UNCOVERED NODES AND 3-CYCLES IN TOURNAMENTS

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Abstract. Let $c(T_n)$ denote the number of 3-cycles in the tournament T_n and let $u(T_n)$ denote the number of nodes i in T_n such that each arc oriented towards i belongs to at least one 3-cycle. We determine the minimum value of $c(T_n)$ when $u(T_n) = n$ and the maximum value of $c(T_n)$ when $u(T_n) = 3$.

1. Introduction. A *tournament* T_n consists of a set of n *nodes* $1, 2, \ldots, n$ such that each pair of distinct nodes i and j is joined by exactly one of the *arcs* $\overrightarrow{ij}$ or $\overrightarrow{ji}$. If the arc $\overrightarrow{ij}$ is in T_n we say that i *beats* j or that j *loses* to i and write $i \to j$. If each node of a subtournament A beats each node of a subtournament B we write $A \to B$. For definitions not given here or for additional material on tournaments, see [12] or [15].

Node i is said to *cover* node j if node i beats every node that node j beats or, equivalently, if $i \to j$ and the arc $\overrightarrow{ij}$ belongs to no 3-cycle. It is not difficult to see that the covering relation thus defined is transitive [10; p. 72]. So, as pointed out in [10], every finite tournament has at least one uncovered node (a result originally proved by another argument in [8; p. 148]). In fact, every strong tournament T_n with $n \geq 3$ nodes has at least three uncovered nodes (*cf.* [17], [11], [15; p. 178], [9] or [10]). We observe that node i is an uncovered node if and only if every arc oriented towards i belongs to at least one 3-cycle or, equivalently, if for any other node j there exists a path from i to j of length at most two; nodes with this property have been called *kings* in several recent papers (*cf.* [9], [2], [18], [7] or [4]).

Miller [10] has shown that if the tournament T_n represents majority preferences between a set of n proposals, then various voting procedures will always select a proposal from the set of uncovered nodes of T_n (see also [16], [1], [14], or [5]) for additional material involving uncovered nodes in this context). Let $u(T_n)$ denote the number of uncovered nodes in the tournament T_n. Miller [10; p. 78] remarked that the size of $u(T_n)$, in a strong tournament T_n, "depends largely on the degree of intransitivity within $[T_n]$, which in turn may be measured by the proportion of all triples of [nodes in T_n] that are cyclic $\cdots$. With some complications, the pattern is this: as intransitivity $\cdots$ declines, $[u(T_n)]$ approaches or equals 3; as it increases, $[u(T_n)]$ approaches or equals $[n]$." Our main object here is to determine the minimum number of 3-cycles possible in a tournament T_n when $u(T_n) = n$ and the maximum number of 3-cycles possible when $u(T_n) = 3$.

2. Preliminary Remarks. The *score* s_i of a node i in a tournament T_n is the number of nodes beaten by i and the *score sequence* of T_n is the sequence $\bar{s} = (s_1, s_2, \ldots, s_n)$; the sum of the scores is clearly $n(n-1)/2$. We let R_n denote any *regular* tournament with n nodes, that is, a tournament in which $s_i = (n-1)/2$ for all nodes if n is odd, or $s_i = n/2 - 1$ for half the nodes and $n/2$ for the other half if n is even.

It is well-known and easy to see that the number $c(T_n)$ of 3-cycles in a tournament T_n with score sequence $\bar{s}$ is given by the formula

$$c(T_n) = \binom{n}{3} - \sum_{1}^{n} \binom{s_i}{2}; \tag{2.1}$$

from this it follows readily that

$$c(T_n) \leq \gamma(n) := \begin{cases} (n^3 - n)/24 & \text{if } n \text{ is odd} \\ (n^3 - 4n)/24 & \text{if } n \text{ is even,} \end{cases} \tag{2.2}$$

with equality holding if and only if T_n is a regular tournament R_n (see, *e.g.*, [12; p. 9] or [15; p. 186]). Landau [8] showed that every node of maximum score in a tournament T_n is an uncovered node. Miller [10; p. 80] pointed out that this and result (2.2) imply that if $c(T_n) = \gamma(n)$ then $u(T_n) = n$ if n is odd, and $n/2 \leq u(T_n) \leq n$ if n is even. When $n = 2m$ the upper bound here can be realized for all $m \geq 3$ and the lower bound can be realized if and only if m is odd; if $n = 4m$ then tournaments T_n such that $c(T_n) = \gamma(n)$ and $u(T_n) = n/2 + 1$ exist for all $m \geq 1$. In Section 4 we show that if $u(T_n) = n$ then the minimum possible value of $c(T_n)$ is a quadratic in n, with leading term $n^2/4$, and we determine the minimal tournaments.

In the other direction, if T_n is a strong tournament with $n \geq 3$ nodes, then $c(T_n) \geq n-2$ [6; p. 306]. Burzio and Demaria [3] (see also [13]) have recently characterized the tournaments for which equality holds here and, as it turns out, all these tournaments have exactly three uncovered nodes. In Section 5 we show that if $u(T_n) = 3$, then the maximum possible value of $c(T_n)$ is a cubic in n, whose first two terms are $(n^3 - n^2)/24$, and we characterize the maximal tournaments.

3. An Inequality. Let $\overline{s} = (s_1, s_2, \ldots, s_n)$ be the score sequence of a tournament T_n such that $u(T_n) = n$. We may suppose the nodes are labelled so that

$$(3.1) \qquad s_1 \leq s_2 \leq \cdots \leq s_n.$$

Let S_j and e_j be such that $S_j = s_1 + \cdots + s_j = j(j-1)/2 + e_j$ for $1 \leq j \leq n$; then $e_j \geq 0$ since there are $j(j-1)/2$ arcs joining any j nodes. Suppose that $j + e_j \leq n-1$ and that $e_j \leq j-1$ for some j. Then there would be a node $p \in \{j+1, \ldots, n\}$ that beats all nodes in $\{1, \ldots, j\}$ and a node $q \in \{1, \ldots, j\}$ that loses to all nodes in $\{j+1, \ldots, n\}$; but then node p would cover node q, contrary to the assumption that $u(T_n) = n$. Consequently, if $u(T_n) = n$, then

$$(3.2) \qquad S_j \geq \binom{j}{2} + \min\{j, n-j\} = \binom{j+1}{2} + \min\{0, n-2j\}$$

for $1 \leq j \leq n-1$, and

$$(3.3) \qquad S_n = \binom{n}{2}.$$

We now derive an inequality involving such sequences $\overline{s}$ that we shall use in the next section to determine the minimum value of $c(T_n)$ if $u(T_n) = n$.

LEMMA 1. *Let $\overline{s} = (s_1, \ldots, s_n)$ denote a sequence of $n \geq 5$ integers satisfying conditions (3.1)-(3.3). Let $\overline{x} = (1, 2, \ldots, m-1, m, m, m, m+1, \ldots, n-2)$ if $n = 2m+1 \geq 5$ and let $\overline{y} = (1, 2, \ldots, m-1, m, m, m, m, m+2, \ldots, n-2)$ and $\overline{z} = (1, 2, \ldots, m-3, m-1, m-1, m-1, m-1, m, \ldots, n-2)$ if $n = 2m \geq 8$; finally, let $\overline{y} = (1, 2, 3, 3, 3, 3)$ and $\overline{z} = (2, 2, 2, 2, 3, 4)$ if $n = 6$. Then*

$$(3.4) \qquad f(\overline{s}) := \sum_{1}^{n} \binom{s_i}{2} \leq \binom{n-1}{3} + \begin{cases} (n-1)(n-3)/4 & \text{if } n = 2m+1 \\ n(n-4)/4 & \text{if } n = 2m, \end{cases}$$

with equality holding if and only if $\overline{s} = \overline{x}$ when n is odd or $\overline{s} = \overline{y}$ or $\overline{z}$ when n is even.

PROOF: It is easy to verify that equality holds in (3.4) when $\overline{s} = \overline{x}$, $\overline{y}$, or $\overline{z}$. Let us assume that $\overline{s}$ is a maximal sequence, *i.e.*, a sequence for which $f(\overline{s})$ assumes its maximum value over the set of all sequences satisfying (3.1)-(3.3). Suppose there exists a least integer k such that strict inequality holds in (3.2) when $j = k$. We assume, initially, that $1 \leq k \leq m$ so that

$$S_k = \binom{k+1}{2} + \alpha \tag{3.5}$$

where $\alpha \geq 1$. Then $s_1 = 1 + \alpha$ if $k = 1$; and if $k \geq 2$ then it follows from the definition of k and the relation $S_j = S_{j-1} + s_j$ that $s_j = j$ for $1 \leq j \leq k-1$, and that

$$s_k = k + \alpha > s_{k-1}. \tag{3.6}$$

Let h denote the largest integer such that $s_{k+1} = \cdots = s_{k+h}$; then

$$s_{k+h} < s_{k+h+1} \tag{3.7}$$

if $k + h < n$. We now show that – apart from one exceptional case –

$$S_{k+u} > \binom{k+u+1}{2} + \min\{0, n - 2k - 2u\} \tag{3.8}$$

for $1 \leq u \leq h-1$, assuming that $h \geq 2$.

We observe that if $s_{k+1} = \sigma$, then

$$S_{k+u} = S_k + u\sigma = \binom{k+1}{2} + \alpha + u\sigma,$$

so (3.8) holds if and only if

$$u\{\sigma - k - (u+1)/2\} + \alpha + \max\{0, 2k + 2u - n\} > 0. \tag{3.9}$$

It follows from (3.2), (3.5), and the definition of h that

$$\begin{aligned} w\sigma = S_{k+w} - S_k &\geq \binom{k+w+1}{2} + \min\{0, n - 2k - 2w\} - \binom{k+1}{2} - \alpha \\ &= w\{k + (w+1)/2\} - \alpha - \max\{0, 2k + 2w - n\} \end{aligned}$$

for $1 \leq w \leq h$. So, in particular,

$$\sigma \geq k + (h+1)/2 - \alpha/h - h^{-1} \cdot \max\{0, 2k + 2h - n\}. \tag{3.10}$$

Moreover, if $m+1 \le k+h$ and $v := m+1-k$, then

$$\sigma \ge k + (v+1)/2 - v^{-1} \cdot (2m+2-n+\alpha). \tag{3.11}$$

We now apply these estimates in (3.9), considering three cases separately.

Case 1: $\boldsymbol{k+u<k+h\le m}$. Let L denote the left hand side of inequality (3.9). In this case it follows from (3.10) that

$$L \ge u\{(h-u)/2 - \alpha/h\} + \alpha \ge u/2 + \alpha/h > 0,$$

as required.

Case 2: $\boldsymbol{k+u \le m < k+v = m+1 \le k+h}$. Notice that $v \ge u+1 \ge 2$ here. In this case it follows from (3.11) that

$$L \ge u\{(v-u)/2 - v^{-1} \cdot (2m+2-n+\alpha)\} + \alpha \ge u/2 + \alpha/v - u(2m+2-n)/v.$$

If $n = 2m+1$, then

$$L \ge u/2 + \alpha/v - u/v \ge \alpha/v > 0,$$

as required. If $n = 2m$ and $v \ge 4$, which is certainly the case if $u \ge 3$, then

$$L \ge u/2 + \alpha/v - 2u/v \ge \alpha/v > 0.$$

Moreover, it follows from the inequality $\sigma = s_{k+1} \ge s_k = k + \alpha$ that

$$L \ge u\{\alpha - (u+1)/2\} + \alpha = (u+1)(\alpha - u/2),$$

so $L > 0$ if $u = 1$ or $u = 2$ and $\alpha \ge 2$. Thus we find that $L > 0$ here *except* when $n = 2m$, $\alpha = 1$, $u = 2$, $v = 3$, and $s_{k+1} = k+1$; in this exceptional case $k = m+1-v = m-2$ and $(s_1, \dots, s_{m+1}) =$ $(1, 2, \dots, m-3, m-1, m-1, m-1, m-1)$.

Case 3: $\boldsymbol{m+1 \le k+u < k+h}$. In this case it follows from (3.10) that

$$\begin{aligned} L &\ge u\{(h-u)/2 - (\alpha+2k-n)/h\} + (\alpha+2k-n) \\ &= h^{-1}(h-u)\{\alpha + 2u + 2k - n + u(h-4)/2\} \\ &\ge h^{-1}(h-u)\{\alpha + 1 + u(h-4)/2\}, \end{aligned}$$

so L is certainly positive if $h \geq 4$; and if $2 \leq h \leq 3$, then

$$\begin{aligned} L &\geq h^{-1}(h-u)\{\alpha+1+(h-1)(h-4)/2\} \\ &= h^{-1}(h-u)\{\alpha+(h-2)(h-3)/2\} \geq \alpha/h > 0, \end{aligned}$$

as required. Thus it follows that inequalities (3.9) and (3.8) hold apart from the one exceptional case.

Suppose we are not in this exceptional case and let $\bar{r} = (r_1, \ldots, r_n)$ denote the integer sequence in which $r_i = s_i$ except that $r_k = s_k - 1$ and $r_{k+h} = s_{k+h}+1$. Then $\bar{r}$ clearly satisfies condition (3.3); and $\bar{r}$ also satisfies conditions (3.1) and (3.2) in view of inequalities (3.6) and (3.7) and relations (3.5) and (3.8), respectively. But

$$\begin{aligned} f(\bar{r}) - f(\bar{s}) &= \binom{s_{k+h}+1}{2} - \binom{s_{k+h}}{2} + \binom{s_k - 1}{2} - \binom{s_k}{2} \\ &= s_{k+h} - s_k + 1 \geq 1, \end{aligned} \tag{3.12}$$

contrary to the assumption that $\bar{s}$ is a maximal sequence.

It follows, therefore, that if $\bar{s}$ is a maximal sequence then either strict equality holds in (3.2) for $1 \leq j \leq m$ and

$$(s_1, \ldots, s_m) = (1, 2, \ldots, m), \tag{3.13}$$

or $\bar{s}$ involves the exceptional case encountered earlier and $n = 2m$ and

$$(s_1, \ldots, s_{m+1}) = (1, 2, \ldots, m-3, m-1, m-1, m-1, m-1). \tag{3.14}$$

(This last sequence is to be interpreted as $(2,2,2,2)$ if $m = 3$.) And, similarly, it follows by duality that if $\bar{s}$ is a maximal sequence, then either $(s_{n-m}, \ldots, s_n) = (n-m-1, n-m, \ldots, n-2)$ — that is,

$$(s_{m+2}, \ldots, s_n) = (m, m+1, \ldots, n-2) \tag{3.15}$$

if $n = 2m+1$ or

$$(s_{m+1}, \ldots, s_n) = (m-1, m, \ldots, n-2) \tag{3.16}$$

if $n = 2m$ — or $n = 2m$ and

$$(s_{m-1}, \ldots, s_n) = (m, m, m, m, m+2, \ldots, n-2). \tag{3.17}$$

If $n = 2m+1$ and we combine (3.13) and (3.15), we find that s_{m+1} must equal m, in view of (3.1), so $\overline{s} = \overline{x}$. If $n = 2m$ and we combine the only compatible alternatives, namely, (3.13) and (3.17) or (3.14) and (3.16), we find that $\overline{s} = \overline{y}$ or $\overline{z}$. This suffices to complete the proof of the lemma.

4. Minimal Tournaments T_n with $u(T_n) = n$. If $u(T_6) = 6$ for a tournament T_6, then T_6 must have score sequence $(2,2,2,3,3,3)$; this follows from an argument that will be given later. There are five non-isomorphic tournaments with this score sequence (*cf.* [12; p. 95]) and of these only the following three have the property that $u(T_6) = 6$: (i) the tournament T_6 consisting of two disjoint 3-cycles (A,B,C) and (c,b,a) such that $(A,B,C) \to (c,b,a)$ except that $a \to A$, $b \to B$, and $c \to C$; (ii) the tournament T_6 with nodes $1,2,\ldots,6$ in which $j \to i$ if $j > i$ except that $1 \to 5$, $1 \to 6$, $2 \to 4$, and $4 \to 6$; and (iii) the dual of the tournament described in (ii). Let M_1 denote the trivial tournament with just one node and let M_6 denote any one of the three tournaments just described. More generally, if $n = 3$ or 5 or $n \geq 7$ let M_n denote any tournament obtained from any tournament M_{n-2} by adjoining two nodes p and q such that $p \to q$, $q \to M_{n-2}$, and $M_{n-2} \to p$. It is not difficult to verify that $u(M_n) = n$ for any such tournament M_n. We now show that among all tournaments T_n such that $u(T_n) = n$ these are the minimal tournaments, that is, the tournaments with the minimum number of 3-cycles.

THEOREM 1. *Let T_n be a tournament with $n \neq 2,4$ nodes such that $u(T_n) = n$. Then*

$$c(T_n) \geq \begin{cases} (n-1)^2/4 & \text{if } n \text{ is odd} \\ (n^2-2n+8)/4 & \text{if } n \text{ is even,} \end{cases}$$

with equality holding if and only if T_n is one of the tournaments M_n.

PROOF: We may suppose that $n \geq 5$ since the result certainly holds when $n = 1$ or 3. And, as we saw earlier, we may suppose the score sequence $\overline{s}$ of T_n satisfies conditions (3.1)-(3.3) so that, in particular, $s_1 \geq 1$ and $s_n \leq n-2$.

We consider first the case when $2 \leq s_1 \leq \cdots \leq s_n \leq n-3$. If $n = 6$ then $\overline{s} = (2,2,2,3,3,3)$ so $c(T_6) = 8$, by (2.1), and T_6 is one of the tournaments M_6, in view of the earlier observation. So we may now suppose that $n = 5$ or $n \geq 7$. If n is odd then $\overline{s}$ is certainly not the sequence $\overline{x}$ described in

Lemma 1; then in this case it follows from (2.1) and Lemma 2 that

$$\begin{aligned} c(T_n) &= \binom{n}{3} - f(\bar{s}) \\ &> \binom{n}{3} - f(\bar{x}) \\ &= \binom{n}{3} - \binom{n-1}{3} - (n-1)(n-3)/4 = (n-1)^2/4, \end{aligned}$$

as required.

If $n \geq 8$ is even then $\bar{s}$ is neither of the sequences $\bar{y}$ or $\bar{z}$ described in Lemma 1, so $f(\bar{s}) < f(\bar{y}) = f(\bar{z})$. Now $\bar{s}$ can be transformed into one of the sequences $\bar{y}$ or $\bar{z}$ by a series of exchanges each of which involves replacing two elements s_k and s_{k+h} by $s_k - 1$ and $s_{k+h}+1$, respectively, where k and h are as defined in the proof of Lemma 1. Each such exchange increases the value of the sum $f(\bar{s})$ by $s_{k+h} - s_k + 1 \geq 1$. Thus it follows from (3.12) that $f(\bar{s}) + 1 \leq f(\bar{y}) = f(\bar{z})$ with equality holding only if $\bar{s}$ can be transformed into $\bar{y}$ or $\bar{z}$ by a *single* exchange that involves replacing two *equal* elements s_k and s_{k+h} by $s_k - 1$ and $s_{k+h} + 1$. Now $y_1 = z_1 = 1$ and $y_n = z_n = n-2$; hence, if $\bar{s}$ can be so transformed, it must be that $k = 1$, $k + h = n$, $s_1 = 2$, and $s_n = n - 3$. But we are assuming that $n \geq 8$ here, so s_1 cannot equal s_n and, consequently, $\bar{s}$ cannot be so transformed into $\bar{y}$ or $\bar{z}$ by a single exchange.

We conclude, therefore, that if n is even, $n \geq 8$, and $2 \leq s_1 \leq s_n \leq n-3$, then $f(\bar{s}) + 1 < f(\bar{y}) = f(\bar{z})$. So in this case it follows from (2.1) and Lemma 1 that

$$\begin{aligned} c(T_n) &= \binom{n}{3} + f(\bar{s}) \\ &> 1 + \binom{n}{3} - f(\bar{y}) \\ &= 1 + \binom{n}{3} - \binom{n-1}{3} - n(n-4)/4 \\ &= (n^2 - 2n + 8)/4, \end{aligned}$$

as required.

It remains to consider the case when $s_1 = 1$ or $s_n = n-2$. If there is a node p of score 1 let q denote the node that loses to p and let T_{n-2} denote the subtournament determined by the remaining nodes, so that $p \to q$

and $T_{n-2} \to p$. If node q lost to some node w of T_{n-2} then w would cover node p, contrary to our hypothesis; it follows, therefore, that $q \to T_{n-2}$ so q has score $n-2$. Similarly, if we initially assume there is a node of score $n-2$ we find that the node that beats this node has score 1. Thus if $s_1 = 1$ or $s_n = n-2$ there exist nodes p and q of score 1 and $n-2$, respectively, such that T_n has the structure described above; and, in this case, it follows readily that

$$c(T_n) = n - 2 + c(T_{n-2}). \tag{4.1}$$

The subtournament T_{n-2} is such that $u(T_{n-2}) = n-2$; for if in T_{n-2} some node v covers some node w, then v clearly covers w in T_n as well, contrary to our hypothesis. We now observe that there is no tournament T_4 such that $u(T_4) = 4$ (since there is no tournament T_2 such that $u(T_2) = 2$). Consequently, the only tournaments T_6 such that $u(T_6) = 6$ are those with score sequence $(2,2,2,3,3,3)$ that were discussed earlier. So, in completing the argument for the case when $s_1 = 1$ and $s_n = n-2$, we may assume that $n = 5$ or $n \geq 7$ and that the required result has already been proved when n is replaced by $n-2$. Thus it follows from (4.1), the fact that $u(T_{n-2}) = n-2$, and the induction hypothesis, that

$$c(T_n) \geq n - 2 + \begin{cases} (n-3)^2/4 = (n-1)^2/4 & \text{if } n \text{ is odd} \\ (n-2)(n-4)/4 + 2 = (n^2 - 2n + 8)/4 & \text{if } n \text{ is even,} \end{cases}$$

with equality holding if and only if T_{n-2} is one of the tournaments M_{n-2}; that is, if and only if T_n is one of the tournaments M_n. This suffices to complete the proof of the theorem.

5. Minimizing Certain Sums.

In the next section we shall make use of the following slight extension of a familiar result on the minimum value of a sum $\sum_1^n \binom{w_i}{2}$, subject to the condition that the w_i's are non-negative integers having a fixed sum.

LEMMA 2. *Let $J, K, j,$ and k be given positive integers such that*

$$\lceil J/j \rceil \leq \lfloor K/k \rfloor. \tag{5.1}$$

For any integer D *such that* $0 \le D \le J$, *let* $h(D)$ *denote the minimum value of the sum*

$$f(\overline{w}) = \sum_{1}^{j+k} \binom{w_i}{2}$$

over all sequences $\overline{w} = (w_1, \ldots, w_{j+k})$ *of* $j+k$ *non-negative integers such that*

$$w_1 + \cdots + w_j = J - D \tag{5.2}$$

and

$$w_{j+1} + \cdots + w_{j+k} = K + D. \tag{5.3}$$

Then $h(D)$ *is a strictly increasing function of* D; *furthermore, if* $\overline{w}$ *satisfies conditions (5.2) and (5.3), then* $f(\overline{w}) = h(D)$ *if and only if the integers* $w_1, \ldots, w_j$ *are as nearly equal as possible and the integers* $w_{j+1}, \ldots, w_{j+k}$ *are as nearly equal as possible.*

PROOF: If $1 \le D \le J$ let $\overline{w}$ be a sequence that satisfies (5.2) and (5.3) and is such that $f(\overline{w}) = h(D)$. We may suppose that

$$w_1 \le \lfloor (J-D)/j \rfloor \le \lceil J/j \rceil - 1 \quad \text{and} \quad w_{j+k} \ge \lceil (K+D)/k \rceil \ge \lfloor K/k \rfloor + 1$$

so that $w_{j+k} \ge w_1 + 2$, by (5.1). Let $\overline{w}'$ denote the sequence that differs from $\overline{w}$ only in that $w_1' = w_1 + 1$ and $w_{j+k}' = w_{j+k} - 1$. Then $\overline{w}'$ satisfies (5.2) and (5.3) with D replaced by $D-1$. Moreover,

$$h(D) = f(\overline{w}) = f(\overline{w}\,') + w_{j+k} - w_1 - 1 \ge f(\overline{w}\,') + 1 \ge h(D-1) + 1.$$

This proves the first part of the required conclusion; and the last part follows readily upon considering the subsequences $(w_1, \ldots, w_j)$ and $(w_{j+1}, \ldots, w_{j+k})$ separately.

6. Maximal Tournaments T_n with $u(T_n) = 3$. If $n \ge 3$ let T_X, T_Y, and T_Z denote (possibly empty) tournaments with X, Y and Z nodes such that $X + Y + Z = n - 3$. Let $Q = Q(T_X, T_Y, T_Z)$ denote the tournament consisting of disjoint copies of T_X, T_Y, and T_Z plus three additional nodes

$x, y,$ and z such that $x \to T_X$, $y \to T_Y$, $z \to T_Z$, $\{x\} \cup T_X \to \{y\} \cup T_Y$, $\{y\} \cup T_Y \to \{z\} \cup T_Z$, and $\{z\} \cup T_Z \to \{x\} \cup T_X$. The only uncovered nodes in any such tournament Q are the nodes $x, y,$ and z. We now show that among the class of tournaments T_n such that $u(T_n) = 3$, the maximal tournaments – that is, the tournaments with the maximum number of 3-cycles – are a certain subset of these tournaments Q.

THEOREM 2. *Let* T_n *be a tournament with* $n \geq 3$ *nodes such that* $u(T_n) = 3$. *Then* $c(T_n) \leq C(n)$, *where*

$$24C(n) = \begin{cases} n(n^2 - n + 2) & n = 6m \\ (n-1)(n^2-1) & n = 6m+1 \\ n(n-2)(n+1) & n = 6m+2 \\ n^3 - n^2 - n + 9 & \textit{if} \quad n = 6m+3 \\ (n+2)(n^2 - 3n + 4) & n = 6m+4 \\ (n-1)(n^2-1) & n = 6m+5. \end{cases}$$

Furthermore, $c(T_n) = C(n)$ *if and only if* T_n *is a tournament of the form* $Q(R_X, R_Y, R_Z)$ *where (i)* $R_X, R_Y,$ *and* R_Z *are regular tournaments with* $X, Y,$ *and* Z *nodes, (ii)* $X + Y + Z = n - 3$, *and (iii)* $X, Y,$ *and* Z *differ from each other by at most one.*

PROOF: The theorem certainly holds when $n = 3$ or 4, so we may assume that $n \geq 5$. Let $x, y,$ and z denote the uncovered nodes of the tournament T_n. We may suppose that $x \to y$ and $y \to z$. If $x \to z$ then, since x does not cover z, there must be a fourth node v such that $z \to v$ and $v \to x$; but then none of the nodes $x, y,$ or z would cover v which, since the covering relation is transitive, contradicts the assumption that $u(T_n) = 3$. Consequently, $z \to x$ and the uncovered nodes $x, y,$ and z form a 3-cycle, (x, y, z) say.

Each node $v \notin \{x, y, z\}$ is covered by at least one of the three uncovered nodes $x, y,$ and z. If node x, say, covers such a node v then $x \to v$ and, in addition, $z \to v$; for, $z \to x$ and if $v \to z$, then x would not cover v. Thus each such node v loses to at least two of the nodes $x, y,$ and z, namely, a node that covers v and the immediate predecessor of the covering node in the 3-cycle (x, y, z).

Let T_X denote the (possibly empty) subtournament of T_n determined by those nodes v such that (i) v is covered by x and hence loses both to x and to z, the predecessor of x in the 3-cycle (x, y, z), but (ii) v beats y, the successor of x in the 3-cycle (x, y, z). Let T_Y and T_Z be similarly defined with respect to nodes y and z and, finally, let T_D denote the subtournament determined by the remaining nodes v that lose to all three nodes $x, y,$ and z. Then $X+Y+Z+D=n-3$ since each node $v \notin \{x,y,z\}$ belongs to exactly one of these four subtournaments.

If there were nodes $v \in T_X$ and $w \in T_Y$, say, such that $w \to v$, then (w,v,y) would be a 3-cycle containing the arc $\overrightarrow{yw}$, contrary to the assumption that y covers w. Consequently, $T_X \to T_Y$ and, similarly, $T_Y \to T_Z$ and $T_Z \to T_X$. The foregoing observations imply that T_n contains a subtournament $Q(T_X, T_Y, T_Z)$ where $X+Y+Z=n-3-D$ plus the (disjoint) subtournament T_D of nodes that lose to all three of the nodes $x, y,$ and z.

Let $s_x, s_y,$ and s_z denote the scores of the nodes $x, y,$ and z in the tournament T_n and let $s_1, s_2, \ldots, s_{n-3}$ denote the scores of the remaining nodes. It follows from what we have deduced about the structure of T_n that

$$\begin{aligned} s_x + s_y + s_z &= (n-2-Z)+(n-2-X)+(n-2-Y) \\ &= 3(n-2)-(X+Y+Z) = 2n-3+D; \end{aligned} \tag{6.1}$$

furthermore,

$$s_1 + \cdots + s_{n-3} = \binom{n}{2} - s_x - s_y - s_z = \binom{n-2}{2} - D. \tag{6.2}$$

It is not difficult to verify that the sequence $\overline{s} = (s_1, \ldots, s_{n-3}, s_x, s_y, s_z)$ satisfies the hypothesis of Lemma 2 with $J=(n-2)(n-3)/2$, $K=2n-3$, $j=n-3$, and $k=3$. Hence we conclude that a lower bound for the sum

$$f(\overline{s}) = \sum_{1}^{n-3} \binom{s_i}{2} + \binom{s_x}{2} + \binom{s_y}{2} + \binom{s_z}{2}$$

is obtained by evaluating the right hand side when the integers s_x, s_y, s_z are as nearly equal as possible and the integers $s_1, \ldots, s_{n-3}$ are as nearly equal as possible, subject to conditions (6.1) and (6.2) with $D=0$.

More specifically, suppose that $n=6m+5$; then $2n-3=12m+7$ and

$$\binom{s_x}{2} + \binom{s_y}{2} + \binom{s_z}{2} \geq 2\binom{4m+2}{2} + \binom{4m+3}{2} = (n-2)(2n-5)/3,$$

with equality holding if and only if $s_x, s_y,$ and s_z equal $4m+2$, $4m+2$, and $4m+3$ or, equivalently, $X, Y,$ and Z equal $2m$, $2m+1$, and $2m+1$ (in some order). Furthermore, $n-2=6m+3$ and $n-3=6m+2$, so

$$\sum_{1}^{n-3}\binom{s_i}{2} \geq (3m+1)\binom{3m+1}{2}+(3m+1)\binom{3m+2}{2}=(n-3)^3/8,$$

with equality holding if and only if half the scores $s_1,\ldots,s_{n-3}$ equal $3m+1$ and the other half equal $3m+2$. Thus it follows from (2.1) that if $n=6m+5$, then

$$\begin{aligned} c(T_n) &= \binom{n}{3} - f(\overline{s}) \\ &\leq \binom{n}{3} - (n-2)(2n-5)/3-(n-3)^3/8 \\ &= (n^2-1)(n-1)/24 = C(6m+5). \end{aligned}$$

Moreover, equality holds if and only if $D=0$ and T_n is a tournament of the form $Q(T_X,T_Y,T_Z)$ where $X,Y,$ and Z equal $2m$, $2m+1$, and $2m+1$; and half the nodes in the subtournaments T_X, T_Y, and T_Z have score $3m+1$ in T_n and the other half have score $3m+2$. It is not difficult to see that this last condition on the scores is satisfied if and only if all the subtournaments T_X, T_Y, and T_Z are regular. This suffices to prove the required result when $n=6m+5$, and the same type of argument covers the cases $n\equiv 0,1,$ or $3 \pmod 6$ as well. (The cases $n\equiv 0$ or $3 \pmod 6$, are particularly easy.)

If, however, $n\equiv 2$ or $4 \pmod 6$, then the foregoing argument yields an upper bound for $c(T_n)$ that is not best possible. For, to realize the bound in these cases, the nodes in the subtournaments T_X, T_Y, and T_Z would all have to have the same score and this is not possible here. Thus we need some additional arguments in these two remaining cases.

Suppose that $n=6m+2$ where $m\geq 1$. If $D\geq 1$ for the tournament T_n, then it follows readily from Lemma 2 that

$$\begin{aligned} f(\overline{s}) &\geq \binom{4m}{2}+2\binom{4m+1}{2}+\binom{3m-1}{2}+(6m-2)\binom{3m}{2} \\ &= n(n-2)(3n-5)/24+1, \end{aligned}$$

so

$$c(T_n)=\binom{n}{3}-f(\overline{s})\leq n(n-2)(n+1)/24-1=C(6m+2)-1.$$

Similarly, if $n = 6m + 4$ and $D \geq 1$, we find that $c(T_n) \leq C(6m+4) - 1$. So we may assume henceforth that $n = 6m + 2$ or $6m + 4$, where $m \geq 1$, and that $D = 0$, that is, that T_n is a tournament of the form $Q(T_X, T_Y, T_Z)$ where $X + Y + Z = n - 3$. We next dispose of the possibility that $\max\{X, Y, Z\}$ exceeds $4m$, say.

If n is even – in particular, if $n \equiv 2$ or $4 \pmod 6$ – then

$$\sum_{1}^{n-3} \binom{s_i}{2} \geq (n-3)\binom{(n-2)/2}{2} = (n-2)(n-3)(n-4)/8.$$

Now suppose that $n = 6m + 2$, where $m \geq 1$, so that $X + Y + Z = 6m - 1$ and $s_x + s_y + s_z = 2n - 3 = 12m + 1$. If $\max\{X, Y, Z\} \geq 4m + 1$, then $\min\{s_x, s_y, s_z\} \leq 2m - 1$ and

$$\binom{s_x}{2} + \binom{s_y}{2} + \binom{s_z}{2} \geq \binom{2m-1}{2} + 2\binom{5m+1}{2} = (9n^2 - 32n + 40)/12,$$

appealing to Lemma 2 again. Hence, in this case,

$$\begin{aligned} c(T_n) &= \binom{n}{3} - f(\overline{s}) \\ &\leq \binom{n}{3} - (n-2)(n-3)(n-4)/8 - (9n^2 - 32n + 40)/12 \\ &= (n^3 - 3n^2 - 6n - 8)/24 \\ &= C(6m+2) - (n^2 + 2n + 4)/12 < C(6m+2). \end{aligned}$$

Similarly, we find that if $n = 6m + 4$, where $m \geq 1$, and $\max\{X, Y, Z\} \geq 4m + 1$, then

$$c(T_n) \leq (n^3 - 3n^2 + 10n - 8)/24 = C(6m+4) - (n-2)(n-4)/12 < C(6m+4).$$

Thus we may further assume, from now on, that $\max\{X, Y, Z\} \leq 4m \leq 2(n-2)/3$.

It follows readily from the definition of $Q(T_X, T_Y, T_Z)$ that

$$c(T_n) = (X+1)(Y+1)(Z+1) + c(T_X) + c(T_Y) + c(T_Z).$$

We mentioned earlier, in (2.2), that $c(T_N) \leq \gamma(N)$, where $24\gamma(N) = N^3 - N$ or $N^3 - 4N$ according as N is odd or even, with equality holding if and only if T_N is a regular tournament R_N. Consequently,

$$c(T_n) \le \Gamma(X, Y, Z)$$

where

$$\Gamma(X, Y, Z) = (X+1)(Y+1)(Z+1) + \gamma(X) + \gamma(Y) + \gamma(Z),$$

with equality holding if and only if T_n is of the form $Q(R_X, R_Y, R_Z)$. So it remains to determine the values of $X, Y,$ and Z, where $X + Y + Z = n - 3$, for which the function $\Gamma(X, Y, Z)$ attains its maximum value. We need consider only the cases when $Z \le Y \le X \le 2(n-2)/3$.

Notice that it follows from the definition of the function $\gamma(N)$ that

$$((N-1)^2 - 1)/8 \le \gamma(N) - \gamma(N-1) \le (N^2 - 1)/8 \tag{6.3}$$

for $N = 1, 2, \ldots,$ with equality holding on the left or the right according as N is even or odd. Now suppose that $X > Z + 1$. Then

$$\begin{aligned}\Delta := &\, \Gamma(X-1, Y, Z+1) - \Gamma(X, Y, Z) \\ = &\, (X - Z - 1)(Y+1) + \gamma(X-1) - \gamma(X) + \gamma(Z+1) - \gamma(Z) \\ \ge &\, (X - Z - 1)(Y+1) - (X^2 - Z^2)/8 \\ = &\, (X - Z - 1)\{(Y+1) - (X + Z + 1)/8\} - Z/4 - 1/8,\end{aligned}$$

where we have used relation (6.3) in the third line.

If $X = Z + 2$, then

$$\begin{aligned}\Delta &\ge Y + 1 - (2Z+3)/8 - Z/4 - 1/8 \\ &= Y - Z/2 + 1/2 \ge Z/2 + 1/2 > 0.\end{aligned}$$

Next we combine the inequalities $Z \le Y$, $Y \ge (n-3-X)/2$, and $X \le 2(n-2)/3$, and find that

$$\begin{aligned}Y + 1 - (X + Z + 1)/8 &\ge (7Y + 7 - X)/8 \\ &\ge \{7(n-1-X) - 2X\}/16 \\ &\ge \{7(n-1) - 6(n-2)\}/16 = (n+5)/16.\end{aligned}$$

Consequently, if $X \geq Z+3$, then

$$\begin{aligned}\Delta &\geq (n+5)/8 - Z/4 - 1/8 \\ &\geq (n+4)/8 - (n-3)/12 = n/24 + 3/4 > 0.\end{aligned}$$

Thus, if $X > Z+1$, then $\Gamma(X-1, Y, Z+1) > \Gamma(X,Y,Z)$. This implies that if $n \equiv 2$ or $4 \pmod 6$ and $\max\{X,Y,Z\} \leq 2(n-2)/3$, then the maximum value of $\Gamma(X,Y,Z)$ occurs when X, Y, and Z are as nearly equal as possible. It is easy to verify that this maximum value of $\Gamma(X,Y,Z)$ equals $C(n)$ when $n \equiv 2$ or $4 \pmod 6$ (and, in fact, for all n), so this suffices to complete the proof of the theorem.

Let M_n and Q_n denote any of the minimal and maximal tournaments considered in Theorems 1 and 2, respectively. We remark in closing that if $n = 3$ or $n \geq 5$, then there exists a tournament T_n such that $c(T_n) = c(M_n)$ but for which $u(T_n) = 3$ whereas $u(M_n) = n$. Furthermore, if $n \geq 18$ (and perhaps for some smaller values also), then there exist tournaments T_n such that $c(T_n) = c(Q_n)$ but $u(T_n) = n-3$ whereas $u(Q_n) = 3$.

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