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Constructive lower bounds for Ramsey numbers from linear graphs

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Abstract

Giraud (1968) demonstrated a process for constructing cyclic Ramsey graph colourings, starting from a known cyclic 'prototype' colouring, adding edges of a single new colour, and producing a larger cyclic pattern. This paper describes an extension of that construction which allows any number of new colours to be introduced simultaneously, by using two multicolour prototypes, each of which is a linear Ramsey graph. The resulting colouring is also linear, which allows the process to be applied iteratively. It is then proved that a simple formula resulting from the new construction provides improved lower bounds for many Ramsey numbers. Giraud's recursive formula is proved for all linear cases, as a corollary.

The formula resulting from the new construction is applied to produce new lower bounds for several particular Ramsey numbers, including $R_5(4) \geq 4176$, $R_4(5) \geq 3282$, $R_5(5) \geq 33495$ and $R_4(6) \geq 20202$. For some larger $R_r(3)$, the construction produces new lower bounds that improve over the construction described by Chung (1973) — including $R_{12}(3) \geq 575666$.

The paper goes on to explore the general limits, implied by the formula, for lower bounds for the Ramsey numbers $R_r(k)$. Specific lower bounds are derived in the form $\lim_{r\to\infty} R_r(k)^{1/r} \geq g_k$.

1 Introduction

This paper addresses the properties of undirected loopless graphs with edge-colourings in an arbitrary number of colours. Based on a new construction, it presents a formula leading to new lower bounds for multicolour classical Ramsey numbers.

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The construction described by Giraud [5] generates cyclic Ramsey graph colourings. Starting from a cyclic 'prototype' colouring, the construction adds one colour to a linear extension of the prototype colouring. In the same paper, Giraud derived a recursive formula from this construction, establishing lower bounds for certain Ramsey numbers. This formula is referenced in [10] and a simplified proof is given there.

The current research was motivated by the observation that in many of the cases under study, when the Giraud construction was based on a prototype graph with several colours, it produced a much larger proportion of edges in the new colour than in each of the existing colours. Intuitively, this seemed to make it unlikely that that construction would produce highly efficient colourings in those cases.

Early in the development of the proofs in this paper, it was found that they apply to general linear colourings: no condition was necessary that the prototype graphs should be cyclic.

In Section 3, a new construction is demonstrated, which allows the simultaneous addition of an arbitrary number of new colours to an extension of any linear graph. The constructed colouring derives from **two** 'prototype' linear graphs. It is established that the maximum size of any complete graph K_n in any colour contained in the extended graph is the same as the maximum size of any K_n in the original colour in the associated prototype graph; and that the resulting graph is also linear. A simple formula is then derived, leading to lower bounds for Ramsey numbers, based on these linear (or separately, cyclic) graphs. Giraud's well-known recursive formula, now based on linear graphs, is proved as a corollary.

In Section 4, some results of applying the formula to specific graphs are listed numerically, giving new lower bounds for several particular 'small' Ramsey numbers.

In Section 5, the formula is applied iteratively to generate several series of graphs which provide limiting lower bounds for 'diagonal' Ramsey graphs — which, for a fixed r, avoid including copies of K_r in all colours. These lower bounds improve significantly over previous limiting values for many $r > 3$.

Brief conclusions are presented in Section 6.

2 Notation

In this paper,

K_n denotes the general complete graph with order n.

If U is a complete graph with m vertices $\{u_0, \ldots, u_{m-1}\}$, then:

(i) a *colouring* of U is a mapping of the edges (u_i, u_j) of U into a set of colours $\{c_s \mid 1 \leq s \leq r\}$;

(ii) the *distance* between two vertices u_i, u_j, or, equivalently, the *length* of the edge (u_i, u_j) connecting them, is defined as $|j - i|$;

(iii) a colouring of U is *linear* if and only if the colour of any edge (u_i, u_j) depends only on the length of that edge; and

(iv) a colouring of U is *cyclic* if and only if (a) it is linear, and (b) the colour of any edge (u_i, u_j) is equal to the colour of every edge of length $(m - |j - i|)$.

A Ramsey graph $U(k_1, \ldots, k_r; m)$ is a complete graph of order m with a colouring such that for each s, where $1 \leq s \leq r$, there exists no complete monochromatic subgraph K_q of U in the colour c_s for any $q \geq k_s$.

The Ramsey number $R(k_1, \ldots, k_r)$ is defined as the smallest value of m for which no Ramsey graph $U(k_1, \ldots, k_r; m)$ exists. When $k_1 = k_2 = \cdots = k_r = k$, this is usually written $R_r(k)$.

3 New Construction

Let $U(k_1, \ldots, k_q; m)$ and $V(k_{q+1}, \ldots, k_{q+r}; n)$ be two linear Ramsey graphs, with no colours in common. Accordingly, let the set of colours of U be $\{c_s \mid 1 \leq s \leq q\}$ and let the set of colours of V be $\{c_{q+s} \mid 1 \leq s \leq r\}$.

Because the colourings are linear, they can be expressed as functions only of the length of the edges, so that we may write the colour of any edge of length i as $c(i)$. The set of lengths of all the edges of U consists of the integers $\{i \mid 1 \leq i \leq m - 1\}$. A linear colouring gives rise to a natural partition of that set of integers into subsets L_s containing the lengths of edges of each colour c_s. That is, for $1 \leq s \leq q$, using the colours of U:

$$L_s = \{i \mid c(i) = c_s\}.$$

and, for $1 \leq s \leq r$, we can partition the set of integers $\{i \mid 1 \leq i \leq n - 1\}$ into the following subsets, using the colours of V:

$$L_{q+s} = \{i \mid c(i) = c_{q+s}\}.$$

We define a new graph W with vertices $(w_0, \ldots, w_{M-1})$ for $M = 2mn - m - n + 1$. The set of lengths of all the edges of W consists of the integers $\{i \mid 1 \leq i \leq M - 1\}$, which can be partitioned and coloured in two stages as follows:

For $1 \leq s \leq q$, let:

$$A_s = \{i + \lambda(2m - 1) \mid i \in L_s,\ 0 \leq \lambda \leq n - 1\}. \tag{3.1}$$

Edges of W with a length $l \in A_s$ are then coloured as c_s.

For $1 \leq s \leq r$, let:

$$A_{q+s} = \{i + (\lambda - 1)(2m - 1) \mid i \in \{m, \ldots, 2m - 1\},\ \lambda \in L_{q+s}\}. \tag{3.2}$$

Edges of W with a length $l \in A_{q+s}$ are then coloured as $c_{q+\lambda}$, where λ is the integer part of $(l - m)/(2m - 1) + 1$.

The new graph W is seen to be linearly coloured.

It is observed that the subgraph of W induced by the first m vertices of the new graph is isomorphic to U. Alternatively, it may be said that they constitute the 'first copy of U' within W. This subgraph is referred to below as $\widehat{U}$.

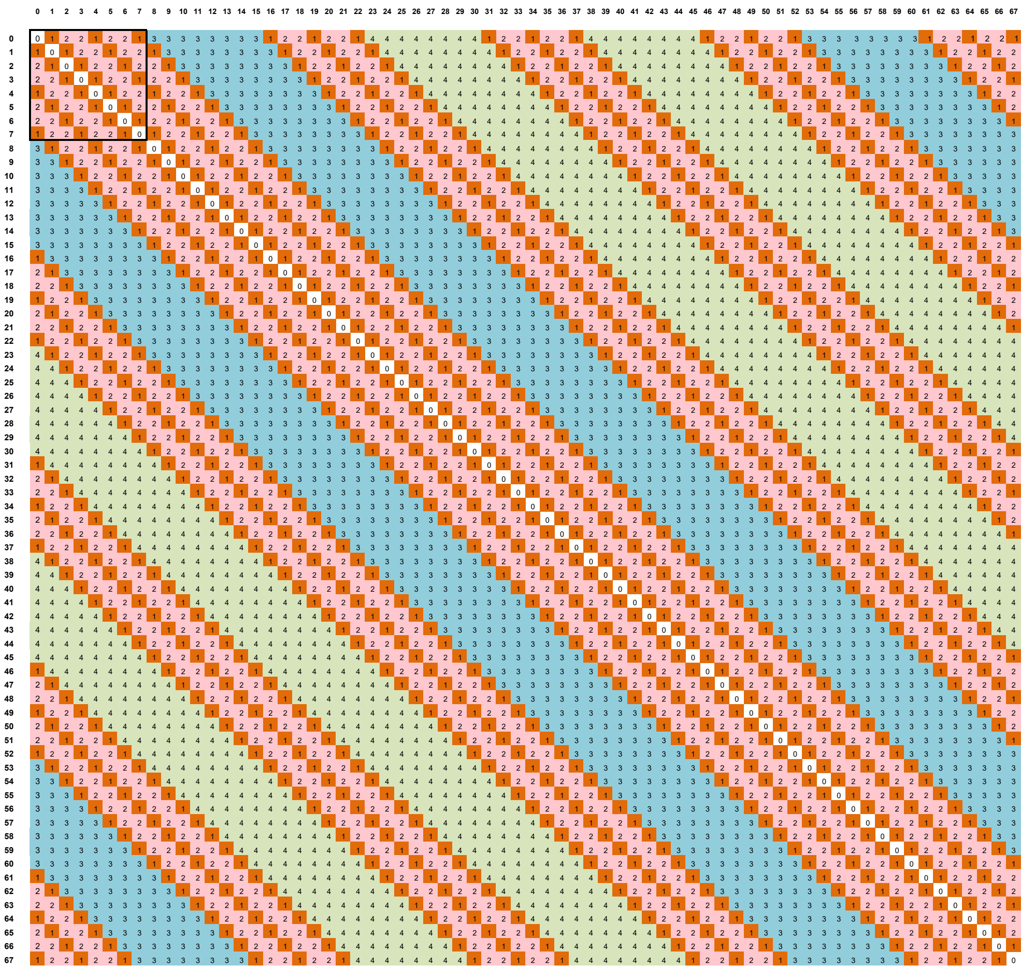

Figure 1: Example adjacency matrix.

Figure 1 illustrates the form of the adjacency matrix of W, when $U = U(3, 4; 8)$ and $V = V(3, 3; 5)$. The edges in all the colours form monochromatic diagonals (as a consequence of linearity), and these are grouped into noticeable wider solid 'bands' for the colours derived from V (i.e. c_{q+s} where $1 < s \leq r$). In the lower left triangle, there are seen to be n $(= 5)$ linearly-extended copies of U and $n - 1$ $(= 4)$ solid bands comprising edges of colours derived from V. Here $\widehat{U}$ is represented within the box at top left.

Theorem 3.1 *(Construction Theorem)*
Let $U(k_1, \ldots, k_q; m)$ and $V(k_{q+1}, \ldots, k_{q+r}; n)$ be two linear (cyclic) Ramsey graphs with distinct colours. If W is a copy of the complete graph $K_{(2mn-m-n+1)}$ and is coloured as described above, then it is a linear (cyclic) Ramsey graph of the form $W(k_1, \ldots, k_{q+r}; 2mn - m - n + 1)$.

Proof: Choose any colour c_s such that $1 \leq s \leq q$. That colour derives from U, and neither U nor $\widehat{U}$ has any subgraph isomorphic to K_j for $j \geq k_s$.

Assume that in W, there is a copy of K_j with $j = k_s$, in colour c_s, with vertices $w_{t_0}, \ldots, w_{t_{j-1}}$. The indices may be assumed to be increasing, without loss of generality. We observe that if we reduce the indices of these vertices modulo $(2m-1)$ then the resulting set of vertices induces a complete graph in colour c_s within $\widehat{U}$.

In doing this it is not necessary to assume that U is cyclic. This can be seen by considering two vertices w_a, w_b where $a < m$ and $a < b$, and $b = b' + \lambda(2m-1)$. If the edge (w_a, w_b) is of a colour derived from U, then from the construction, we must have $a < b'$. Therefore $c\,|b' - a| = c\,|b - a| = c(b - a)$.

It remains to show that this 'reduced' copy graph also has order j. In fact, this must be the case: if, when the indices are reduced modulo $(2m-1)$ as proposed, any two of those vertices were to map onto the same 'image' vertex, then they would be joined in W by an edge with a length that is a multiple of $(2m-1)$ — which can be seen (from the construction) to be of a colour derived from V, which is a contradiction. So if those vertices form part of the assumed copy of K_j, they are mapped to distinct vertices in $\widehat{U}$, and there is a copy of K_j in colour c_s within $\widehat{U}$. This gives us a second contradiction: so W contains no copy of K_j in colour c_s.

Now choose any colour c_{q+s} such that $1 \leq s \leq r$. This colour clearly derives from V.

Assume that W has a subgraph in colour c_{q+s} that is isomorphic to K_j with $j = k_{q+s}$, on vertices $(w_{t_0}, \ldots, w_{t_{j-1}})$ where $t_0 < t_1 < \cdots < t_{j-1}$. We note (from the linearity of W) that we can reduce the indices of all these vertices by subtracting t_0 without loss of generality. The first vertex is then w_0.

We can now define a partition the vertices of W into subsets according to their distance from w_0. The first subset is $S_0 = \{w_0\}$.

The remaining subsets are of the following form, for $1 \leq \lambda \leq n-1$:

$$S_\lambda = \{w_t \mid t = i + (\lambda - 1)(2m-1),\ m \leq i \leq 2m-1\}. \tag{3.3}$$

We see that any vertex of our assumed subgraph must be a member of one of these subsets, if it is connected to w_0 by an edge with colour derived from V. The maximum length of an edge joining two vertices within any of these subsets is $m-1$. Thus, from the construction, they cannot be connected by an edge with a colour derived from V. Therefore any two distinct vertices of our copy of K_j are contained in distinct subsets S_λ and S_μ.

Now we consider any distinct pair of subsets $\{S_\lambda, S_\mu\}$, with $\lambda < \mu$. From the construction, we can calculate that the distance between a vertex in S_λ and a vertex in

S_μ is within the range $(\mu - \lambda)(2m - 1) - (m - 1)$ to $(\mu - \lambda)(2m - 1) + (m - 1)$. Also from the construction, we see that although the edges connecting a vertex in S_λ to a vertex in S_μ may be of several colours, only one of those colours is derived from V. And again from the construction, we know that the single colour identified in this way is the same as the colour of the edge (v_λ, v_μ) in V.

We can hence identify the subsets S_λ with the vertices of V, in the natural way: $S_\lambda \mapsto v_\lambda$, for $0 \leq \lambda \leq n - 1$. It follows immediately from that mapping that if there is a monochromatic copy of K_j in W with each vertex contained in a distinct subset S_λ, then there must be a monochromatic copy of K_j in the same colour contained within V. This again gives a contradiction, which proves the theorem. □

The following corollary is a direct consequence of the theorem, noting that if U and V are cyclic, then W is also cyclic:

Corollary 3.2 *(Lower Bounds from Constructed Linear or Cyclic Graphs)*
If $L_1 = L(k_1, \ldots, k_q)$ and $L_2 = L(k_{q+1}, \ldots, k_{q+r})$ are the maximal orders of all the linear Ramsey graphs $U(k_1, \ldots, k_q; m)$ and $V(k_{q+1}, \ldots, k_{q+r}; n)$ respectively, then $R(k_1, \ldots, k_{q+r}) > L(k_1, \ldots, k_{q+r}) \geq 2L_1L_2 - L_1 - L_2 + 1$. The same formula applies when all the graphs are cyclic.

In a particular case, if we take $U(k_1, \ldots, k_q; m)$ as any linear Ramsey graph and V as a complete monochromatic graph of order $k_{q+1} - 1$, then:

$$|W| \;=\; 2|U|(k_{q+1} - 1) - |U| - (k_{q+1} - 1) + 1 \;=\; |U|(2k_{q+1} - 3) - k_{q+1} + 2.$$

There are clearly no copies of K_{q+1} in W, which leads directly to:

Corollary 3.3 *(Giraud's Formula for Linear Graphs)*
If $L_q = L(k_1, \ldots, k_q)$ and $L_{q+1} = L(k_1, \ldots, k_q, k_{q+1})$ are the maximal orders of the related linear Ramsey graphs, then $L_{q+1} \geq L_q(2k_{q+1} - 3) - k_{q+1} + 2$.

4 Some New Lower Bounds

Table 1 shows the highest orders of linear Ramsey graphs known to the author (in black), along with the orders of some new linear Ramsey graphs derived from the new construction above (in larger red **bold** font). All the red numbers are believed to represent Ramsey graphs with orders larger than previously published. The grey area indicates that the orders resulting from the new construction are exceeded by other published results for graphs that are not known to be linear. Numbers above 5 million have been excluded, but can be calculated easily from the formula.

A few of the smaller 'known' cyclic and linear graphs in Table 1 were derived by the current author. The majority were sourced, at least conceptually, from papers mentioned in the bibliography or Radziszowski's Dynamic Survey [9].

r \ m	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8
2	5	17	41	101	113	281	373
3	14	127	414	1069	1778		
4	45	633	**3281**	**20201**		**157361**	**277513**
5	161	**4175**	**33494**	**214769**		**2045687**	**4162688**
6	537	**32005**	**341965**	**4060301**			
7	1681	**160023**	**2712974**				

Table 1: Highest orders of linear Ramsey graphs known to the author (black) and orders of constructed graphs (red **bold**), avoiding K_m in r colours.

Almost all the graphs for $r = 2$ are Paley graphs, as described in [8], although notably the cyclic graph of order 101 avoiding K_6 was identified much earlier by Kalbfleisch [7]. Another exception is the cyclic graph of order 113 avoiding K_7, which is mentioned in [6].

The linear graphs avoiding K_3 for $r = 3$ and 4 were generated by basic computer search. The cyclic graph in five colours avoiding K_3 is described in [3]. The graphs avoiding K_3 in six colours (non-cyclic) and seven colours (cyclic) are described in [4].

The cyclic graphs in three colours avoiding K_4 and K_6 are described by means of 'cyclotomic' associations in [8]. The non-cyclic linear graph of order 414 in three colours avoiding K_5 is described in [10]. The cyclic graph in four colours avoiding K_4 is derived in the same paper, using the Giraud construction. The cyclic graph in three colours avoiding K_7 is a product graph derived by the author.

A number of new lower bounds including $R_5(4) \geq 4176$, $R_4(5) \geq 3282$, $R_5(5) \geq 33495$ and $R_4(6) \geq 20202$ can be deduced directly from Table 1.

In the same way, we can deduce that $R_{10}(3) \geq 51522$, $R_{11}(3) \geq 172218$, and $R_{12}(3) \geq 575666$. These limits are an improvement over those derived from the ingenious construction described by Chung [2], when both calculations are based on the data in Table 1. Chung's construction nevertheless gives better values for $R_8(3)$ and $R_9(3)$.

We note that the new construction can be applied in a variety of ways to generate Ramsey graphs with desired properties. In constructing the new graphs it proved necessary to choose components in such a way as to maximise the order, by partitioning the k_s appropriately between U and V.

If we write the construction in the form $U \otimes V = W$, then since the formula for $|W|$ is symmetrical in m and n, we have $|W| = |U \otimes V| = |V \otimes U|$. Hence the operations can be applied in any sequence, producing a graph of the same order in each case, so that $|(W \otimes U) \otimes V| = |U \otimes (V \otimes W)|$ etc. However, there is no implication that the resulting graphs are necessarily isomorphic.

The formula in the construction for the number of vertices of W is $M = 2mn - m - n + 1$. It follows that $(2M - 1) = (2m - 1)(2n - 1)$. From this, it is easy also to deduce that:

$$2\,|U \overbrace{\otimes V \otimes V \cdots \otimes V \otimes V}^{p\ times}| - 1 \;=\; (2m-1)(2n-1)^p. \tag{4.1}$$

This observation leads naturally to the ideas of the next section.

5 New Limiting Lower Bounds

It is shown above that the new construction can be applied repeatedly without limit, while preserving linearity. This property gives rise to new limiting lower bounds for Ramsey graphs in r colours where all the k_s are constant and equal to k. The relevant Ramsey numbers are denoted by $R_r(k)$.

After adjusting terminology, Giraud [5] gave the following results for $r \geq 2$:

$$R_r(4) \geq 5^{r-2}.(16.5) + 1.5\,;$$
$$R_r(5) \geq 7^{r-2}.(36.5) + 1.5\,;$$
$$R_r(6) \geq 9^{r-2}.(100.5) + 1.5\,.$$

More generally it has been established that for $k \geq 3$ there is a constant m_k such that $R_r(k) \geq m_k(2k-3)^{r-2}$. It is simple to deduce that:

$$\lim_{r\to\infty} R_r(4)^{1/r} \geq 5\,;$$
$$\lim_{r\to\infty} R_r(5)^{1/r} \geq 7\,;$$
$$\lim_{r\to\infty} R_r(6)^{1/r} \geq 9\,.$$

Improved limits in similar form can be derived from repeated application of the new construction in the manner suggested by equation (4.1) above.

Note first that from Theorem 3.1 if $1 \leq n \leq m$, we have:

$$|W|/|U| = (2mn - m - n + 1)/m = 2n - 1 - \delta \tag{5.1}$$

where $0 \leq \delta < 1$.

Suppose we have available a finite sequence of linear Ramsey graphs, $U_1(k), \ldots,$ $U_l(k)$, where each $U_t(k)$ is coloured with t colours, and for each graph all the k_r are equal to k. Such a sequence always exists, since it can be produced by applying the construction above. We select the value of t, which we call t_0, which maximises $g_k = (2|U_t(k)| - 1)^{1/t}$.

We can then construct $U_{s+t_0}(k)$, for each of $s = 1, \ldots t_0$, using the new construction, taking V in the construction equal to $U_{t_0}(k)$ in all cases. By repeatedly applying $U_{t_0}(k)$ as V, we can produce an infinite sequence of graphs $U_{s+jt_0}(k)$ of increasing size, for $s = 1, \ldots, t_0$ and $j = 1, 2, 3 \ldots$. The orders of these graphs each provide a lower bound for the corresponding Ramsey numbers, for all positive integers.

The ratios of the orders of successive graphs within each of these sequences are given by Equation (5.1) above. The orders of the graphs increase rapidly with increasing j. As $j \to \infty$, the deltas approach zero and the ratios converge to g_k. It follows directly that $\lim_{r\to\infty} R_r(k)^{1/r} \geq g_k$.

Calculations based on the data in Table 1 are shown in Table 2.

r \ m	3	4	5	6	7	8	9
1	3.000	5.000	7.000	9.000	11.000	13.000	15.000
2	3.000	5.745	9.000	**14.1774**	15.000	**23.6854**	**27.2947**
3	3.000	**6.3247**	**9.3865**	12.881	**15.2621**		
4	3.071						
5	3.172						
6	**3.1996**						
7	3.190						

Table 2: Factors g_k calculated from the data in Table 1.

The figures that determine maximal g_k are shown in red **bold**. These results indicate that:

$$\lim_{r\to\infty} R_r(3)^{1/r} \geq 3.199\ldots$$

$$\lim_{r\to\infty} R_r(4)^{1/r} \geq 6.324\ldots$$

$$\lim_{r\to\infty} R_r(5)^{1/r} \geq 9.386\ldots$$

$$\lim_{r\to\infty} R_r(6)^{1/r} \geq 14.177\ldots$$

$$\lim_{r\to\infty} R_r(7)^{1/r} \geq 15.262\ldots$$
$$\lim_{r\to\infty} R_r(8)^{1/r} \geq 23.685\ldots$$
$$\lim_{r\to\infty} R_r(9)^{1/r} \geq 27.294\ldots$$

…and so on.

It is noted that [1] provides a slightly stronger limit in one special case, as follows:

$$\lim_{r\to\infty} R_r(5)^{1/r} \geq (89)^{1/2} = 9.433\ldots.$$

For other values of k greater than 3, these limits constitute a significant improvement over the formula identified by Giraud [5] and Abbott and Hanson [1], and referred to as formula 6.2(c) in Radziszowski's Dynamic Survey [9].

The noticeably 'low' results for $R_r(7)^{1/r}$ and $R_r(9)^{1/r}$ reflect the relatively low orders of the best available linear graphs (1778 and 373 respectively). This may indicate a useful area for further work. Using graphs mentioned in Table 1, it is also possible to construct a product graph in 5 colours avoiding K_9 with order 5392534 in a manner derived from [1], although this does not improve the limit shown above.

6 Conclusions

It is interesting to note that the author has constructed a $(5, 5, 5, 5; 3281)$ colouring as described above, using two identical cyclic 2-colourings of K_{41} as prototypes, which contains an equal number of edges and an equal number of copies of K_3 and K_4 in all four colours. This elegant and highly symmetrical result is obviously very much in line with the original motivation of the research.

However, the results quoted or derived in this paper seem to indicate a somewhat richer scope for improving the lower bounds of classical Ramsey numbers by expanding our knowledge of linear graph colourings that are non-cyclic. The power of modern computers seems to give greater scope for the exploration of less symmetrical graph colourings than was practical previously.

The new lower bounds that result from the new construction are not expected to be restricted to those listed here, since many more prototype graphs might have been considered.

Equally, one might expect the limits $\lim_{r\to\infty} R_r(k)^{1/r}$ shown here to be improved in the near future, since each one rests on the existence of a particular linear graph colouring.

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