IN SEARCH OF 4 — (12, 6, 4) DESIGNS: PART II

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Abstract

In a 4–(12, 6, 4) design a block is either disjoint from one other block or it has five points in common with one other block. For a 4–(12, 6, 4) design with a pair of blocks of the second type it is shown that another thirty blocks of the design can be completed in a unique way and these thirty blocks contain a copy of a 3–(10, 4, 1) design.

1. Introduction

A $t-(v,k,\lambda)$ design is a collection of subsets, called *blocks*, of a set S with v elements, called *points*, such that every t-subset of S is contained in precisely λ blocks. If s is a whole number such that $0 \leq s \leq t$, then a t–design is also an s–design with, of course, a different λ value. Thus the 5–(12, 6, 1) design, which is unique in structure, is also a 4–(12, 6, 4) design. But the converse is not true; there are 4–(12, 6, 4) designs which are not 5–(12, 6, 1) designs. Nine non-isomorphic such designs are constructed by Breach, Elmes, Sharry and Street [1].

For any $t-(v,k,\lambda)$ design let λ_i be the number of times each i-subset of the v points appears in the design. Thus $\lambda_0 = b$ is the number of blocks; $\lambda_1 = r$ is the number of replicas of each point; and $\lambda_t = \lambda$. For a 4–(12, 6, 4) design we have

$$\lambda_0 = b = 132, \quad \lambda_1 = r = 66, \quad \lambda_2 = 30, \quad \lambda_3 = 12, \quad \lambda_4 = 4.$$

Let B be any block of the 12 points of a 4–(12, 6, 4) design and let b_i be the number of blocks intersecting B in exactly i points. In [1] it is shown that only two solution sets are possible. They are as follows.

	b_0	b_1	b_2	b_3	b_4	b_5	b_6
Type I	1	0	45	40	45	0	1
Type II	0	5	35	50	40	1	1

The blocks of either type occur in pairs. A block of Type I is disjoint from just one other block. A block of Type II has five points in common with just one other block and intersects all other blocks. Two blocks of Type II with five points in common are said to be *friendly* blocks. In this paper the point set for a 4–(12, 6, 4) design is the set {**1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c**}.

THEOREM 1: *If* [**123456**] *and* [**123457**] *are friendly blocks in a* 4–(12, 6, 4) *design, then the replicas of the seven distinct points from these blocks can be distributed in a unique way to give the skeleton as presented in* Table I.
Proof : See Part I [2]. □

A
123456
123457

B
1234 . .
1234 . .
1235 . .
1235 . .
1245 . .
1245 . .
1345 . .
1345 . .
2345 . .
2345 . .

C	
12367 .	14567 .
1236 . .	1456 . .
1236 . .	1456 . .
1237 . .	1457 . .
1237 . .	1457 . .
123 . . .	145 . . .
12467 .	23467 .
1246 . .	2346 . .
1246 . .	2346 . .
1247 . .	2347 . .
1247 . .	2347 . .
124 . . .	234 . . .
12567 .	23567 .
1256 . .	2356 . .
1256 . .	2356 . .
1257 . .	2357 . .
1257 . .	2357 . .
125 . . .	235 . . .
13467 .	24567 .
1346 . .	2456 . .
1346 . .	2456 . .
1347 . .	2457 . .
1347 . .	2457 . .
134 . . .	245 . . .
13567 .	34567 .
1356 . .	3456 . .
1356 . .	3456 . .
1357 . .	3457 . .
1357 . .	3457 . .
135 . . .	345 . . .

D	
1267 . .	2467 . .
126 . . .	246 . . .
127 . . .	247 . . .
12	24 . . . .
1367 . .	2567 . .
136 . . .	256 . . .
137 . . .	257 . . .
13	25 . . . .
1467 . .	3467 . .
146 . . .	346 . . .
147 . . .	347 . . .
14	34 . . . .
1567 . .	3567 . .
156 . . .	356 . . .
157 . . .	357 . . .
15	35 . . . .
2367 . .	4567 . .
236 . . .	456 . . .
237 . . .	457 . . .
23	45 . . . .

E
167 . . .
167 . . .
267 . . .
267 . . .
367 . . .
367 . . .
467 . . .
467 . . .
567 . . .
567 . . .

F
16 . . . .
17 . . . .
26 . . . .
27 . . . .
36 . . . .
37 . . . .
46 . . . .
47 . . . .
56 . . . .
57 . . . .

Table 1 : The skeleton for a 4–(12, 6, 4) design. This is also the paradigm for the distribution of the points from a friendly pair of blocks.

2. Some Definitions and Basic Rules

The skeleton, Table 1, gives the distribution of the seven points from any pair of friendly blocks and so is the *paradigm* to be followed by all such friendly pairs. For each block of a friendly pair, the point which is not on the other member of the pair is called the *prong* of the block. Thus, in the statement of Theorem 1 the blocks have prongs **6** and **7**. Together these form the *prong pair* **67**. An examination of the skeleton shows that, relative to given friendly pair of blocks, the prongs have special properties. Variations on these are used later in this paper so they will here be stated as a set of **Prong Laws**:

(i) *If a block B intersects one of a friendly pair of blocks in just one point, then B contains the prong of the other block of the friendly pair;*

(ii) *One-point intersections between Type II blocks never occur on the prongs of those blocks; (note, Type I blocks never intersect another block in one point);*

(iii) *Prongs are never* orphans, *that is to say, the prongs of a friendly pair never appear unless they are accompanied by at least one of the non-prong points from the same friendly pair. This applies even when both prongs appear together.*

From the block intersection numbers we encapsulate a useful principle known as the **Rule of Five**, or RF for short:

Given three blocks in the design, if two of them intersect in five points then the third cannot intersect either of these in five points.

A *fragment* is always defined relative to a given set of points X and is a subset of X that appears on a block as part of no larger subset of X, thus it is locally maximal. Relative to the set of points on the blocks [**123456**], [**123457**], Table 1 gives the distribution of all the fragments from this set. For example, while the triple **123** occurs twelve times in all in the design, relative to the blocks of section A, the fragment **123** occurs only once, and that is in the last block of the first subsection of section C. The 4–fragment **1236,** however occurs twice, and that is in the same subsection of section C. The third prong law could be restated in terms of fragments by saying that the prongs of a friendly pair of blocks never form one-point or 2–point fragments relative to those blocks. The notion of fragments is particularly useful when discussing the positioning of points none of which is on an already established friendly pair. For example, the completion of the skeleton to a 4–(12, 6, 4) design requires the insertion of 1–, 2– , 3– and 4–fragments relative to the set {**8, 9, a, b, c**}.

It will be shown that each triple from {**8, 9, a, b, c**} appears just once in section E of the skeleton, and that a cyclic arrangement of the blocks of section E, with no two of the blocks intersecting in five points, is possible. To this end we need to know the distribution of n–tuples and n–fragments from {**8, 9, a, b, c**} within each section of the skeleton.

A *swapmap* is a special mapping of a 4–(12, 6, 4) design onto itself which preserves a specified pair of friendly blocks and maps the remaining blocks onto each other in pairs. To apply a swapmap to the skeleton, replace each block by its complement with respect to the point set and then perform transpositions according to the permutation **(18)(29)(3a)(4b)(5c)(67)**. Although other swapmaps are possible, this is the only one used in this paper.

3. The Vital Statistics of a 4–(12, 6, 4) Design

We compile these statistics through a sequence of lemmas.

LEMMA 1: *In section* F *of the skeleton the five quadruples from* {**8, 9, a, b, c**} *each appear twice, once on a block with* **6** *and once on a block with* **7**.

Proof: If **89ab**, say, appeared on three blocks in section F then at least two of these blocks would have to contain **6**, say. Thus there would be a friendly pair patterned after [**1689ab**], [**2689ab**]. The prongs of this pair are **1** and **2**. But then the prong laws are broken through one-point intersections on **1** and **2** with the second block of section A. Thus **89ab** cannot appear thrice nor can it appear twice with **6**. Consequently the lemma is proved. □

Another Proof: Applying a swapmap carries section F onto itself. Relative to the set {**1, 2, 3, 4, 5, 6, 7**} each 2–fragment containing **6** is associated with a unique 5–fragment relative to {**6, 7, 8, 9, a, b, c**} and containing **6**. Likewise for fragments containing **7**. Thus every quadruple from {**8, 9, a, b, c**} appears twice in section F, once with **6** and once with **7**. □

LEMMA 2: *Each triple from* {**8, 9, a, b, c**} *appears once in section* E *of the skeleton.*

Proof: A swapmap carries the blocks of section E onto the last blocks in the subsections of section D, and vice versa. The last blocks in the subsections of section D between them contain each pair from {**1, 2, 3, 4, 5**} just once, and by the swapmap each of these pairs is associated with a unique triple from {**8, 9, a, b, c**}. □

LEMMA 3: *In section* D *each quadruple from* {**8, 9, a, b, c**} *appears twice.*

Proof: Either; note that each relevant quadruple occurs four times in the 4–(12, 6, 4) design and two of these appearances are in section F: or; use a swapmap as in Lemma 2. □

LEMMA 4: *In section* D *each 3–fragment relative to* {**8, 9, a, b, c**} *appears twice, once with* **6** *and once with* **7**.

Proof: With **x**, **y**, **z** $\in$ {**8, 9, a, b, c**}, each quadruple **xyz6** appears four times in the design. By Lemma 1, two of these appearances are in section F; a third is in section E, by Lemma 2. Therefore the remaining appearance must be in section D, and likewise for quadruples with **7**. □

LEMMA 5: *In section* D *each 2–fragment relative to* {**8, 9, a, b, c**} *appears just once.*

Proof : Apply a swapmap and note that the first block in each subsection of section D goes into the last block in some subsection of section C. Then note that these latter blocks contain each 3–fragment relative to {**1, 2, 3, 4, 5**} just once and use the principle applied in Lemma 2. □

LEMMA 6: *In section* C *each triple from* {**8, 9, a, b, c**} *appears just once.*
Proof: As in Lemma 5 with the mapping taken the other way. □

LEMMA 7: *In section* C *each one-fragment relative to* {**8, 9, a, b, c**}*appears twice.*
Proof: A swapmap takes the leading blocks of the subsections of section C onto the blocks of section B, and vice versa. Section B contains each quadruple in {**1, 2, 3, 4, 5**} twice, so each one-fragment from {**8, 9, a, b, c**} appears twice in section C.

LEMMA 8: *In section* B *each pair from* {**8, 9, a, b, c**} *appears just once.*
Proof: This is implied by Lemma 7 and the principle used in Lemma 2. □

LEMMA 9: *In section* C, *relative to* {**8, 9, a, b, c**} *each 2–fragment occurs four times, twice with* **6** *and twice with* **7**.
Proof: Each triple **xy6**, with **x**, **y** ∈ {**8, 9, a, b, c**}, occurs twelve times in the design; three times in section F, three times in section E, four times in section D; and so twice in section C. A similar argument applies to the triple **xy7**. □

The distributions of the bits and pieces from {**8, 9, a, b, c**} within each section of the skeleton are now completely determined, although, of course the precise structuring of each section is yet to be done. Table 2 gives a summary of this information.

Type of Configuration		**#Occurences of each in each section**				
	Section	**B**	**C**	**D**	**E**	**F**
Points	**8, 9, a, b, c.**	**4**	**24**	**24**	**6**	**8**
Pairs	**89, 8a, 8b, 8c, 9a, 9b, 9c, ab, ac, bc.**	**1**	**7**	**13**	**3**	**6**
Triples	**89a, 89b, 89c, 8ab, 8ac, 8bc, 9ab, 9ac, 9bc, abc.**	**0**	**1**	**6**	**1**	**4**
Quadruples	**89ab, 89ac, 89bc, 8abc, 9abc.**	**0**	**0**	**2**	**0**	**2**
1-fragments	**8, 9, a, b, c.**	**0**	**2**	**0**	**0**	**0**
2-fragments	**89, 8a, 8b, 8c, 9a, 9b, 9c, ab, ac, bc.**	**1**	**4**	**1**	**0**	**0**
3-fragments	**89a, 89b, 89c, 8ab, 8ac, 8bc, 9ab, 9ac, 9bc, abc.**	**0**	**1**	**2**	**1**	**0**

Table 2 : The distribution within the skeleton of n-tuples and n-fragments relative to the set {8, 9, a, b, c}.

4. Some Block Intersection Theorems

The aim is to establish that section E of the skeleton can be completed essentially in only one way. To this end we prove some block intersection theorems.

THEOREM 2: *The blocks* [**2345** . .], [**2345** . .] *of section* B *of the skeleton have five points in common if and only if the blocks* [**167** . . .], [**167** . . .] *of section* E *have five points in common.*

Proof: Suppose in section B there are the blocks [**234589**] and [**23458a**]. These being friendly blocks, the blocks [**167** . . .], [**167** . . .] cannot each have a disjoint mate, and so these blocks must also be of Type II. By applying the prong laws and the paradigm (Table 1) to the friendly pair [**234589**], [**23458a**], and by using the permutations (**9a**) and (**bc**), it is found that the pair [**167** . . .], [**167** . . .] can be completed in just two non-equivalent ways. These are represented by the (vertical) pairs:

[**1678ac**]	and	[**16789b**]
[**16789b**]		[**1678ab**].

(a) Suppose the first of these ways is valid. Then, in section F, consider the blocks [**16** . . . .], [**17** . . . .]. Relative to the assumed blocks of section B, these two blocks cannot contain either of the fragments **89** or **8a**, since the mandatory one-time appearances of these have been bespoken by the blocks [**1678ac**], [**16789b**]. But [**16** . . . .] and [**17** . . . .] intersect [**234589**] and [**23458a**] in at least one point. Therefore, to conform to the prong laws, there must be blocks [**1689a** .], [**1789a** .]. But to preserve the RF with the blocks of section E we must then have [**1689ab**] and [**1789ac**] (**6** and **7** are equivalent at this stage). This creates the friendly pairs [**1689ab**], [**1789ab**] and [**1789ac**], [**1678ac**], which, by the RF, cannot co-exist. Therefore no pair of points from either of the triples **8ac** or **89b** can appear in any block of section D containing **167**. This leaves available for the completion of the four **167**–blocks of section D the four pairs **9a**, **9c**, **ab**, **bc**. Each can be used once only. Therefore section D contains a block [**1267bc**] say. But this intersects both [**234589**] and [**23458a**] in the single point **2**, which is not possible by the paradigm.

Therefore the existence of the pair [**234589**], [**23458a**], intersecting in five points, requires the existence of the pair [**16789b**], [**1678ab**] also intersecting in five points.

(b) Suppose section E has the friendly pair [**16789b**] and [**1678ab**]. Now suppose the blocks [**2345** . .], [**2345** . .] do not have five points in common. Then, given the permutations (**9a**) and (**8b**), there is essentially only one way of completing them in accordance with the prong laws, namely; [**234589**] and [**2345ab**].

In section F, to avoid five-point intersections with the blocks of section E there must be blocks [**1689ac**] and [**179abc**]. (If **89ac** appeared in both the there would be a bad repeated one-point intersection on **a** in the block [**2345ab**].)

The blocks [**16789b**], [**1678ab**] induce the fragment **189ab** just once. With **2**, **3**, **4**, **5** equivalent here, section D must have a block [**1589ab**] say, and, consequently, blocks

[**12** . . . **c**], [**13** . . . **c**], [**14** . . . **c**]. The blocks [**16789b**], [**1678ab**] also induce the fragments **189b** and **18ab** each twice. Each of these can appear at most once in section C since each triple from {**8, 9, a, b, c**} is in section C exactly once. Therefore section D must have blocks [**1389bc**] and [**148abc**] say.

The block [**1689ac**], in section F, is of Type II. Although its friendly mate is not known, it has one-point intersections on **1** and **a** and consequently a fragment **1a** which appears just once on a block not containing any of **6, 8, 9, c**. The fragment **1a** is not in section B since the pair **ab** is on a block of section B not containing **1**. Therefore somewhere in section C there is just one block [**1xy7ab**] where **x y** $\in$ {**2, 3, 4, 5**}. Likewise, from [**179abc**] of section F, section C has just one block [**1xy689**] where **x**, **y** $\in$ {**2, 3, 4, 5**}.

Now, from section E, [**16789b**] and [**1678ab**] generate fragments **17ab** and **1689** each of which is to appear twice. One each of these appearances is in section C. The others must be in section D which therefore must contain [**1x689c**] and [**1y7abc**]. But these intersect [**1689ac**] and [**179abc**] respectively in five points. Therefore, in section D, the block [**12** . . . **c**] can only be completed with either **89b** or **8ac**.

In section F the quadruple **19ac** appears twice. It cannot appear in section D unless **6** or **7** is in the block; but then the RF would be violated. Therefore **19ac** must appear twice in section C of this 4–(12, 6, 4) design. But triples from {**8, 9, a, b, c**} each occur just once in section C, so the assumption about the blocks of section B is false. Thus the condition of the theorem is both necessary and sufficient. □

It will now be shown that when section E has a friendly pair of blocks the design cannot be completed.

THEOREM 3: *A block from section* C *not containing* **6** *or* **7** *cannot have five points in common with any block from section* D *that does not contain* **6** *or* **7**.

Proof: Suppose, without loss of generality, that the blocks [**123** . . .], from section C, and [**12** . . . .], from section D, have five points in common. Then **3** must be one of the prongs of the resulting friendly pair. According to the paradigm (Table 1), relative to these blocks, the fragment **123** occurs just once elsewhere in the design. But the blocks of section A contain this fragment twice since [**123** . . .] and [**12** . . . .] are to be completed using {**8, 9, a, b, c**}. Hence the supposition is false. □

Now make a swapmap (as described in Section 2). Then Theorem 3 translates into Theorem 4.

THEOREM 4: *No block from section* D *containing* **6** *or* **7** *can intersect a block of section* E *in five points.* □

THEOREM 5: *No two blocks of section* E *can have five points in common.*

Proof: Suppose that section E contains the friendly blocks [**16789b**], [**1678ab**] with prongs **9** and **a**. Then to preserve the RF, the first two blocks of section F must have the form [**16** . **9ac**] and [**17** . **9ac**]. By Theorem 2, in section B the two blocks with **2345** have five points in common. Neither can contain **c** since the prong laws would be broken by the completed blocks of saection E. The two blocks [**2345 . .**], [**2345 . .**] cannot contain **9a**, the prong pair of [**16789b**] and [**1678ab**]. If they contain **8** and **b** each once then these form a prong pair (the blocks must intersect in five points) which makes an illegal appearance in one of [**16789b**], [**1678ab**]. Thus, **8** and **b** being equivalent, there must be blocks [**23458a**], [**23458 .**] in section B, **a** and **9** being equivalent. But then **a** is a prong and the blocks [**16 . 9ac**], [**17 . 9ac**] must both contain **8** if the prong laws are to be obeyed. Then [**23458 .**] is forced to contain **9**, again by the prong laws. We now have the three pairs of friendly blocks:

[**16789b**];	[**1689ac**];	[**23458a**].
[**1678ab**]	[**1789ac**]	[**234589**]

The pair [**16789b**], [**1678ab**] engender the fragment **189ab**, so, **2**, **3**, **4**, **5** being equivalent, there must be a block [**1589ab**] in section D. The fragment **189ab** occurs once only, so there must be blocks [**12 . . . c**], [**13 . . .. c**], [**14 . . . c**] in section D. All three of these must also contain **b**, by the RF applied to the pair [**1689ac**], [**1789ac**]. This pair induces fragments **189a, 189c, 18ac, 19ac** each twice. As triples from {**8, 9, a, b, c**} appear just once in section C, each of these four fragments must occur at least once in section D, the only other place for them. The fragment **189a** is already in position on the block [**1589ab**]. Therefore there are three more blocks [**1289bc**], [**138abc**], [**149abc**], **2**, **3** and **4** being equivalent up to now. By Theorem 3, the block [**123 . . .**] in section C must contain **9a**. A similar argument applies to the last block in each subsection of section C that contains **1**. Thus there are blocks [**123 . 9a**], [**124 . 8a**], [**125 . ac**], [**134 . 89**], [**135 . 9c**], and [**145 . 8c**].

Relative to the friendly pair [**16789b**], [**1678ab**] there are fragments **18ab** and **189b**. According to the paradigm each of these appears twice. Two of these four appearances are accounted for by the blocks [**138abc**] and [**149abc**]. The only places for the other two must be in section C. Therefore there must be blocks [**1248ab**] and [**13489b**]. Likewise, relative to the pair [**1689ac**], [**1789ac**] the fragment **189a** must occur twice and one of these occurences is in the block [**1589ab**]. The other occurence forces, in section C, the block [**12389a**]. Then by Theorem 3 in section D there must be blocks [**23 . . bc**], [**24 . . 9c**], [**34 . . ac**]. Thus we have the configuration of Table 3.

In section D the quadruple **89ab** must occur twice. Therefore the block [**1589ab**] must be friends with one of [**25 . . . .**], [**35 . . . .**], [**45 . . . .**]. Suppose we have [**1589ab**] and [**4589ab**]. Then this friendly pair, with prongs **1** and **4**, must have the fragment **145** twice, and indeed in section A there are the two blocks [**514263**], [**514273**]. Relative to [**1589ab**] and [**4589ab**], these blocks of section A are the analogs of [**16789b**], [**1678ab**] of section E, and analogous to the blocks [**1689ac**], [**1789ac**] of section F, there must be a pair of blocks [**51267c**], [**54267c**]; (see Table 4). Down to this stage the permutation (23)(9a) is an automorphism.

Neither of the friendly pair [**12567c**], [**24567c**] can have five points in common with any of the blocks [**1267 . .**], [**1367 . .**], [**1467 . .**], [**1567 . .**]. But, by Theorem

A
123456
123457

B
1234 . .
1234 . .
1235 . .
1235 . .
1245 . .
1245 . .
1345 . .
1345 . .
23458a
234589

C	
12367 .	14567 .
1236 . .	1456 . .
1236 . .	1456 . .
1237 . .	1457 . .
1237 . .	1457 . .
12389a	145 . 8c
12467 .	23467 .
1246 . .	2346 . .
1246 . .	2346 . .
1247 . .	2347 . .
1247 . .	2347 . .
1248ab	234 . . .
12567 .	23567 .
1256 . .	2356 . .
1256 . .	2356 . .
1257 . .	2357 . .
1257 . .	2357 . .
125 . ac	235 . . .
13467 .	24567 .
1346 . .	2456 . .
1346 . .	2456 . .
1347 . .	2457 . .
1347 . .	2457 . .
13489b	245 . . .
13567 .	34567 .
1356 . .	3456 . .
1356 . .	3456 . .
1357 . .	3457 . .
1357 . .	3457 . .
135 . 9c	345 . . .

D	
1267 . .	2467 . .
126 . . .	246 . . .
127 . . .	247 . . .
1289bc	24 . . 9c
1367 . .	2567 . .
136 . . .	256 . . .
137 . . .	257 . . .
138abc	25 . .
1467 . .	3467 . .
146 . . .	346 . . .
147 . . .	347 . . .
149abc	34 . . ac
1567 . .	3567 . .
156 . . .	356 . . .
157 . . .	357 . . .
1589ab	35 . . . .
2367 . .	4567 . .
236 . . .	456 . . .
237 . . .	457 . . .
23 . . bc	45 . . . .

E
16789b
1678ab
267 . . .
267 . . .
367 . . .
367 . . .
467 . . .
467 . . .
567 . . .
567 . . .

F
1689ac
1789ac
26 . . . .
27 . . . .
36 . . . .
37 . . . .
46 . . . .
47 . . . .
56 . . . .
57 . . . .

Table 3 : The situation at the end of the third paragraph of the proof of Theorem 5.

A	E	F	(A)	(E)	(F)
123456	16789b	1689ac	89ab51	514263	51267c
123457	1678ab	1789ac	89ab54	514273	54267c
	. . . . . .	. . . . . .		. . . . . .	. . . . . .

Table 4 : The analogy used in the proof of Theorem 5.

4, the only pairs that can complete these blocks are **8c**, **9a**, **9c**, **ac**, **bc** and each of these can be used only once. Hence one of [**1267 . .**] and [**1567 . .**] contains **c** and so breaks the RF with the pair [**12567c**], [**24567c**] from section C. Permutations on **2**, **3**, **4** do not affect the nature of the argument. Therefore no two blocks of section E have five points in common. □

5. Some Order at Last

THEOREM 6: *In section* E *of the skeleton the set of blocks is always isomorphic to that generated by the permutation* (**12345**)(**89abc**) *acting on the pair* [**1678ab**] *and* [**16789c**].

Proof: By Theorem 5, the pair [**167** . . .], [**167** . . .] between them contain all five points of {**8, 9, a, b, c**} and just one of those points twice. The pair can be taken to be [**1678ab**] and [**16789c**]. Now **8** appears six times in section E. If **8** appears in both of [**267** . . .], [**267** . . .] there must be a pair, [**367** . . .], [**367** . . .] say, without **8** and therefore intersecting in five points contrary to Theorem 5. Thus the five pairs of blocks of section E each have a different point from {**8, 9, a, b, c**} appearing twice.

Without loss of generality, the pair [**2679 . c**], [**2679 . 8**] can be assumed since the triple **89c** may not be used again in section E. Now **a** and **b** are equivalent at this stage so these blocks may be completed as [**2679ac**], [**2679a8**]. Apart from permutations on **3**, **4** and **5**, which are equivalent, the completion of the remaining six blocks of section E is forced, thus yielding the arrangement in the statement of the theorem. □

THEOREM 7: *In section* D *of the skeleton, when section* E *is completed according to* Theorem 6, *the blocks containing* **67** *are generated by* (**12345**)(**89abc**) *acting on* [**1267ac**] *and* [**1367bc**].

Proof: By Theorem 4, the block [**1267** . .] of section D cannot intersect any of [**1678ab**], [**16789c**], [**2679bc**], [**2679a8**] in five points. Thus **ac** is the only pair available to complete the block. Likewise [**1367** . .] can be completed only by the pair **bc**; with similar arguments applied in a cyclic fashion. □

Thus far, given the blocks in section A, we have been able to standardize and complete twenty other blocks. The next two theorems enable a further ten blocks to be completed in a unique way.

THEOREM 8: *In the skeleton as standardized by* Theorems 6 *and* 7, *the last block in each subsection of section* C *cannot intersect any block of section* B *in five points.*

Proof: In section C suppose the block [**123** . . .] has a friendly mate in section B. Then the prong of that mate must be either **4** or **5**. Blocks that intersect [**123** . . .] in just one point must either all contain **4** or all contain **5**, but not both. There are ten triples from the set {**8, 9, a, b, c**} each of which can be used, in turn, to complete the block [**123** . . .].

The triple **9bc** creates a block [**1239bc**] which is disjoint from [**4567a8**] so there is nothing further to prove. For any other triple from {**8, 9, a, b, c**}, **89a** say, the argument is a variation of the following. The block [**12389a**] intersects [**567c8b**] in one point so its mate's prong must be **5**. But [**12389a**] intersects [**467bca**] in one point so its mate's prong must be **4**. This is impossible, so [**12389a**] does not have a friendly mate in section B. Again, [**12389c**] say, intersects each of [**467b89**] and [**2567ab**] in one point so its friendly mate in section B would have to have both **4** and **5** as prongs, etc. Having dealt with the blocks [**123** . . .] and [**124** . . .] in this fashion, we can use the permutation (**12345**)(**89abc**) to dispose of the rest. □

THEOREM 9: *In the skeleton as standardized by* Theorems 6 *and* 7, *no block from section* C *containing* **67** *can have five points in common with any block of section* D *that contains* **67**.
Proof: Apply a swapmap to Theorem 8. □

This result forces a unique completion for the first block in each subsection of section C. Thus thirty-two blocks in the design are now completed and the situation displayed in Table 5 has been reached. In fact it can be shown that a further ten blocks of the skeleton are determined , these being all those blocks of section D which contain **7** but not **6**. The effort involved in showing this, however, is considerable and, in view of the fact that a manageable algorithm for reducing the number of completions of the skeleton to a small number of cases is at hand (see Part III [3]), it will not be discussed further here.

6. An Unexpected Embedding

In Table 5, which is the new improved skeleton, if the thirty blocks that contain **67** are selected and the points **6** and **7** are deleted from those blocks, then there remain the thirty blocks of a 3–(10, 4, 1) design. This is most unexpected, for the process of restricting on two points of 4–design normally leads to a 2–design. It is to be remarked that any pair of friendly blocks will have associated with it a copy of the 3–(10, 4, 1) design, so a 4–(12, 6, 4) design can have many 3–(10, 4, 1) designs embedded in it. We conclude with a theorem that embodies the current result.

THEOREM 10: *Every* 4–(12, 6, 4) *design has embedded in it a* 3–(10, 4, 1) *design.*
Proof: If the 4–(12 6, 4) design does not have pairs of friendly blocks then it is a 5–(12, 6, 1) design (see Part I [2]). The restriction on a pair of points is then a 3–(10, 4, 1) design. If the design does have a friendly pair of blocks then Table 5 provides the proof. □

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A
123456
123457

B
1234 . .
1234 . .
1235 . .
1235 . .
1245 . .
1245 . .
1345 . .
1345 . .
2345 . .
2345 . .

C	
123679	14567c
1236 . .	1456 . .
1236 . .	1456 . .
1237 . .	1457 . .
1237 . .	1457 . .
123 . . .	145 . . .
12467b	23467a
1246 . .	2346 . .
1246 . .	2346 . .
1247 . .	2347 . .
1247 . .	2347 . .
124 . . .	234 . . .
125678	23567c
1256 . .	2356 . .
1256 . .	2356 . .
1257 . .	2357 . .
1257 . .	2357 . .
125 . . .	235 . . .
134678	245679
1346 . .	2456 . .
1346 . .	2456 . .
1347 . .	2457 . .
1347 . .	2457 . .
134 . . .	245 . . .
13567a	34567b
1356 . .	3456 . .
1356 . .	3456 . .
1357 . .	3457 . .
1357 . .	3457 . .
135 . . .	345 . . .

D	
1267ac	24678c
126 . . .	246 . . .
127 . . .	247 . . .
12	24 . . . .
1367bc	2567ab
136 . . .	256 . . .
137 . . .	257 . . .
13	25 . . . .
14679a	34679c
146 . . .	346 . . .
147 . . .	347 . . .
14	34 . . . .
15679b	356789
156 . . .	356 . . .
157 . . .	357 . . .
15	35 . . . .
23678b	4567a8
236 . . .	456 . . .
237 . . .	457 . . .
23	45 . . . .

E
1678ab
16789c
2679bc
2679a8
367ac8
367ab9
467b89
467bca
567c9a
567c8b

F
16 . . . .
17 . . . .
26 . . . .
27 . . . .
36 . . . .
37 . . . .
46 . . . .
47 . . . .
56 . . . .
57 . . . .

Table 5 : The new improved skeleton for a 4–(12, 6, 4) design. The blocks containing the pair 67 form a 3–(10, 4, 1) design on the other ten points.

7. References

[1]. D. R. Breach, Sylvia H. Elmes, Martin J. Sharry and Anne Penfold Street, *The curious property of* 4–(12, 6, 4) *designs*, submitted.

[2]. D. R. Breach, Martin J. Sharry and Anne Penfold Street, *In search of* 4–(12, 6, 4) *designs*; Part I, this volume, 237-245.

[3]. D. R. Breach and Anne Penfold Street, *Insearch of* 4–(12, 6, 4) *designs,* Part III, this volume, 259-273.

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