 1-vertices. By Claim 2, the vertex v is adjacent to a 4^+ -vertex.
 If v is adjacent to a 4_1 -vertex, then the final charge is at least $\frac{5}{2} \times 2 - 7 + 3 = 1$ by (R3). So we may assume that v is not adjacent to 4_1 -vertices.
 If v is adjacent to a 4^+ -vertex u and a 2-vertex w , then according to Claim 4 and excluded case, the vertex u is a 5^+ -vertex, and then the final charge of v is $\frac{5}{2} \times 2 - 7 + 2 = 0$ by (R6).

If v is adjacent to a 4-vertex u and a 3-vertex w , then according to Claim 5, 6 and excluded cases, the vertex u is a 4_2 -vertex and w is a 3_1 -vertex, and then the final charge is $\frac{5}{2} \times 2 - 7 + \frac{3}{2} + \frac{1}{2} = 0$ by (R4) and (R7).

If v is adjacent to a 5^+ -vertex u and a 3_1 -vertex w , then the final charge is $\frac{5}{2} \times 2 - 7 + \frac{3}{2} + \frac{1}{2} = 0$ by (R8).

If v is adjacent to a 5^+ -vertex u and a 3_2 -vertex w , then the final charge is $\frac{5}{2} \times 2 - 7 + 2 = 0$ by (R9).

Now, we assume that v is adjacent to two 4^+ -vertices. If v is adjacent to two 4_2 -vertices, then the final charge is $\frac{5}{2} \times 2 - 7 + \frac{3}{2} + \frac{3}{2} = 1$ by (R4). If v is adjacent to a 4_2 -vertex and a 4_3 -vertex, then the final charge is $\frac{5}{2} \times 2 - 7 + \frac{3}{2} + 1 = \frac{1}{2}$ by (R4) and (R5). If v is adjacent to two 4_3 -vertices, then the final charge is $\frac{5}{2} \times 2 - 7 + 1 + 1 = 0$ by (R5). If v is adjacent to a 5^+ -vertex, then the final charge is at least $\frac{5}{2} \times 2 - 7 + 1 + 1 = 0$ by (R10).

- **Let v be a 3-vertex.** By Claim 1, the vertex v is not adjacent to 1-vertices. If v is a 3_0 -vertex, then the final charge is $\frac{5}{2} \times 3 - 7 = \frac{1}{2}$. If v is a 3_1 -vertex, then the final charge is at least $\frac{5}{2} \times 3 - 7 - \frac{1}{2} = 0$ by (R8) and (R7). If v is a 3_2 -vertex, then the final charge is $\frac{5}{2} \times 3 - 7 = \frac{1}{2}$ by (R9).
- **Let v be a 4-vertex.** If v is adjacent to a 1-vertex, then according to Claim 1 and (R2), the final charge is $\frac{5}{2} \times 4 - 7 - \frac{5}{2} = \frac{1}{2}$. So we may assume that v is not adjacent to 1-vertices. If v is a 4_1 -vertex, then the final charge is $\frac{5}{2} \times 4 - 7 - 3 = 0$ by (R3). If v is a 4_2 -vertex, then the final charge is $\frac{5}{2} \times 4 - 7 - 2 \times \frac{3}{2} = 0$ by (R4). If v is a 4_3 -vertex, then the final charge is $\frac{5}{2} \times 4 - 7 - 3 \times 1 = 0$ by (R5). No 4-vertex is adjacent to four 2-vertices. If v is adjacent to four 3^+ -vertices, then the final charge is $\frac{5}{2} \times 4 - 7 = 3$.
- **Let v be a k -vertex with $k \geq 5$.** If v is adjacent to at least three 3^+ -vertices, then the final charge is at least $\frac{5}{2}k - 7 - \frac{5}{2}(k - 3) = \frac{1}{2}$. By Claim 3, we may assume that v is adjacent to one or two 3^+ -vertices. If v is adjacent to exactly one 3^+ -vertex, then v has the final charge at least $\frac{5}{2}k - 7 - (k - 1) = \frac{3}{2}k - 6 > 0$ by Claim 7 and (R10).

Assume that v is adjacent to exactly two 3^+ -vertices. Let ℓ be the number of adjacent 1-vertices. If $0 \leq \ell \leq k - 6$, then the final charge of v is at least $\frac{5}{2}k - 7 - \frac{5}{2}\ell - 2(k - 2 - \ell) = \frac{1}{2}(k - \ell) - 3 \geq 0$. By Claim 8, we have that $\ell = k - 5$, and v is adjacent to exactly three 2-vertices, one of which is adjacent to a 4^+ - or 3_1 -vertex. Thus, the final charge is at least $\frac{5}{2}k - 7 - \frac{5}{2}(k - 5) - \frac{3}{2} - 2 - 2 = 0$ by (R2), (R8), (R9) and (R10). □

Corollary 2. *If G is a planar graph with maximum degree at least 4 and girth at least 7, then $\chi'_s(G) \leq 3\Delta(G)$.*

4 Concluding remark

We could not find an example regarding the tightness of the upper bounds we have shown. So we expect that the upper bounds might be further improved.

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