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Rainbow connections of graph joins

Fendy Septyanto Kiki A. Sugeng

Department of Mathematics
Universitas Indonesia
Depok
Indonesia
fendy.septyanto41@sci.ui.ac.id kiki@sci.ui.ac.id

In memory of Mirka Miller

Abstract

The rainbow connection number (respectively, strong rainbow connection number) of a graph G, denoted by $\mathrm{rc}(G)$ (respectively, $\mathrm{src}(G)$), is the smallest number of colors needed to color the edges of G so that any two vertices are connected by a path (respectively, geodesic) whose edges all have different colors. In this paper, we examine the rc and src of a general graph join. We prove lower and upper bounds for the rc and src of the join of two graphs in terms of several parameters of the individual graphs: the number of vertices of degree 0, maximum degree, independent domination number, clique number, and independence number. These bounds are best possible, in the sense that equality is satisfied for infinitely many non-isomorphic graphs.

1 Introduction

All graphs in this paper are finite, undirected, and simple. By a k-coloring (or simply "coloring") on a graph G we mean any map $\gamma : E(G) \to \{1, \ldots, k\}$. We write

$$x \overset{i}{-} y$$

to mean xy is an edge with color $\gamma(xy) = i$. A path is called *rainbow* if all of its edges have different colors. A coloring is called *rainbow* if any two vertices are connected by a rainbow path. A simple way to achieve this is by coloring each edge differently (we call this a *trivial coloring*), but usually there is a more efficient way. The *rainbow connection number* of a graph G, denoted by $\mathrm{rc}(G)$, is defined as the smallest k for which G has a rainbow k-coloring.

A geodesic is a shortest path between two vertices. A coloring is called *strong rainbow* if any two vertices are connected by a rainbow geodesic. The *strong rainbow connection number* of G, denoted by $\mathrm{src}(G)$, is the smallest k for which G has a strong rainbow k-coloring. Any trivial coloring is strong rainbow, so $\mathrm{src}(G)$ always exists. In fact, as observed in [2], we have

$$\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq \mathrm{src}(G) \leq |E(G)|.$$

The concepts of rainbow connection were introduced by Chartrand et al. in 2008 [2]. For a detailed survey on rc and src, the reader is referred to [6]. Some authors (e.g. [1, 4]) have studied the effect that some graph operations (such as Cartesian product, direct product, lexicographic product, and strong product) have on rc and src. Typically, a bound is proved for the rc (or src) of the result of a graph operation in terms of the parameters of the individual graphs. In this paper we examine the rc and src of a graph join. Recall that the *join* of two vertex-disjoint graphs G and H is a new graph $G + H$ obtained by connecting each vertex of G to each vertex of H by a new edge.

It is immediate to see that $\mathrm{rc}(G + H) \leq \max\{\mathrm{rc}(G), \mathrm{rc}(H)\}$ if G and H are connected, because we can simply color each graph independently, and color the "crossing" edges in any manner. Note that even if G or H is disconnected, $G + H$ is always connected. In such a case $\mathrm{rc}(G + H)$ exists but the previous bound fails, since rc is only defined for connected graphs. Thus we need more general bounds for $\mathrm{rc}(G+H)$. It turns out that information about vertices of degree 0 play an important role.

Since $\mathrm{diam}(G + H) \leq 2$, the distance between vertices in each graph is typically destroyed in the join graph. However, there is a useful way to exploit distance in the individual graphs : if two vertices in one of the two graphs have distance at least 3 in that graph, then any geodesic between them in $G + H$ must pass through the other graph. We will use this observation to construct a strong rainbow coloring on the join graph by locally defining several strong rainbow colorings in one graph and tying them together by using an independent dominating set in the other graph. Thus an upper bound for $\mathrm{src}(G + H)$ is obtained. We also use this idea to obtain lower bounds on $\mathrm{src}(G + H)$.

2 Main results

We first recall some definitions and notation. Following [7], we denote by $n_i(G)$ the number of vertices of degree i in a graph G. Following [3], we define the square of G as a new graph G^2 with $V(G^2) = V(G)$ such that two vertices are adjacent in G^2 if and only if they have distance at most 2 in G.

Following [5], we use the following definitions. A set of pairwise adjacent vertices which is maximal with respect to this property is called a *clique*. The *clique number* of G, denoted by $\omega(G)$, is defined as the largest size of a clique in G. A set of pairwise non-adjacent vertices is called an *independent set*. The *independence number* of G, denoted by $\beta_0(G)$, is defined as the largest size of an independent set in G. A set of vertices for which any other vertex has at least one neighbor in the set is called a *dominating set*. The *independent domination number* of G, denoted by $i(G)$, is defined as the smallest size of an independent dominating set in G.

2.1 Bounds for rc

Our first result is a lower bound and an upper bound for the rc of a graph join, in terms of the number of vertices of degree 0 in the individual graphs.

Theorem 2.1 *Let G and H be vertex-disjoint graphs, with $E(G) \neq \emptyset$. Then*

$$\min\left\{3, n_0(G)^{\frac{1}{|V(H)|}}\right\} \leq \mathrm{rc}(G+H) \leq \max\{3, n_0(G)\}.$$

. If $E(G) = \emptyset$, then the lower bound continues to hold. If $E(G) \neq \emptyset$ and $E(H) = \emptyset$, then the lower bound can be improved to $\min\left\{4, n_0(G)^{\frac{1}{|V(H)|}}\right\}$.

Proof: Let $I := \{v_1, \ldots, v_{n_0(G)}\}$ be the set of isolated vertices in G (if any). We first prove the upper bound. Let F be a spanning forest for $G \backslash I$ with a bipartition $V(F) = W_1 \cup W_2$. Let $k := \max\{3, n_0(G)\}$. Define a k-coloring on $G + H$ as follows.

1. Each $H{-}W_1$ edge is colored 1.
2. Each $H{-}W_2$ edge is colored 2.
3. Each $H{-}v_i$ edge is colored i.
4. All other edges are colored 3.

Now we prove that this is a rainbow coloring. Let $x, y \in V(G+H)$ be non-adjacent. We will produce a rainbow path in $G + H$ from x to y.

Case 1 : $x, y \in I$.

Let $x = v_i$, $y = v_j$, $i < j$, and choose any $h \in V(H)$. Then $v_i \overset{i}{-} h \overset{j}{-} v_j$ is rainbow.

Case 2 : $x \in I$ and $y \in W_1 \cup W_2$.

Say $x = v_i$ and $y \in W_1$. Choose any $h \in V(H)$ and $c \in W_2 \cap N_G(y)$. If $i \neq 1$, then $v_i \overset{i}{-} h \overset{1}{-} y$ is rainbow. If $i = 1$, then $v_i \overset{1}{-} h \overset{2}{-} c \overset{3}{-} y$ is rainbow.

Case 3 : $x, y \in W_1$ or $x, y \in W_2$.

Say $x, y \in W_1$. Let $h \in V(H)$ and $c \in W_2 \cap N_G(y)$. Then $x \overset{1}{-} h \overset{2}{-} c \overset{3}{-} y$ is rainbow.

Case 4 : $x \in W_1$ and $y \in W_2$.

Choose any $h \in V(H)$. Then $x \overset{1}{-} h \overset{2}{-} y$ is rainbow.

Case 5 : $x, y \in V(H)$.

Choose any edge uv with $u \in W_1$ and $v \in W_2$ (this is possible since $E(G) \neq \emptyset$). Then $x \overset{1}{-} u \overset{3}{-} v \overset{2}{-} y$ is rainbow. The proof of the upper bound is completed.

Next, we prove the lower bound without assuming $E(G) \neq \emptyset$. Let $b := |V(H)|$ and $k := \min\left\{2, \left\lceil \sqrt[b]{n_0(G)} \right\rceil - 1\right\}$. Suppose $\mathrm{rc}(G+H) \leq k$. Then $G+H$ has a rainbow k-coloring γ. Let $V(H) = \{x_1, \ldots, x_b\}$. For each $v_i \in I$ we define

$$\mathrm{code}(v_i) := (\gamma(v_i x_1), \gamma(v_i x_2), \ldots, \gamma(v_i x_b)) \tag{2.1}$$

Each code is a b-tuple of numbers taken from $1, \ldots, k$, so there are at most k^b different codes. Note that $k^b < n_0(G) = |I|$. Thus, some two vertices in I have the same code, say $\mathrm{code}(v_i) = \mathrm{code}(v_j)$ for some $1 \leq i < j \leq n_0(G)$. Let L be a rainbow path in $G+H$ from v_i to v_j. Since $d_{G+H}(v_i, v_j) = 2$, the length of L is at least 2. So $L : v_i - x_u - \cdots - x_v - v_j$ for some $x_u, x_v \in V(H)$ (since v_i and v_j are isolated in G). If L has length at least 3, then not all of its edges can have different colors (since $k \leq 2$). So L has length exactly 2, and $x_u = x_v$. Since L is rainbow, $\gamma(v_i x_u) \neq \gamma(v_j x_u)$. This contradicts $\mathrm{code}(v_i) = \mathrm{code}(v_j)$, and the lower bound is proved.

Finally, suppose that $E(G) \neq \emptyset$ and $E(H) = \emptyset$. Redefine

$$k := \min\left\{3, \left\lceil \sqrt[b]{n_0(G)} \right\rceil - 1\right\}$$

and repeat the previous paragraph up to the conclusion that $L : v_i - x_u - \cdots - x_v - v_j$ for some $x_u, x_v \in V(H)$. If $x_u = x_v$ then we get a contradiction as before. Now assume $x_u \neq x_v$. Because $E(H) = \emptyset$, the vertices x_u, x_v are not adjacent. So the length of the rainbow path L is at least 4, contradicting the fact that the number of colors is $k \leq 3$. With this, the proof of the lower bound is finished. □

Tight example For any G and H with $n_0(H) \leq 3$ and $n_0(G) > 2^{|V(H)|}$, we have $\mathrm{rc}(G+H) = 3$ by Theorem 2.1 (switching the role of G and H for the upper bound).

Alternatively, we may simply take H to be a singleton and $n_0(G) = 3$ or $n_0(G) = 4$, where both sides of the theorem are the same. Interestingly, if $H = K_1$ and $n_0(G) \geq 3$, then only the right hand side of Theorem 2.1 becomes an equality.

Corollary 2.2 *If $E(G) \neq \emptyset$ and $n_0(G) \geq 3$, then $\mathrm{rc}(G + K_1) = n_0(G)$.*

PROOF: By Theorem 2.1 it remains to show $\mathrm{rc}(G+K_1) \geq n_0(G)$. Note that $n_1(G+K_1) = n_0(G)$. By a lemma of [7], we have $\mathrm{rc}(G) \geq n_1(G)$ for any graph G. So we obtain $\mathrm{rc}(G+K_1) \geq n_1(G+K_1) = n_0(G)$. □

2.2 Bounds for src

Our second result is a lower bound and an upper bound for the src of a graph join, in terms of independence number, clique number, maximum degree, and independent domination number.

In the proof, we need the following two facts from [2]. The "spanning subgraph bound" states that if G has a connected spanning subgraph H with $\mathrm{diam}(H) = 2$, then $\mathrm{src}(G) \leq \mathrm{src}(H)$. The second fact is the src of a complete bipartite graph, $\mathrm{src}(K_{q,p}) = \lceil \sqrt[q]{p} \rceil$ provided that $1 \leq q \leq p$.

Theorem 2.3 *Let G and H be vertex-disjoint graphs. Then*

$$\begin{aligned}\max\left\{\beta_0(G^2), \frac{|V(G)|}{\omega(G^2)}\right\}^{\frac{1}{|V(H)|}} &\leq \operatorname{src}(G+H)\\ &\leq \max\left\{\Delta(G), \left\lceil i(G)^{\frac{1}{|V(H)|}}\right\rceil, \left\lceil |V(H)|^{\frac{1}{i(G)}}\right\rceil\right\}.\end{aligned}$$

Proof: We first prove the upper bound. Let $X \subseteq V(G)$ be an independent dominating set in G with $|X| = i(G)$, and let U be the subgraph of $G+H$ induced by X and H. Then U is the join of $\langle X\rangle$ (i.e. the subgraph of G induced by X) and H. Since it is a graph join, U has a spanning subgraph which is complete bipartite $K_{q,p}$ with $p := \max\{i(G), |V(H)|\}$ and $q := \min\{i(G), |V(H)|\}$. By the spanning subgraph bound,

$$\operatorname{src}(U) \leq \operatorname{src}(K_{q,p}) = \lceil \sqrt[q]{p}\,\rceil \tag{2.2}$$

Let γ be a strong rainbow $\lceil \sqrt[q]{p}\,\rceil$-coloring on U, and let $k := \max\{\Delta(G), \lceil \sqrt[q]{p}\,\rceil\}$. We extend γ to a new coloring $\gamma^* : E(G+H) \to \{1,\ldots,k\}$ in several steps as follows.

1. Color the edges of U according to γ.
2. For each $t \in X$, let G_t be the G-neighborhood star around t, i.e. $V(G_t) = \{t\} \cup N_G(t)$ and $E(G_t) = \{tx : x \in N_G(t)\}$. Put a strong rainbow $\deg_G(t)$-coloring on G_t. This is well-defined, because X being independent is equivalent to $E(G_t) \cap E(G_u) = \emptyset$ for all $t,u \in X$ with $t \neq u$.
3. Next, put color 1 on all the previously uncolored edges in G.
4. For each $v \in V(G)\backslash X$ choose one (arbitrary but fixed) $t_v \in X$ that is adjacent to v; this is possible since X is dominating. For any crossing edge vy, with $v \in V(G)\backslash X$ and $y \in V(H)$, put $\gamma^*(vy) := \gamma(t_v y)$. This is possible since $t_v y \in E(U)$ and the edges in U have been colored in the first step.

After the fourth step, all edges of $G+H$ have been colored. Now we prove that γ^* is strong rainbow. Let $x, y \in V(G+H)$ be non-adjacent vertices. So $d_{G+H}(x,y) = 2$. We will show that there is a rainbow geodesic in $G+H$ from x to y.

Case 1 : $x, y \in V(U)$.

In this case $d_U(x,y) = d_{G+H}(x,y) = 2$. So any rainbow geodesic in U from x to y is also a rainbow geodesic in $G+H$.

Case 2 : $x \in X$ and $y \in V(G)\backslash X$.

Since x, y are non-adjacent, $t_y \neq x$. Let L be a rainbow geodesic in U from x to t_y, which exists because $x, t_y \in X \subseteq V(U)$. Since $diam(U) \leq 2$ and X is independent, $d_U(x,t_y) = 2$. So $L : x \overset{i}{-} h \overset{j}{-} t_y$ for some $h \in V(H)$, $i,j \in \{1,\ldots,k\}$, $i \neq j$. We can form the path $x - h - y$, by the definition of graph join. Since $\gamma^*(yh) = \gamma(t_y h) = j$, the path $x \overset{i}{-} h \overset{j}{-} y$ is a rainbow geodesic.

Case 3 : $x, y \in V(G)\backslash X$.

If $t_x = t_y = t$ then $x, y \in G_t$ and we are done, since γ^* was defined locally on G_t as a strong rainbow coloring. So we suppose $t_x \neq t_y$. Since $diam(U) \leq 2$ and X is independent, $d_U(t_x, t_y) = 2$. Let $L : t_x \overset{i}{-} h \overset{j}{-} t_y$ be a rainbow geodesic in U, where $h \in V(H)$, $i, j \in \{1, \ldots, k\}$, $i \neq j$. As in Case 2, we have a rainbow geodesic $x \overset{i}{-} h \overset{j}{-} y$. The proof of the upper bound is now completed.

Next, we prove the lower bounds separately. Let $b := |V(H)|$ and $k := \left\lceil \sqrt[b]{\frac{|V(G)|}{\omega(G^2)}} \right\rceil - 1$. Suppose there is a strong rainbow k-coloring γ on $G + H$. Let $V(H) = \{x_1, \ldots, x_b\}$. For each $v \in V(G)$, define

$$\text{code}(v) := (\gamma(vx_1), \gamma(vx_2), \ldots, \gamma(vx_b)). \tag{2.3}$$

Note that $|V(G)| > \omega(G^2)k^b$. Since there are at most k^b different codes, there must be at least $\omega(G^2) + 1$ vertices in G with the same code. Let X be one such set of vertices. Since $|X| > \omega(G^2)$, X cannot induce a complete subgraph in G^2. So there are $v, w \in X$ with $d_{G^2}(v, w) \geq 2$, i.e. $d_G(v, w) \geq 3$. Let L be a rainbow geodesic in $G + H$ from v to w. Since v and w are non-adjacent and $\text{diam}(G + H) = 2$, the length of L is exactly 2. So $L : v - x - w$ for some $x \in V(G + H)$. But $x \notin V(G)$ since $d_G(v, w) \geq 3$. So $x \in V(H)$. Since L is rainbow, $\gamma(vx) \neq \gamma(wx)$. This contradicts $\text{code}(v) = \text{code}(w)$, and the bound $\text{src}(G + H) \geq \left\lceil \sqrt[b]{\frac{|V(G)|}{\omega(G^2)}} \right\rceil$ is proved.

Now let $k := \left\lceil \sqrt[b]{\beta_0(G^2)} \right\rceil - 1$ and suppose there is a strong rainbow k-coloring γ on $G + H$. Define code as before. Since $\beta_0(G^2) > k^b$, there must be an independent set X in G^2 with $|X| > k^b$. There are at most k^b different codes, so there must be $v, w \in X$ with $v \neq w$ and $code(v) = code(w)$. Since X is independent in G^2, we have $d_{G^2}(v, w) \geq 2$, i.e. $d_G(v, w) \geq 3$. The same line of reasoning as in the previous paragraph follows through to produce a contradiction. □

Tight example If $G = \overline{K_r}$ is the complement of the complete graph K_r for some r, then $\beta_0(G^2) = i(G) = r$ and $\Delta(G) = 0$. So, for any graph H with $|V(H)| \leq r$ we have $\text{src}(\overline{K_r} + H) = \lceil \sqrt[|V(H)|]{r} \rceil$. This generalizes the src of complete bipartite graph.

As a side note, Theorem 2.3 has the following corollary.

Corollary 2.4 *For any graph G, the following holds.*

$$\max\left\{\beta_0(G^2), \frac{|V(G)|}{\omega(G^2)}\right\} \leq \max\{\Delta(G), i(G)\}.$$

PROOF: Take $H = K_1$ in Theorem 2.3. □

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