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Edge irregular reflexive labeling of prisms and wheels*

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In memory of Mirka Miller

Abstract

For a graph G we define a k-labeling ρ such that the edges of G are labeled with integers $\{1, 2, \dots, k_e\}$ and the vertices of G are labeled with even integers $\{0, 2, \dots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. The labeling ρ is called an *edge irregular reflexive k-labeling* if distinct edges have distinct weights, where the edge weight is defined as the sum of the label of that edge and the labels of its ends. The smallest k for which such a labeling exists is called the *reflexive edge strength* of G.

In this paper we give exact values for the reflexive edge strength for prisms, wheels, baskets and fans.

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1 Introduction

Regular graphs have been an area of interest almost as long as graphs have been studied. However, as a consequence of the Handshaking Lemma, no simple graphs can be completely irregular. That is, no simple graph can have each vertex bearing a distinct degree. Multigraphs however can display this property. In [4] the authors asked, "In a loopless multigraph, determine the fewest parallel edges required to ensure that all vertices have distinct degree." For convenience, the problem was recast in terms of a weighted simple graph with the edge weights representing the number of parallel edges. The degree of a vertex was then determined by adding the weights of the edges incident to that vertex. Then the problem became:

> "Assign positive values to the edges of a simply connected graph of order at least 3 in such a way that the graph becomes irregular. What is the minimum value of the largest label over all such irregular assignments?"

The minimum value of the largest label is known as the *irregularity strength* of a graph.

The concept of *reflexive irregular multigraphs* originated in [8] as a natural consequence of irregular multigraphs by allowing for loops. Following the initiative of [4] and rewording the problem as a graph labeling exercise, *irregular reflexive labelings* include vertex labels representing degrees contributed by the loops. The weight of a vertex v, denoted by $wt(v)$, is now determined by summing the incident edge labels and the label of v.

Previously, Bača et al. [2] proposed an *irregular total labeling* in which the vertices were labelled by positive integers. For some of the results in irregular total labelings, refer to [1, 5, 6, 7, 9]. The difference between this idea and reflexive labeling is threefold:

1. The concept is consistent with the genesis of the problem by considering multigraphs with loops.
2. The vertex labels must be even non-negative integers, representing the fact that each loop contributes 2 to the vertex degree.
3. Vertex label 0 is permissible as representing a loopless vertex.

As in the case of irregular total labelings, this new scheme allows for considering not just vertex weights but also edge weights. An edge weight is the sum of the edge label and the labels of the vertices incident to the edge. Thus we are able to propose *vertex irregular reflexive labelings* and *edge irregular reflexive labelings*.

A labeling g is said to be an *edge irregular reflexive k-labeling* (respectively, *vertex irregular reflexive k-labeling*) if, for distinct edges e and f, $wt_g(e) \neq wt_g(f)$ (respectively, for distinct vertices u and v, $wt_g(u) \neq wt_g(v)$). The smallest value of k for which such labelings exist is called the *reflexive edge strength* (respectively, *reflexive vertex strength*).

In this paper we find exact values of reflexive edge strength for prisms, wheels, baskets and fans. Due to the similarity of the labeling schemes, many of the theorems and proofs presented here will be similar to those given in the papers [2, 3]; however, respective strengths (on the same graph) may be different. For example, $\text{res}(\text{D}_{10}) = 10$, while in [3] it was shown that $\text{tes}(\text{D}_{10}) = 11$.

When proving the subsequent result, we will frequently use the following lemma that was proved in [8].

Lemma 1.1 [8] *For every graph G,*

$$\text{res}(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil & \text{if } |E(G)| \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1 & \text{if } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

The lower bound for $\text{res}(G)$ follows from the fact that the minimal edge weight under an edge irregular reflexive labeling is 1 and the minimum of the maximal edge weights, that is, $|E(G)|$, can be achieved only as the sum of 3 numbers from which at least 2 are even.

2 Edge Irregular Reflexive Labeling of Prisms

The prism D_n, $n \geq 3$, is a trivalent graph which can be defined as the Cartesian product $P_2 \square C_n$ of a path on two vertices with a cycle on n vertices. We denote the vertex set and the edge set of D_n by $V(D_n) = \{x_i, y_i : i = 1, 2, \dots, n\}$ and $E(D_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i : i = 1, 2, \dots, n\}$, where indices are taken modulo n.

Theorem 2.1 *For $n \geq 3$,*

$$\text{res}(D_n) = \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Proof: As the prism D_n has $3n$ edges, immediately from Lemma 1.1 we get that $\text{res}(D_n) \geq n+1$ for n odd and $\text{res}(D_n) \geq n$ for n even.

Let $k = n$ for n even and let $k = n+1$ for n odd. We define the labeling f of D_n such that:

$$\begin{aligned} f(x_i) &= 0 && i = 1, 2, \dots, n, \\ f(y_i) &= k && i = 1, 2, \dots, n, \\ f(x_i x_{i+1}) = f(y_i y_{i+1}) &= i && i = 1, 2, \dots, n-1, \\ f(x_1 x_n) = f(y_1 y_n) &= n, && \\ f(x_i y_i) &= i && i = 1, 2, \dots, n. \end{aligned}$$

Evidently f is a k-labeling. The edge weights of the edges in D_n under the labeling f are the following:

$$\begin{aligned} wt_f(x_ix_{i+1}) &= 0 + i + 0 = i && \text{for } i = 1, 2, \ldots, n-1,\\ wt_f(x_1x_n) &= 0 + n + 0 = n, && \\ wt_f(x_iy_i) &= 0 + i + k = k + i && \text{for } i = 1, 2, \ldots, n,\\ wt_f(y_iy_{i+1}) &= k + i + k = 2k + i && \text{for } i = 1, 2, \ldots, n-1,\\ wt_f(y_1y_n) &= k + n + k = 2k + n. && \end{aligned}$$

This means that for n odd, $\{wt_f(e) : e \in E(D_n)\} = \{1, 2, \ldots, n, n+2, n+3, \ldots, 2n+1, 2n+3, 2n+4, \ldots, 3n+2\}$, and for n even, $\{wt_f(e) : e \in E(D_n)\} = \{1, 2, \ldots, 3n\}$. Thus the edge weights are distinct, that is, f is a edge irregular reflexive k-labeling of a prism D_n. □

3 Edge Irregular Reflexive Labeling of Wheels

The wheel W_n, $n \geq 3$, is a graph obtained by joining all vertices of C_n to a further vertex called the centre. We denote the vertex set and the edge set of W_n by $V(W_n) = \{x, x_i : i = 1, 2, \ldots, n\}$ and $E(W_n) = \{x_ix_{i+1}, xx_i : i = 1, 2, \ldots, n\}$, where indices are taken modulo n. We prove the following result for wheels.

Theorem 3.1 *For $n \geq 3$,*

$$\operatorname{res}(W_n) = \begin{cases} 4 & \text{if } n = 3,\\ \lceil \frac{2n}{3} \rceil & \text{if } n \equiv 0, 2 \pmod 3 \text{ and } n \geq 5,\\ \lceil \frac{2n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod 3). \end{cases}$$

PROOF: According to the fact that the wheel W_n has $2n$ edges, using Lemma 1.1 we obtain the following lower bound for the wheel: $\operatorname{res}(W_n) \geq k = \lceil \frac{2n}{3} \rceil$ if $n \equiv 0, 2 \pmod 3$ and $\operatorname{res}(W_n) \geq k = \lceil \frac{2n}{3} \rceil + 1$ if $n \equiv 1 \pmod 3$. It is easy to see that k is even.

Thus $\operatorname{res}(W_3) \geq 2$. If $\operatorname{res}(W_3) \leq 3$ then the vertices of W_3 can be labeled only with 0's or 2's and the set of all possible edge weights is a subset of the set $\{1, 2, \ldots, 7\}$. As the edge weights 1 and 2 could be realizable only as 1=0+1+0; 2=0+2+0, it follows that five edges have end vertex labeled with 0. However, this is a contradiction as the edge weights 6 and 7 can be realizable only as 6=2+2+2 and 7=2+3+2. Thus $\operatorname{res}(W_3) \geq 4$. The corresponding labelings for W_n, $n = 3, 4, 5, 6$, are illustrated in Figure 1.

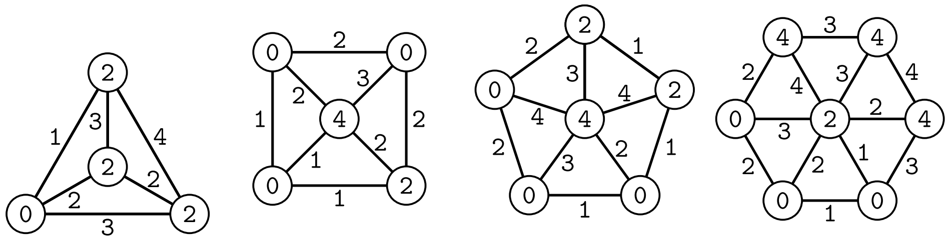

Figure 1: The reflexive edge irregular k-labeling of wheels W_n, $n = 3, 4, 5, 6$.

For $n \geq 7$ we define a total labeling of W_n such that

$$f(x) = k,$$

$$f(x_i) = \begin{cases} 0 & i = 1, 2, \dots, k-1, \\ 2 & i = k, \\ k & i = k+1, k+2, \dots, n-1, \\ k-2 & i = n, \end{cases}$$

$$f(xx_i) = \begin{cases} i & i = 1, 2, \dots, k-1, \\ k-2 & i = k, \\ n-2k+1+i & i = k+1, k+2, \dots, n-1, \\ 5 & i = n, \end{cases}$$

$$f(x_i x_{i+1}) = \begin{cases} i & i = 1, 2, \dots, k-2, \\ k-3 & i = k-1, \\ k-1 & i = k, \\ i-k+3 & i = k+1, k+2, \dots, n-2, \\ 4 & i = n-1, \\ 2 & i = n. \end{cases}$$

Evidently, for $n \geq 7$, f is a k-labeling. Now we calculate the edge weights.

$$\begin{aligned}
wt_f(x_i x_{i+1}) =& 0 + i + 0 = i && \text{for } i = 1, 2, \ldots, k-2,\\
wt_f(x_{k-1} x_k) =& 0 + (k-3) + 0 = k-1, &&\\
wt_f(x_k x_{k+1}) =& 2 + (k-1) + k = 2k+1, &&\\
wt_f(x_i x_{i+1}) =& k + (i-k+3) + k = k+3+i &&\\
& && \text{for } i = k+1, k+2, \ldots, n-2,\\
wt_f(x_{n-1} x_n) =& k + 4 + (k-2) = 2k+2, &&\\
wt_f(x_n x_1) =& (k-2) + 2 + 0 = k, &&\\
wt_f(x x_i) =& k + i + 0 = k+i && \text{for } i = 1, 2, \ldots, k-1,\\
wt_f(x x_k) =& k + (k-2) + 2 = 2k, &&\\
wt_f(x x_i) =& k + (n-2k+1+i) + k = n+1+i &&\\
& && \text{for } i = k+1, k+2, \ldots, n-1,\\
wt_f(x x_n) =& k + 5 + (k-2) = 2k+3. &&
\end{aligned}$$

It is easy to check that the edge weights are from the set $\{1, 2, \ldots, 2n\}$. This concludes the proof. $\square$

A *fan* F_n is obtained from wheel W_n if one rim edge, say $x_1 x_n$, is deleted. A *basket* B_n is obtained by removing a spoke, say xx_n, from wheel W_n. Before we give the exact value of reflexive edge strength of fans and baskets, we give the following lemma.

Lemma 3.2 *Let e be an arbitrary edge in G. Then*

$$\operatorname{res}(G - \{e\}) \;\leq\; \operatorname{res}(G).$$

Proof: The proof is trivial. $\square$

Theorem 3.3 *For $n \geq 3$,*

$$\operatorname{res}(F_n) = \operatorname{res}(B_n) = \begin{cases} 3 & \text{if } n = 3,\\ 4 & \text{if } n = 4,\\ \lceil \frac{2n}{3} \rceil & \text{if } n \geq 5. \end{cases}$$

Proof: Using similar arguments as in the proof of Theorem 3.1, we find that $\operatorname{res}(F_3) \geq 3$ and $\operatorname{res}(F_4) \geq 4$. The corresponding labeling for F_3 can be obtained from the labeling illustrated for W_3; see Figure 1, when the edge labeled with 4 is deleted. For the fan F_4 we can delete an arbitrary rim edge.

For $n \geq 5$, combining Lemmas 1.1 and 3.2 we have $\operatorname{res}(F_n) = \operatorname{res}(W_n) = \lceil \frac{2n}{3} \rceil$ for $n \equiv 0, 2 \pmod 3$ and $\lceil \frac{2n}{3} \rceil \leq \operatorname{res}(F_n) \leq \lceil \frac{2n}{3} \rceil + 1$ for $n \equiv 1 \pmod 3$.

Let $n \equiv 1 \pmod 3$, $n \geq 7$. Then the number $k = \lceil \frac{2n}{3} \rceil$ is odd. We define a k-labeling of F_n such that:

$$f(x) = k - 1,$$

$$f(x_i) = \begin{cases} k-1 & i = 1, \\ k-3 & i = 2, \\ 0 & i = 3, 4, \dots, k, \\ 2 & i = k+1, \\ k-1 & i = k+2, k+3, \dots, n, \end{cases}$$

$$f(xx_i) = \begin{cases} 4 & i = 1, \\ 5 & i = 2, \\ i-2 & i = 3, 4, \dots, k, \\ k-3 & i = k+1, \\ 2i-2k+1 & i = k+2, k+3, \dots, n, \end{cases}$$

$$f(x_ix_{i+1}) = \begin{cases} 4 & i = 1, \\ 2 & i = 2, \\ i-2 & i = 3, 4, \dots, k-1, \\ k-4 & i = k, \\ k-2 & i = k+1, \\ 2i-2k+2 & i = k+2, k+3, \dots, n-1. \end{cases}$$

It is not difficult to check that the edge weights are distinct numbers from the set $\{1, 2, \dots, 2n-1\}$.

The proof for the basket B_n can be done analogously as for the fan. Evidently $V(B_n) = V(F_n)$ and $E(B_n) = E(F_n) \cup \{x_1x_n\} - \{xx_n\}$. For $n \equiv 1 \pmod 3$, $n \geq 7$, the following $\lceil \frac{2n}{3} \rceil$-labeling g of B_n, defined such that $g(y) = f(y)$ for $y \in V(F_n)$ or $y \in E(F_n) - \{xx_n\}$ and $g(x_1x_n) = f(xx_n)$, has the desired properties. $\square$

4 Conclusion

In this paper we have described the reflexive edge strength for several classes of graphs. For further investigation we suggest solving the corresponding problem for the reflexive vertex strength of these graphs.

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