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Total Roman domination number of trees

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Abstract

A *total Roman dominating function* on a graph G is a function $f : V(G) \to \{0,1,2\}$ satisfying the following conditions: (i) every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$, and (ii) the subgraph of G induced by the set of all vertices of positive weight has no isolated vertices. The weight of a total Roman dominating function f is the value $f(V(G)) = \Sigma_{u\in V(G)} f(u)$. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a total Roman dominating function of G. In [Ahangar, Henning, Samodivkin and Yero, *Appl. Anal. Discrete Math.* **10** (2016), 501–517], it was recently shown that for any graph G without isolated vertices, $\gamma_{tR}(G) \le 2\gamma_t(G)$ where $\gamma_t(G)$ is the total domination number of G, and they posed the problem of characterizing the graphs G with $\gamma_{tR}(G) = 2\gamma_t(G)$. In this paper we provide a constructive characterization of trees T with $\gamma_{tR}(T) = 2\gamma_t(T)$.

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1 Introduction

Throughout this paper, G is a simple graph with no isolated vertices, with vertex set $V(G)$ and edge set $E(G)$ (briefly, V, E). The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$. A *leaf* of G is a vertex with degree one in G, a *support vertex* is a vertex adjacent to a leaf, a *strong support vertex* is a support vertex adjacent to at least two leaves, and an *end support vertex* is a support vertex all of whose neighbors with the exception of at most one are leaves, and an *end strong support vertex* is a strong support vertex all of whose neighbors with the exception of at most one are leaves. For every vertex $v \in V(G)$, the set of all leaves adjacent to v is denoted by L_v. The *double star* $DS_{q,p}$, where $q \geq p \geq 1$, is the graph consisting of the union of two stars $K_{1,q}$ and $K_{1,p}$ together with an edge joining their centers. A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider*. We denote by P_n the path on n vertices. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is the greatest distance between two vertices of G. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v; moreover, $D(v)$ denotes the set of descendants of v, and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

A subset S of vertices of G is a *total dominating set* if $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. A total dominating set with cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. The total domination number was introduced by Cockayne, Dawes and Hedetniemi [9] and is now well-studied in graph theory. The literature on this subject has been surveyed and detailed in the book by Henning and Yeo [15].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF is the value $\omega(f) = f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. in [10] and was inspired by the work of ReVelle and Rosing [17] and Stewart [18]. It is worth mentioning that since 2004, a hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [14]; Roman $\{2\}$-domination [8]; maximal Roman domination [1]; mixed Roman domination [2]; double Roman domination [6]; and recently, total Roman domination was introduced by Liu and Chang [16].

For a Roman dominating function f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$ (or

$f = (V_0^f, V_1^f, V_2^f)$ to refer to f). We note that $\omega(f) = |V_1| + 2|V_2|$.

A *total Roman dominating function* of a graph G with no isolated vertex, abbreviated TRDF, is a Roman dominating function f on G with the additional property that the subgraph of G induced by the set of all vertices of positive weight under f has no isolated vertex. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a TRDF on G. A TRDF with minimum weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$-function. The concept of total Roman domination in graphs was introduced by Liu and Chang [16] and has been studied in [3, 4, 5]. The authors in [3] observed that for any graph G with no isolated vertex,

$$\gamma_{tR}(G) \leq 2\gamma_t(G), \tag{1}$$

and they posed the following problem.

Problem: Characterize the graphs G satisfying $\gamma_{tR}(G) = 2\gamma_t(G)$.

A graph G for which $\gamma_{tR}(G) = 2\gamma_t(G)$ is defined in [3] to be a *total Roman graph*. The authors in [3] presented the following trivial necessary and sufficient condition for a graph to be a total Roman graph.

Proposition A. *Let G be a graph with no isolated vertices. Then G is a total Roman graph if and only if there exists a $\gamma_{tR}(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$.*

Finding a nontrivial necessary and sufficient condition for a graph to be a total Roman graph, or characterizing the total Roman graphs, remains an open problem. Let T_1 be a tree obtained from a star $K_{1,r}$ ($r \geq 2$) by adding at least two pendant edges at every vertex of the star, and let T_2 be a tree obtained from a star $K_{1,r}$ ($r \geq 2$) by adding at least two pendant edges at every vertex of the star except its center. Clearly, T_1 is a total Roman graph and T_2 is not a total Roman graph, while both of T_1, T_2 have a unique γ_{tR}-function. Thus, characterizing the total Roman graphs G, even when G has a unique γ_{tR}-function, is not easy.

In this paper, we provide a constructive characterization of trees T with $\gamma_{tR}(T) = 2\gamma_t(T)$ which settles the above problem for trees.

We make use of the following results in this paper.

Observation 1. *If T is a star of order at least two, then $\gamma_{tR}(T) < 2\gamma_t(T)$.*

Observation 2. *Let v be a strong support vertex in a graph G. Then there exists a $\gamma_{tR}(G)$-function f such that $f(v) = 2$.*

Proof. Let v be a strong support vertex and v_1, v_2 be leaves adjacent to v. Assume that f is a $\gamma_{tR}(G)$-function. To totally Roman dominate v_1 we must have $f(v) \geq 1$. If $f(v) = 2$, then we are done. Let $f(v) = 1$. Then to Roman dominate v_1, v_2 we must have $f(v_1) = f(v_2) = 1$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 2, g(v_1) = 1$, $g(v_2) = 0$ and $g(x) = f(x)$ otherwise, is a $\gamma_{tR}(G)$-function with the desired property. $\square$

Observation 3. *Let G be a connected graph different from a star, let v be an end strong support vertex in G, and let w be the neighbor of v which is not a leaf. Then there exists a $\gamma_{tR}(G)$-function f such that $f(v) = 2$ and $f(w) = 1$.*

Proof. Since v is a strong support vertex, we deduce from Observation 2 that there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that $f(v) = 2$. Since the induced subgraph $G[V_1 \cup V_2]$ has no isolated vertices, we have $(V_1 \cup V_2) \cap N(v) \neq \emptyset$. If $w \in (V_1 \cup V_2) \cap N(v)$, then we are done. Assume that $w \notin (V_1 \cup V_2) \cap N(v)$. Then $(V_1 \cup V_2) \cap L_v \neq \emptyset$. Let $z \in (V_1 \cup V_2) \cap L_v$. Clearly $z \in V_1$ and the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(z) = 0, g(w) = 1$ and $g(x) = f(x)$ otherwise, is a $\gamma_{tR}(G)$-function with the desired property. □

Observation 4. *If u_1, u_2 are two adjacent support vertices in a graph G, then there exists a $\gamma_{tR}(G)$-function f such that $f(u_1) = f(u_2) = 2$.*

Proof. Let u_1, u_2 be two adjacent support vertices and let v_i be a leaf adjacent to u_i for $i = 1, 2$. Assume that f is a $\gamma_{tR}(G)$-function. As above, we have $f(u_i)+f(v_i) \geq 2$ for $i = 1, 2$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(u_1) = g(u_2) = 2, g(v_1) = g(v_2) = 0$ and $g(x) = f(x)$ otherwise, is a $\gamma_{tR}(G)$-function with the desired property. □

Observation 5. *Let H be a subgraph of a graph G such that G and H have no isolated vertices. If $\gamma_{tR}(H) = 2\gamma_t(H)$, $\gamma_t(G) \leq \gamma_t(H) + s$ and $\gamma_{tR}(G) \geq \gamma_{tR}(H) + 2s$ for some non-negative integer s, then $\gamma_{tR}(G) = 2\gamma_t(G)$.*

Proof. Since $\gamma_{tR}(G) \leq 2\gamma_t(G)$, we deduce from the assumptions that

$$\gamma_{tR}(G) \geq \gamma_{tR}(H) + 2s = 2\gamma_t(H) + 2s \geq 2\gamma_t(G)$$

and this leads to the result. □

Observation 6. *Let H be a subgraph of a graph G such that G and H have no isolated vertices. If $\gamma_{tR}(G) = 2\gamma_t(G)$, $\gamma_t(G) \geq \gamma_t(H) + s$ and $\gamma_{tR}(G) \leq \gamma_{tR}(H) + 2s$ for some non-negative integer s, then $\gamma_{tR}(H) = 2\gamma_t(H)$.*

Proof. By the assumptions and the fact $\gamma_{tR}(H) \leq 2\gamma_t(H)$, we have

$$\gamma_{tR}(G) \leq \gamma_{tR}(H) + 2s \leq 2\gamma_t(H) + 2s \leq 2\gamma_t(G) = \gamma_{tR}(G)$$

and this leads to the result. □

2 A characterization of trees T with $\gamma_{tR}(T) = 2\gamma_t(T)$

In this section, we give a constructive characterization of all trees T satisfying $\gamma_{tR}(T) = 2\gamma_t(T)$. We start with three definitions.

Definition 1. Let v be a vertex of a tree T. A function $f : V(T) \to \{0, 1, 2\}$ is said to be an *almost total Roman dominating function* (almost TRDF) with respect to v, if the following two conditions are fulfilled: (i) every vertex $x \in V(T) - \{v\}$ for which $f(x) = 0$ is adjacent to at least one vertex $y \in V(T)$ for which $f(y) = 2$ and (ii) every vertex $x \in V(T) - \{v\}$ for which $f(x) \geq 1$ is adjacent to at least one vertex $y \in V(T)$ for which $f(y) \geq 1$. Let

$$\gamma_{tR}(T, v) = \min\{\omega(f) \mid f \text{ is an almost TRDF with respect to } v\}.$$

Definition 2. Let v be a vertex of a tree T. A *nearly total Roman dominating function* (nearly TRDF) with respect to v, is an almost total Roman dominating function f with an additional property that $f(v) \geq 1$ or $f(v) + f(u) \geq 2$ for some $u \in N(v)$. Let

$$\gamma_{tR}(T; v) = \min\{\omega(f) \mid f \text{ is a nearly TRDF with respect to } v\}.$$

Since any total Roman dominating function on T is an almost TRDF and a nearly TRDF with respect to each vertex of T, $\gamma_{tR}(T, v)$ and $\gamma_{tR}(T; v)$ are well defined and $\gamma_{tR}(T, v) \leq \gamma_{tR}(T)$ and $\gamma_{tR}(T; v) \leq \gamma_{tR}(T)$ for each $v \in V(T)$. Now let

$$W_T^1 = \{v \in V(T) | \gamma_{tR}(T, v) = \gamma_{tR}(T)\}$$

and

$$W_T^2 = \{v \in V(T) | \gamma_{tR}(T; v) = \gamma_{tR}(T)\}.$$

Definition 3. For a tree T and each vertex $v \in V(T)$, we say v has property P in T if for any $\gamma_{tR}(T)$-function f we have $f(v) \neq 2$. Define

$$W_T^3 = \{v | v \text{ has property } P \text{ in } T\}.$$

In order to presenting our constructive characterization, we define a family of trees as follows. Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees for some $k \geq 1$, where T_1 is P_4 and $T = T_k$. If $k \geq 2$, T_{i+1} can be obtained from T_i by one of the following operations.

Operation $\mathcal{O}_1$: If $x \in V(T_i)$ is a support vertex and there is a $\gamma_{tR}(T)$-function f with $f(x) = 2$, then $\mathcal{O}_1$ adds a vertex y and an edge xy to obtain T_{i+1}.

Operation $\mathcal{O}_2$: If $x \in V(T_i)$ has degree at least two and x is adjacent to an end strong support vertex, then $\mathcal{O}_2$ adds a path yz and joins x to y to obtain T_{i+1}.

Operation $\mathcal{O}_3$: If $x \in V(T_i)$ is a support vertex and x is at distance 2 from some leaves, then $\mathcal{O}_3$ adds a path yz and joins x to y to obtain T_{i+1}.

Operation $\mathcal{O}_4$: If $x \in W_{T_i}^1$ and x is at distance 1 or 2 from a support vertex, then $\mathcal{O}_4$ adds a path P_4 and joins x to a support vertex of it to obtain T_{i+1}.

Operation $\mathcal{O}_5$: If $x \in W_{T_i}^2 \cap W_{T_i}^3$, then $\mathcal{O}_5$ adds a double star $DS_{q,1}$ $(q = 1, 2)$ and joins x to the leaf adjacent to the support vertex of degree 2 in $DS_{q,1}$ to obtain T_{i+1}.

Operation $\mathcal{O}_6$: If $x \in W^2_{T_i} \cap W^3_{T_i}$, then $\mathcal{O}_6$ adds the graph F_t (see Figure 1) and the edge xz to obtain T_{i+1}.

Operation $\mathcal{O}_7$: If $x \in V(T_i)$, then $\mathcal{O}_7$ adds a double star $DS_{2,1}$ and joins x to a leaf adjacent to the support vertex of degree 3 to obtain T_{i+1}.

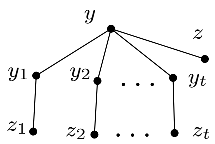

Figure 1: The graph F_t used in Operation $\mathcal{O}_6$

The proof of the first lemma is trivial and is therefore omitted.

Lemma 2.1. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$.*

Since $\gamma_{tR}(DS_{q,p}) = 2\gamma_t(DS_{q,p})$ and $DS_{q,p}$ $(q \geq 2)$ is obtained from P_4 only by Operation $\mathcal{O}_1$, it follows that this operation is necessary to construct the family $\mathcal{T}$.

Lemma 2.2. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$.*

Proof. Let $w \in V(T_i)$ be an end strong support vertex adjacent to x and let the Operation $\mathcal{O}_2$ add a path yz and join x to y. Clearly, any total dominating set of T_i containing no leaf can be extended to a total dominating set of T_{i+1} by adding y. So $\gamma_t(T_{i+1}) \leq \gamma_t(T_i) + 1$.

Now let f be a $\gamma_{tR}(T_{i+1})$-function such that $f(x)$ is as large as possible. Clearly, $f(y) \geq 1$ and $f(y) + f(z) \geq 2$. By Observation 3, we may assume that $f(w) = 2$ and $f(x) \geq 1$. Thus the function f, restricted to T_i, is a total Roman dominating function of T_i of weight $\gamma_{tR}(T_{i+1}) - 2$ and hence

$$\gamma_{tR}(T_{i+1}) = \omega(f) \geq 2 + \omega(f|_{T_i}) \geq 2 + \gamma_{tR}(T_i).$$

It follows from Observation 5 that $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$. □

Lemma 2.3. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$.*

Proof. Let $\mathcal{O}_3$ add a path yz and the edge xy. Since x is a support vertex, adding y to any total dominating set of T_i yields a total dominating set for T_{i+1} and this implies that $\gamma_t(T_{i+1}) \leq \gamma_t(T_i) + 1$.

Now let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(T_{i+1})$-function. Obviously $f(y) + f(z) \geq 2$ and $x, y, w \in V_1 \cup V_2$ where $w \in N_{T_i}(x)$ is a support vertex (note that x is at distance 2 from some leaves and so x is adjacent to a support vertex). Therefore the function f, restricted to T_i, is a total Roman dominating function of T_i and so

$$\gamma_{tR}(T_{i+1}) = \omega(f) \geq 2 + \omega(f|_{T_i}) \geq 2 + \gamma_{tR}(T_i).$$

Now the result follows by Observation 5. □

Since $\gamma_{tR}(F_t) = 2\gamma_t(F_t)$ and F_t $(t \geq 2)$ is obtained from P_4 only by using Operation $\mathcal{O}_3$, $t-1$ times, we conclude that the Operation $\mathcal{O}_3$ is necessary to construct the family $\mathcal{T}$.

Lemma 2.4. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_4$, then $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$.*

Proof. Let $\mathcal{O}_4$ add a path P_4: $y_1y_2y_3y_4$ and join x to y_3. Clearly, any total dominating set of T_i can be extended to a total dominating set of T_{i+1} by adding y_2, y_3, yielding $\gamma_t(T_{i+1}) \leq \gamma_t(T_i) + 2$.

Assume now that $f = (V_0, V_1, V_2)$ is a $\gamma_{tR}(T_{i+1})$-function. By Observation 4, we may assume that $y_2, y_3 \in V_2$. Then the function f, restricted to T_i, is an almost total Roman dominating function of T_i and since $x \in W^1_{T_i}$ we have $\omega(f|_{T_i}) \geq \gamma_{tR}(T_i)$. Hence

$$\gamma_{tR}(T_{i+1}) = \omega(f) \geq 4 + \omega(f|_{T_i}) \geq 4 + \gamma_{tR}(T_i).$$

It follows from Observation 5 that $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$. □

Let T be a tree obtained from three copies of P_4 by adding a new vertex and joining it to exactly one support vertex of each copy of P_4. Clearly, $\gamma_{tR}(T) = 2\gamma_t(T)$ and T is obtained from P_4 by applying Operations $\mathcal{O}_7$ and $\mathcal{O}_4$ respectively. On the other hand, T cannot be obtained by other operations, and so Operation $\mathcal{O}_4$ is necessary to construct the family $\mathcal{T}$.

Lemma 2.5. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_5$, then $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$.*

Proof. Let $\mathcal{O}_5$ add a double star $DS_{q,1}$ with central vertices a, b where $\deg(a) = 2$ and join x to the leaf c adjacent to a. By adding a, b to any total dominating set of T_i we obtain a total dominating set of T_{i+1}, implying that $\gamma_t(T_{i+1}) \leq \gamma_t(T_i) + 2$.

Now let f be a $\gamma_{tR}(T_{i+1})$-function such that $f(b)$ is as large as possible. Then clearly $f(b) = 2$, $f(a) + f(b) \geq 3$ and $f(a) + f(b) + f(c) \geq 4$. If $f(c) \leq 1$, then the function f, restricted to T_i is a nearly total Roman dominating function of T_i, and if $f(c) = 2$, then the function $g : V(T_i) \to \{0, 1, 2\}$ defined by $g(x) = 1$ and $g(u) = f(u)$ for $u \in V(T_i) - \{x\}$, is a nearly total Roman dominating function of T_i. Since $x \in W^2_{T_i}$, we have $\omega(f|_{T_i}) \geq \gamma_{tR}(T_i)$. Thus

$$\gamma_{tR}(T_{i+1}) = \omega(f) \geq 4 + \omega(f|_{T_i}) \geq 4 + \gamma_{tR}(T_i)$$

and the result follows by Observation 5. □

Since $\gamma_{tR}(P_8) = 2\gamma_t(P_8)$ and P_8 is obtained from P_4 only by applying Operation $\mathcal{O}_5$, we deduce that the operation $\mathcal{O}_5$ is necessary to construct the family $\mathcal{T}$.

Lemma 2.6. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_6$, then $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$.*

Proof. Clearly, any total dominating set of T_i can be extended to a total dominating set of T_{i+1} by adding $N[y] - \{z\}$ yielding $\gamma_t(T_{i+1}) \le \gamma_t(T_i) + \deg(y)$.

Let f be a $\gamma_{tR}(T_{i+1})$-function. To totally Roman dominate z_i, we must have $f(y_i) + f(z_i) \ge 2$ for $i = 1, \ldots, t$. If $f(y) = 2$ and $f(z) = 0$, then the function f restricted to T_i is a nearly total Roman dominating function of T_i and since $x \in W^2_{T_i}$ we obtain $\gamma_{tR}(T_{i+1}) = \omega(f) \ge 2\deg(y) + \omega(f|_{T_i}) \ge 2\deg(y) + \gamma_{tR}(T_i)$. If $f(y) = 2$ and $f(z) \ge 1$, then the function $g : V(T_i) \to \{0, 1, 2\}$ defined by $g(x) = \min\{f(x) + 1, 2\}$ and $g(u) = f(u)$ for $u \in V(T_i) - \{x\}$ is a nearly total Roman dominating function of T_i and as above we have $\gamma_{tR}(T_{i+1}) \ge 2\deg(y) + \gamma_{tR}(T_i)$. Let $f(y) = 1$. If $f(z) \ge 1$, then as above we have $\gamma_{tR}(T_{i+1}) \ge 2\deg(y) + \gamma_{tR}(T_i)$. If $f(z) = 0$, then $f|_{T_i}$ is a TRDF of T_i with $f(x) = 2$ and we conclude from $x \in W^3_{T_i}$ that $\omega(f|_{T_i}) > \gamma_{tR}(T_i)$. Hence

$$\gamma_{tR}(T_{i+1}) = \omega(f) \ge 2\deg(y) - 1 + \omega(f|_{T_i}) \ge 2\deg(y) + \gamma_{tR}(T_i).$$

Assume finally that $f(y) = 0$. To totally Roman dominate y, y must have a neighbor with label 2. If $f(z) = 2$, then the function f restricted to T_i is a nearly total Roman dominating function of T_i and since $x \in W^2_{T_i}$ we have $\gamma_{tR}(T_{i+1}) = \omega(f) \ge 2\deg(y) + \omega(f|_{T_i}) \ge 2\deg(y) + \gamma_{tR}(T_i)$. If $f(z) \le 1$, then $f(y_i) = 2$ for some $1 \le i \le t$. If $f(z) = 1$, then as above we obtain $\gamma_{tR}(T_{i+1}) \ge 2\deg(y) + \gamma_{tR}(T_i)$. If $f(z) = 0$, then to dominate z we must have $f(x) = 2$ and hence $f|_{T_i}$ is a TRDF of T_i with $f(x) = 2$. We deduce from $x \in W^3_{T_i}$ that $\omega(f|_{T_i}) > \gamma_{tR}(T_i)$ and so $\gamma_{tR}(T_{i+1}) = \omega(f) \ge 2\deg(y) - 1 + \omega(f|_{T_i}) \ge 2\deg(y) + \gamma_{tR}(T_i)$. It follows from Observation 5 that $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$. □

Let T be the tree obtained from two copies of F_2 by joining the leaves adjacent to the centers of F_2. Obviously, $\gamma_{tR}(T) = 2\gamma_t(T)$ and T is obtained from P_4 by applying Operations $\mathcal{O}_3$ and $\mathcal{O}_6$ respectively. On the other hand, T cannot be obtained by other operations and so Operation $\mathcal{O}_6$ is necessary to construct the family $\mathcal{T}$.

Lemma 2.7. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_7$, then $\gamma_{tR}(T_{i+1}) = 2\gamma_t(T_{i+1})$.*

Proof. Let $\mathcal{O}_7$ add a double star $DS_{2,1}$ with central vertices a, b where $\deg(a) = 3$ and let $\mathcal{O}_7$ join x to a leaf z adjacent to a. By adding a, b to any total dominating set of T_i we obtain a total dominating set of T_{i+1} and so $\gamma_t(T_{i+1}) \le \gamma_t(T_i) + 2$.

Suppose now that f is a $\gamma_{tR}(T_{i+1})$-function such that $f(z)$ is as small as possible. We may assume, without loss of generality, that $f(a) = f(b) = 2$. We claim that $f(z) = 0$. Assume, to the contrary, that $f(z) \ge 1$. If $f(z) = 2$, then it is easy to see that $f(x) = 0$. If $f(w) \ge 1$ for a vertex $w \in N_{T_i}(x)$, then define $g : V(T_{i+1}) \to$

$\{0,1,2\}$ by $g(z)=0, g(x)=1$ and $g(u)=f(u)$ otherwise. Then g is also a total Roman dominating set of T_{i+1} of weight $\omega(f)-1$, a contradiction. If $f(w)=0$ for all $w \in N_{T_i}(x)$, then define $g: V(T_{i+1}) \to \{0,1,2\}$ by $g(z)=0$, $g(x)=g(w)=1$ for some $w \in N_{T_i}(x)$ and $g(u)=f(u)$ otherwise. Then g is a $\gamma_{tR}(T_{i+1})$-function contradicting the choice of f.

Let now $f(z)=1$. If $f(x)=2$, then it is easy to see that $f(w)=0$ for all $w \in N_{T_i}(x)$. Now define $g: V(T_{i+1}) \to \{0,1,2\}$ by $g(z)=0, g(w)=1$ for some $w \in N_{T_i}(x)$ and $g(u)=f(u)$ otherwise. If $f(x)=1$, then it is easy to see that $f(w)=0$ for all $w \in N_{T_i}(x)$. Now define $g: V(T_{i+1}) \to \{0,1,2\}$ by $g(z)=0, g(w)=1$ for some $w \in N_{T_i}(x)$ and $g(u)=f(u)$ otherwise. If $f(x)=0$, then there exists a vertex $w \in N_{T_i}(x)$ such that $f(w)=2$. Now define $g: V(T_{i+1}) \to \{0,1,2\}$ by $g(z)=0, g(x)=1$ and $g(u)=f(u)$ otherwise. Then g is a $\gamma_{tR}(T_{i+1})$-function contradicting the choice of f. Thus $f(z)=0$. Then the function f, restricted to T_i is a total Roman dominating function of T_i and hence $\gamma_{tR}(T_{i+1})=\omega(f)\geq 4+\omega(f|_{T_i})\geq 4+\gamma_{tR}(T_i)$, and the result follows from Observation 5. □

Let T be a tree obtained from P_{10} by adding one pendant edges at every support vertex and leaf. Clearly, $\gamma_{tR}(T)=2\gamma_t(T)$ and T is obtained from P_4 by applying Operations $\mathcal{O}_1, \mathcal{O}_5$ and $\mathcal{O}_7$ respectively. On the other hand, T cannot be obtained by other operations and so Operation $\mathcal{O}_7$ is necessary to construct the family $\mathcal{T}$.

Theorem 2.1. *If $T \in \mathcal{T}$, then $\gamma_{tR}(T)=2\gamma_t(T)$.*

Proof. If T is P_4, then obviously $\gamma_{tR}(T)=2\gamma_t(T)$. Suppose now that $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that T_1 is P_4, and if $k \geq 2$, then T_{i+1} can be obtained from T_i by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_7$ for $i=1,2,\ldots,k-1$. We apply induction on the number of operations used to construct T. If $k=1$, the result is trivial. Assume the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T'=T_{k-1}$. By the induction hypothesis, we have $\gamma_{tR}(T')=2\gamma_t(T')$. Since $T=T_k$ is obtained by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_7$ from T', we conclude from the above lemmas that $\gamma_{tR}(T)=2\gamma_t(T)$. □

Now we are ready to prove our main result.

Theorem 2.2. *Let T be a tree of order $n \geq 4$. Then $\gamma_{tR}(T)=2\gamma_t(T)$ if and only if $T \in \mathcal{T}$.*

Proof. According to Theorem 2.1, we need only to prove necessity. Let T be a tree of order $n \geq 4$ with $\gamma_{tR}(T)=2\gamma_t(T)$. The proof is by induction on n. If $n=4$, then the only tree T of order 4 with $\gamma_{tR}(T)=2\gamma_t(T)$ is $P_4 \in \mathcal{T}$. Let $n \geq 5$ and let the statement hold for all trees of order less than n. Assume that T is a tree of order n with $\gamma_{tR}(T)=2\gamma_t(T)$. By Observation 1, we have $\operatorname{diam}(T)\geq 3$. If $\operatorname{diam}(T)=3$, then T is a double star and T can be obtained from P_4 by applying Operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. Hence let $\operatorname{diam}(T)\geq 4$.

Let $v_1v_2\dots v_k$ ($k \geq 5$) be a diametral path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k. If $\deg_T(v_2) \geq 4$, then clearly $\gamma_{tR}(T-v_1) = 2\gamma_t(T-v_1)$. It follows from the induction hypothesis that $T-v_1 \in \mathcal{T}$ and hence T can be obtained from $T-v_1$ by Operation $\mathcal{O}_1$, implying that $T \in \mathcal{T}$. Let $\deg_T(v_2) \leq 3$. We consider two cases.

Case 1. $\deg_T(v_2) = 3$.
Assume that $L_{v_2} = \{v_1, w\}$.

Subcase 1.1. $\deg_T(v_3) \geq 3$.
First let v_3 be adjacent to a support vertex $z \notin \{v_2, v_4\}$. Suppose $T' = T - T_z$. For any $\gamma_t(T)$-set S containing no leaves we have $z, v_2, v_3 \in S$ and so $S \setminus \{z\}$ is a total dominating set of T' yielding $\gamma_t(T) \geq \gamma_t(T') + 1$. Now let f be a $\gamma_{tR}(T')$-function. Since v_2 is an end strong support vertex and since f is a TRDF of T', we may assume that $f(v_2) = 2$ and $f(v_3) \geq 1$. Clearly f can be extended to a TRDF of T by assigning the weight 2 to z and the weight 0 to the leaves adjacent to z and this implies that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2$. It follows from Observation 6 that $\gamma_{tR}(T') = 2\gamma_t(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_2$ if $\deg_T(z) = 2$ and by Operations $\mathcal{O}_2$ and $\mathcal{O}_1$ when $\deg_T(z) \geq 3$. Hence $T \in \mathcal{T}$.

Now assume that each neighbor of v_3 except v_2, v_4, is a leaf and let $T' = T - v_1$. It is easy to see that $\gamma_t(T) = \gamma_t(T-v_1)$ and $\gamma_{tR}(T) = \gamma_{tR}(T-v_1)$. Hence $\gamma_{tR}(T-v_1) = 2\gamma_t(T-v_1)$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Since v_2, v_3 are support vertices in T', there exists a $\gamma_{tR}(T')$-function f such that $f(v_2) = f(v_3) = 2$. Now T can be obtained from T' by Operation $\mathcal{O}_1$.

Subcase 1.2. $\deg_T(v_3) = 2$.
If v_4 is a support vertex, then let $T' = T - \{v_1, w\}$. It is easy to see that $\gamma_t(T) = \gamma_t(T') + 1$ and $\gamma_{tR}(T) = \gamma_{tR}(T') + 1$. Then $2\gamma_t(T) = \gamma_{tR}(T) \leq \gamma_{tR}(T') + 1 \leq 2\gamma_t(T') + 1 = 2\gamma_t(T) - 1$ which is a contradiction. If v_4 has a children $z \neq v_3$, with depth 1 or 2, then let $T' = T - T_{v_3}$. It is not hard to see that $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. But then $2\gamma_t(T) = \gamma_{tR}(T) \leq \gamma_{tR}(T') + 3 \leq 2\gamma_t(T') + 3 = 2\gamma_t(T) - 1$, a contradiction again. Henceforth, we assume $\deg(v_4) = 2$. Since $\gamma_{tR}(T) = 2\gamma_t(T)$, we have $\mathrm{diam}(T) \geq 5$. Let $T' = T - T_{v_4}$. Clearly, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning the weight 2 to v_2, v_3 and the weight 0 to v_1, v_4, w and so $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 4$. On the other hand, let S be a $\gamma_t(T)$-set containing no leaves. Then $v_2, v_3 \in S$ and the set $S' = S - \{v_2, v_3\}$ if $v_4 \notin S$, and $S' = (S - \{v_2, v_3, v_4\}) \cup \{v_6\}$ if $v_4 \in S$, is a total dominating set of T' yielding $\gamma_t(T) \geq \gamma_t(T') + 2$. By Observation 6 we have $\gamma_{tR}(T') = 2\gamma_t(T')$ and this implies that $\gamma_{tR}(T) = \gamma_{tR}(T') + 4$ and $\gamma_t(T) = \gamma_t(T') + 2$ by the assumption. By the induction hypothesis we have $T' \in \mathcal{T}$. Now we show that $v_5 \in W^2_{T'} \cap W^3_{T'}$. If $v_5 \notin W^2_{T'}$, then let g be a nearly TRDF of T' of weight less than $\gamma_{tR}(T')$ and define $h : V(T) \to \{0, 1, 2\}$ by $h(v_2) = 2, h(v_3) = h(v_4) = 1$, $h(x) = g(x)$ for $x \in V(T')$ and $h(x) = 0$ otherwise. If $v_5 \notin W^3_{T'}$, then let g be a TRDF of T' with $g(v_5) = 2$ and define $h : V(T) \to \{0, 1, 2\}$ by $h(v_2) = 2, h(v_3) = 1$, $h(x) = g(x)$ for $x \in V(T')$ and $h(x) = 0$ otherwise. Clearly h is a TRDF of T with weight $\gamma_{tR}(T) - 1$, a contradiction.

Thus $v_5 \in W^2_{T'} \cap W^3_{T'}$ and so T can be obtained from T' by Operation $\mathcal{O}_5$, implying $T \in \mathcal{T}$.

Case 2. $\deg(v_2) = 2$.
By the choice of the diametral path, we may assume that all support vertices adjacent to v_3 and v_{k-1} have degree 2. We consider the following subcases.

Subcase 2.1. v_3 is a support vertex and v_3 has a support neighbor w other than v_2.
Let $T' = T - \{v_1, v_2\}$. If S is a $\gamma_t(T)$-set containing no leaves, then $v_2, v_3, w \in S$ and so $S \setminus \{v_2\}$ is a total dominating set of T', implying that $\gamma_t(T) \geq \gamma_t(T') + 1$. On the other hand, since any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning the weight 2 to v_2 and the weight 0 to v_1, we have $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2$. By Observation 6 and the induction hypothesis, we obtain $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_3$, and hence $T \in \mathcal{T}$.

Subcase 2.2. $\deg_T(v_3) \geq 3$ and all neighbors of v_3 except v_2, v_4 are leaves.
Let w be a leaf adjacent to v_3. If $\deg(v_3) \geq 4$, then let $T' = T - w$. It is easy to see that $\gamma_t(T) = \gamma_t(T')$ and $\gamma_{tR}(T) = \gamma_{tR}(T')$. Hence $\gamma_{tR}(T') = 2\gamma_t(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Then T can be obtained from T' by Operation $\mathcal{O}_1$. Assume that $\deg_T(v_3) = 3$. We distinguish the following cases.

(a) v_4 is a support vertex.
Let $T' = T - \{v_1, v_2\}$. As above we can see that $\gamma_t(T) = \gamma_t(T') + 1$ and $\gamma_{tR}(T) = \gamma_{tR}(T') + 2$, yielding $\gamma_{tR}(T') = 2\gamma_t(T')$. By the induction hypothesis we have $T' \in \mathcal{T}$ and now T can be obtained by Operation $\mathcal{O}_3$.

(b) $\deg(v_4) = 2$.
By (a) we may assume that v_4 is not a support vertex. Let $T' = T - T_{v_4}$. As in the proof of subcase 1.2, we can see that $T' \in \mathcal{T}$. Then T can be obtained from T' by Operation $\mathcal{O}_7$.

(c) $\deg(v_4) \geq 3$.
By (a) we may assume that v_4 is not a support vertex. Thus v_4 has a children z different from v_2 with depth 1 or 2. Let $T' = T - T_{v_3}$. If S is a $\gamma_t(T)$-set containing no leaves, then clearly $v_2, v_3, z \in S$ and so $S - \{v_2, v_3\}$ is a total dominating set of T', yielding $\gamma_t(T) \geq \gamma_t(T') + 2$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning 2 to v_2, v_3 and the weight 0 to w, v_1, and hence $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 4$. We deduce from Observation 6 that $\gamma_{tR}(T') = 2\gamma_t(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. If $v_4 \notin W^1_{T'}$, then let f be an almost TRDF of T' with respect to v_4 of weight at most $\gamma_{tR}(T') - 1$ and extend f to a TRDF of T by assigning the weight 2 to v_2, v_3 and the weight 0 to w, v_1; this implies that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3 = 2\gamma_t(T') + 3 \leq 2\gamma_t(T) - 1$, a contradiction. Thus $v_4 \in W^1_{T'}$, and now T can be obtained from T' by Operation $\mathcal{O}_4$, yielding $T \in \mathcal{T}$.

Subcase 2.3. $\deg_T(v_3) \geq 3$ and all children of v_3 are support vertices of degree 2. We distinguish three cases.

(i) v_4 is a support vertex.
Suppose $T' = T - v_1$. By adding v_2 to any $\gamma_t(T')$-set we obtain a total dominating set of T and so $\gamma_t(T) \le \gamma_t(T')+1$. On the other hand, if S is a $\gamma_t(T)$-set containing no leaves then $N[v_3] \subseteq S$ and clearly $S - \{v_2\}$ is a total dominating set of T', implying that $\gamma_t(T) \ge \gamma_t(T') + 1$. Thus $\gamma_t(T) = \gamma_t(T') + 1$. Now let f be a $\gamma_{tR}(T')$-function. Since v_3 and its neighbors other than v_2 in T' are support vertices, we may assume that $f(x) = 2$ for each $x \in N_{T'}[v_3] - \{v_2\}$. Then the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(v_3) = 1, g(v_2) = 2, g(v_1) = 0$, and $g(u) = f(u)$ otherwise, is a TRDF of T with weight $\omega(f) + 1$. Hence $\gamma_{tR}(T) \le \gamma_{tR}(T') + 1 \le 2\gamma_t(T') + 1 = 2\gamma_t(T) - 1$, a contradiction.

(ii) v_4 has a child $z \ne v_3$ with depth 1 or 2.
Assume that $T' = T - T_{v_3}$. Any $\gamma_t(T')$-set S can be extended to a total dominating set of T by adding $C(v_3) \cup \{v_3\}$ and so $\gamma_t(T) \le \gamma_t(T') + |C(v_3)| + 1$. On the other hand, if S is a $\gamma_t(T)$-set containing no leaves, then $C(v_3) \cup \{v_3, z\} \subseteq S$, and clearly $S - (C(v_3) \cup \{v_3\})$ is a total dominating set of T', implying that $\gamma_t(T) \ge \gamma_t(T') + |C(v_3)| + 1$. Thus $\gamma_t(T) = \gamma_t(T') + |C(v_3)| + 1$. Clearly, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning the weight 1 to v_3, the weight 2 to the children of v_3 and the weight 0 to the leaves of T_{v_3}, and this implies that $\gamma_{tR}(T) \le \gamma_{tR}(T') + 2|C(v_3)| + 1 \le 2\gamma_t(T') + 2|C(v_3)| + 1 = 2\gamma_t(T) - 1$, a contradiction again.

(iii) $\deg(v_4) = 2$.
If $\mathrm{diam}(T) = 4$, then T is a healthy spider, and we have $\gamma_{tR}(T) = 2\deg(v_3)+1 \le 2(\deg(v_3)+1) - 1 = 2\gamma_t(T) - 1$, which is a contradiction. Let $\mathrm{diam}(T) \ge 5$ and let $T' = T - T_{v_4}$. Assume that S is a $\gamma_t(T)$-set. Then clearly $N[v_3] - \{v_4\} \subseteq S$, and the set $S' = S - N[v_3]$ if $v_4 \notin S$ and $S' = (S - N[v_3]) \cup \{v_6\}$ if $v_4 \in S$, is a total dominating set of T', yielding $\gamma_t(T) \ge \gamma_t(T')+\deg(v_3)$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning the weight 2 to each vertex in $N[v_3] - \{v_4\}$ and the weight 0 to the remaining vertices, and this implies that $\gamma_{tR}(T) \le \gamma_{tR}(T') + 2\deg(v_3)$. It follows from Observation 6 and the induction hypothesis that $T' \in \mathcal{T}$. If $v_5 \notin W^2_{T'}$, then let f be a nearly TRDF of T' of weight at most $\gamma_{tR}(T') - 1$ and define $g : V(T) \to \{0, 1, 2\}$ by $g(u) = f(u)$ for $u \in V(T')$, $g(u) = 1$ for $u \in V(T_{v_4})$. If $v_5 \notin W^3_{T'}$, then let f be a $\gamma_{tR}(T')$-function with $f(v_5) = 2$ and define $g : V(T) \to \{0, 1, 2\}$ by $g(u) = f(u)$ for $u \in V(T')$, $g(v_4) = 0$ and $g(u) = 1$ for $u \in N[v_3] - \{v_4\}$ and $g(u) = 0$ otherwise. In each case, g is a TRDF of T of weight at most $\gamma_{tR}(T') + 2\deg(v_3) - 1$ that leads to a contradiction. Thus $v_5 \in W^2_{T'} \cap W^3_{T'}$ and so T can be obtained from T' by Operation $\mathcal{O}_6$, yielding $T \in \mathcal{T}$.

Subcase 2.4. $\deg(v_3) = 2$.
We claim that $\deg(v_4) = 2$. Assume, to the contrary, that $\deg(v_4) \ge 3$. First assume v_4 is at distance 1 or 2 from a support vertex other than v_2 and let $T' = T - T_{v_3}$. Assume that S is a $\gamma_t(T)$-set containing no leaves. Then $v_2, v_3 \in S$ and clearly $S - \{v_2, v_3\}$ is a total dominating set of T', implying that $\gamma_t(T') \le \gamma_t(T) - 2$. On

the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning the weight 1 to v_3, v_2, v_1 and this implies that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. But then

$$2\gamma_t(T) = \gamma_{tR}(T) \leq \gamma_{tR}(T') + 3 \leq 2\gamma_t(T') + 3 \leq 2\gamma_t(T) - 1$$

which is a contradiction. Now let v_4 be a support vertex and let $T' = T - v_1$. Suppose that S is a $\gamma_t(T)$-set containing no leaves. Then $v_2, v_3, v_4 \in S$, and clearly $S - \{v_2\}$ is a total dominating set of T' yielding $\gamma_t(T') \leq \gamma_t(T) - 1$. On the other hand, let f be a $\gamma_{tR}(T')$-function. Since v_3, v_4 in T' are support vertices, we may assume that $f(v_3) = f(v_4) = 2$. Define $g : V(T) \to \{0, 1, 2\}$ by $g(u) = f(u)$ for $u \in V(T') - \{v_2, v_3\}$, $g(v_3) = 1, g(v_2) = 2$ and $g(v_1) = 0$. Clearly g is a TRDF of T of weight $\gamma_{tR}(T') + 1$. It follows that

$$2\gamma_t(T) = \gamma_{tR}(T) \leq \gamma_{tR}(T') + 1 \leq 2\gamma_t(T') + 1 \leq 2\gamma_t(T) - 1,$$

a contradiction again. This proves our claim. That is, $\deg(v_4) = 2$. Since $\gamma_{tR}(T) = 2\gamma_t(T)$, we have $\operatorname{diam}(T) \geq 6$. Let $T' = T - T_{v_4}$. Any total dominating set of T' can be extended to a total dominating set of T by adding v_2, v_3, and so $\gamma_t(T) \leq \gamma_t(T') + 2$. Let S be a total dominating set of T containing no leaves. Then $v_2, v_3 \in S$ and the set $S' = S \setminus \{v_2, v_3\}$ if $v_4 \notin S$ and $S' = (S \setminus \{v_2, v_3, v_4\}) \cup \{v_6\}$ if $v_4 \in S$ is a total dominating set of T'. Hence $\gamma_t(T) - 2 \geq \gamma_t(T')$ and we have $\gamma_t(T') = \gamma_t(T) - 2$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning the weight 2 to v_2, v_3 and the weight 0 to v_1, v_4, yielding $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 4$. Hence, $2\gamma_t(T) = \gamma_{tR}(T) \leq \gamma_{tR}(T') + 4 \leq 2\gamma_t(T') + 4 = 2\gamma_t(T)$, and this leads to

$$\gamma_{tR}(T) = \gamma_{tR}(T') + 4 \tag{2}$$

and $\gamma_{tR}(T') = 2\gamma_t(T')$. Therefore, by the induction hypothesis, we have $T' \in \mathcal{T}$.

If $v_5 \notin W^2_{T'}$, then let f be a nearly TRDF with respect to v_5 with $w(f) \leq \gamma_{tR}(T') - 1$. If $f(v_5) = 0$, then f is a TRDF of T', which is impossible. Hence $f(v_5) \geq 1$. Then f can be extended to a TRDF of T by assigning the weight 1 to v_4, v_3, v_2, v_1 and hence $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$, which is a contradiction with (2). If $v_5 \notin W^3_{T'}$, then let f be a $\gamma_{tR}(T')$-function with $f(v_5) = 2$, and define $g : V(T) \to \{0, 1, 2\}$ by $g(u) = f(u)$ for $u \in V(T')$, $g(v_4) = 0, g(v_3) = g(v_2) = g(v_1) = 1$. Clearly g is a TRDF of T of weight $\gamma_{tR}(T') + 3$, contradicting (2). Thus $v_5 \in W^2_{T'} \cap W^3_{T'}$ and so T can be obtained from T' by Operation $\mathcal{O}_5$. This completes the proof. □

It is shown in [10] that for every graph G, the Roman domination number of G is bounded above by twice its domination number. Graphs which have Roman domination number equal to twice their domination number are called Roman graphs. A characterization of Roman trees is given in [13]. If T is a tree obtained from a star $K_{1,r}$ $(r \geq 2)$ by adding at least two pendant edges at every vertex of $K_{1,r}$, then clearly T is both Roman and total Roman. On the other hand, P_4 is a total Roman tree which is not a Roman tree and P_5 is a Roman tree which is not a total Roman tree. We conclude this paper with an open problem.

Problem. Characterize the trees T which are both Roman and total Roman.

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