IN SEARCH OF 4 — (12, 6, 4) DESIGNS: PART III

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Abstract

A 4–(12, 6, 4) design that is not also a 5–(12, 6, 1) design must have at least one pair of blocks with five points in common. It is shown that there are just nine non-isomorphic such designs; so, including the 5–(12, 6, 1) design, there are ten 4–(12, 6, 4) designs. These designs are characterised by the orders of their automorphism groups and they all contain a 4–(11, 5, 1) design.

1. Introduction

A $t-(v,k,\lambda)$ design based on a set S of v *points* is a collection of subsets, each of size k, called *blocks*, such that each t–subset of S appears in exactly λ blocks. For an integer s such that $0 < s \leq t$, a t–design is also an s–design with, of course, a different value of λ. Thus the 5–(12, 6, 1) design is also a 4–(12, 6, 4) design. However, there are 4–(12, 6, 4) designs which are not 5–designs. In this paper we continue the work of Part I [4] and Part II [1] and show that there are just nine mutually non-isomorphic such designs. We do this by reducing the number of cases that have to be examined in detail to forty-six which can then be completed by hand. Each of these cases has a unique completion to a 4–(12, 6, 4) design. A computer check shows that these fall into nine equivalence classes and that designs in these classes are characterised by the orders of their automorphism groups.

For any $t-(v,k,\lambda)$ design let λ_i be the number of times each i-subset of the v points appears in the design. Thus $\lambda_0 = b$ is the number of blocks; $\lambda_1 = r$ is the number of replicas of each point; and $\lambda_t = \lambda$. For a 4–(12, 6, 4) design we have

$$\lambda_0 = b = 132, \quad \lambda_1 = r = 66, \quad \lambda_2 = 30, \quad \lambda_3 = 12, \quad \lambda_4 = 4.$$

Let B be any block of the 12 points of a 4–(12, 6, 4) design and let b_i be the number of blocks intersecting B in exactly i points. In Part I it is shown that only two solution sets are possible. They are

	b_0	b_1	b_2	b_3	b_4	b_5	b_6
Type I	1	0	45	40	45	0	1
Type II	0	5	35	50	40	1	1

The blocks of either type occur in pairs. A block of Type I is disjoint from just one other block. A block of Type II has five points in common with just one other block and intersects all other blocks. Two blocks of Type II with five points in common are said to be *friendly* blocks. In this paper the point set for a 4–(12, 6, 4) design is the set {**1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c**}.

A

123456
123457

B

1234 . .
1234 . .
1235 . .
1235 . .
1245 . .
1245 . .
1345 . .
1345 . .
2345 . .
2345 . .

C

123679	14567c
1236 . .	1456 . .
1236 . .	1456 . .
1237 . .	1457 . .
1237 . .	1457 . .
123 . . .	145 . . .
12467b	23467a
1246 . .	2346 . .
1246 . .	2346 . .
1247 . .	2347 . .
1247 . .	2347 . .
124 . . .	234 . . .
125678	23567c
1256 . .	2356 . .
1256 . .	2356 . .
1257 . .	2357 . .
1257 . .	2357 . .
125 . . .	235 . . .
134678	245679
1346 . .	2456 . .
1346 . .	2456 . .
1347 . .	2457 . .
1347 . .	2457 . .
134 . . .	245 . . .
13567a	34567b
1356 . .	3456 . .
1356 . .	3456 . .
1357 . .	3457 . .
1357 . .	3457 . .
135 . . .	345 . . .

D

1267ac	24678c
126 . . .	246 . . .
127 . . .	247 . . .
12	24 . . . .
1367bc	2567ab
136 . . .	256 . . .
137 . . .	257 . . .
13	25 . . . .
14679a	34679c
146 . . .	346 . . .
147 . . .	347 . . .
14	34 . . . .
15679b	356789
156 . . .	356 . . .
157 . . .	357 . . .
15	35 . . . .
23678b	4567a8
236 . . .	456 . . .
237 . . .	457 . . .
23	45 . . . .

E

16789c
1678ab
26789a
2679bc
3678ac
3679ab
46789b
467abc
5678bc
5679ac

F

16 . . . .
17 . . . .
26 . . . .
27 . . . .
36 . . . .
37 . . . .
46 . . . .
47 . . . .
56 . . . .
57 . . . .

Table 1: : The new improved skeleton for a 4–(12, 6, 4) design. The blocks with both 6 and 7 contain a 3–(10, 4, 1) design.

In Part I the unique skeleton created by a pair of friendly blocks is determined. In Part II this skeleton is improved upon by showing that thirty-two of the blocks can be standardized in a unique fashion (see Table 1). These results are summarized in our first theorem.

THEOREM 1: *If* [**123456**] *and* [**123457**] *are a pair of friendly blocks in a* 4–(12, 6, 4,) *design, then the completed and partially completed blocks of the design must follow the pattern given in* Table 1. *Furthermore the blocks with* **67** *contain a* 3–(10, 4, 1) *design on* {**1, 2, 3, 4, 5, 8, 9, a, b, c**}. □

Here we recall the **Rule of Five**, or the RF for short, which says that, given any three blocks of a 4–(12, 6, 4) design, at most two of them can have five points in common. We also recall the **Prong Laws** from Part II . The *prongs* of a pair of friendly blocks are the two points which lie on one but not both of them, so each block of the friendly pair has its prong. From an examination of Table I we deduce the following principles:

(i) *If a block B intersects one of a friendly pair of blocks in just one point, then B contains the prong of the other block of the friendly pair;*

(ii) *One-point intersections, which can occur only between Type II blocks, never occur on the prongs of those blocks.*

(iii) *Prongs are never* orphans, *that is to say, the prongs of a friendly pair never appear unless they are accompanied by at least one of the non-prong points from the same friendly pair. This applies even when both prongs appear together.*

2. A Partial Standardization of Section F

Section F of the skeleton contains each quadruple from {**8, 9, a, b, c**} exactly twice, once on a block with **6** and once on a block with **7** (Part II, Lemma 1). The distribution of these quadruples relative to the already completed blocks is governed by Theorems 2 and 3.

THEOREM 2: *In section* D *of the skeleton, blocks containing neither* **6** *nor* **7** *cannot intersect each other in five points.*

Proof: By Theorem 5 of Part II, no two blocks of section E can have five points in common. Apply a swapmap (see Part II), thus carrying the blocks of section E into those described by the current theorem. □

THEOREM 3: *Consider the two blocks of section* F *that contain the quadruple* **9abc**, *say. Then;* **either** *these two blocks intersect in five points;* **or** *one of them has five points in common with a block from section* E *and the other has five points in common with a block from section* D *not containing* **6** *or* **7**.

Proof: Given the standardized section E, if the two blocks of section F containing **9abc** have five points in common, then, to satisfy the RF with blocks of section E, they must be [**169abc**] and [**179abc**]. Otherwise, the two blocks are patterned after [**179abc**] and [**269abc**]. Then [**269abc**] is friendly with the block [**2679bc**] of section E (in fact, if **x** is any one of **2, 3, 4, 5**, then the block [**x69abc**] in section F is friendly with a block of section E). Suppose that [**179abc**] does not intersect a block of section D not containing **6** or **7** in five points. Then, since the quadruple **9abc** has to appear twice in section D (Part II, Lemma 3), those appearances must be on two of the three blocks [**34** . . . .], [**35** . . . .], [**45** . . . .]. But this situation is prevented by Theorem 2. Therefore [**179abc**] has to be friendly with a block of section D containing neither **6** nor **7**; this block will have to contain **1**. □

COROLLARY: *In section* F *each of the quintuples* **19abc, 28abc, 389bc, 489ac, 589ab** *must appear at least once.*

Proof: Apply the permutation (**12345**)(**89abc**) in Theorem 3. □

This corollary reduces considerably the number of ways of completing section F. There is a further reduction to be had through the application of the automorphism group of section E to section F.

3. The Group of Section E

Section E
1678ab
16789c
2679bc
2679a8
367ac8
367ab9
467b89
467bca
567c9a
567c8b

α = **(12345)(89abc)**
β = **(1243)(89ba)**
γ = **(1325)(8a9c)**
δ = **(1452)(8bc9)**
ϵ = **(1534)(8cab)**
ζ = **(2354)(9acb)**

Table 2: Some of the elements of H, the automorphism group of section E.

With reference to Table I, let H be the group of point permutations that map section E onto itself. This group has order 40. It is the direct product of the group of order 2 generated by the transposition (**67**) and a group of order 20 which fixes **6** and **7**. The group H is also an automorphism group of of the whole of Table I. In Table 2 are listed six of the elements of H, and the last five of these are elements that fix a pair of blocks of section E while cycling through the other four pairs of blocks. The selected elements show that H is 2–transitive on each of the sets {**1, 2, 3, 4, 5**} and {**8, 9, a, b, c**}.

4. The Completion of Section F

Since each of the quintuples **19abc**, **28abc**, **389bc**, **489ac**, **589ab** has to appear at least once in section F, each member of {**1**, **2**, **3**, **4**, **5**} has associated with it a unique quadruple from {**8**, **9**, **a**, **b**, **c**}. When one of these quadruples appears with its associate on a block of section F we shall describe that quadruple as being *at home on* **6** or **7** as the case may be (it can be at home on both). We shall take it to be the normal case when all the relevant quadruples are at home, in which case section F contains five pairs of friendly blocks.

So far **6** and **7** are equivalent so, without loss of generality, it can always be assumed that **9abc** is at home on **7**. Now suppose the other **9abc** is not at home on **6**. Then, by Lemma 1 of Part II, **9abc** must appear with **6** elsewhere in section F. This forces another quadruple from {**8**, **9**, **a**, **b**, **c**} to be not at home on **6**, which in turn forces another displacement; and so on. This process is called *chaining* on **6**. Chaining on either or both of **6** and **7** is possible. Each chain must eventually close up to form a circuit. Unless otherwise indicated, all chains are on **6**.

The possible chain configurations can be displayed as directed graphs of degree two on a set of labelled vertices. Such graphs are called *chain diagrams*. Suppose the five points at the vertices of a regular pentagon are labelled successively with **1**, **2**, **3**, **4**, **5**. Suppose **x**, **y** $\in$ {**1**, **2**, **3**, **4**, **5**}. If the quadruple from {**8**, **9**, **a**, **b**, **c**} that should be at home with **x** is instead on a block of section F containing **y** then draw a directed edge from vertex **x** to vertex **y**. Note that it is possible for **x** and **y** to be the same, in which case the quadruple is at home on both **6** and **7**, giving a pair of friendly blocks, and the chain diagram has a loop on vertex **x**. A loop is not a proper chain. For diagrams with two proper chains it is necessary to indicate on which of **6** and **7** the chains are formed by placing the appropriate digit near the chain. There is the exceptional case corresponding to a 5–(12, 6, 1) design for which the chain diagram has no edges. We treat this as an empty set and give it the symbol $\emptyset$; otherwise the chain diagrams are labelled with roman numerals.

Under the action of the group H the possible completions of section F are placed in fourteen equivalence classes whose chain diagrams are given in Table 3. For most of the cases the direction of the circuits is immaterial and the arrows on the edges are omitted. However, for classes VIII and IX the directions of the circuits do matter and arrows are needed.

5. Not-Gardening and the Completion of Section B

For each way of completing section F there are at most six ways of completing section B and experience has shown that if one of these ways is selected then the completion of the remaining unfinished blocks is uniquely forced. We demonstrate with a selected case (see Table 4) in which section F belongs to Class III.

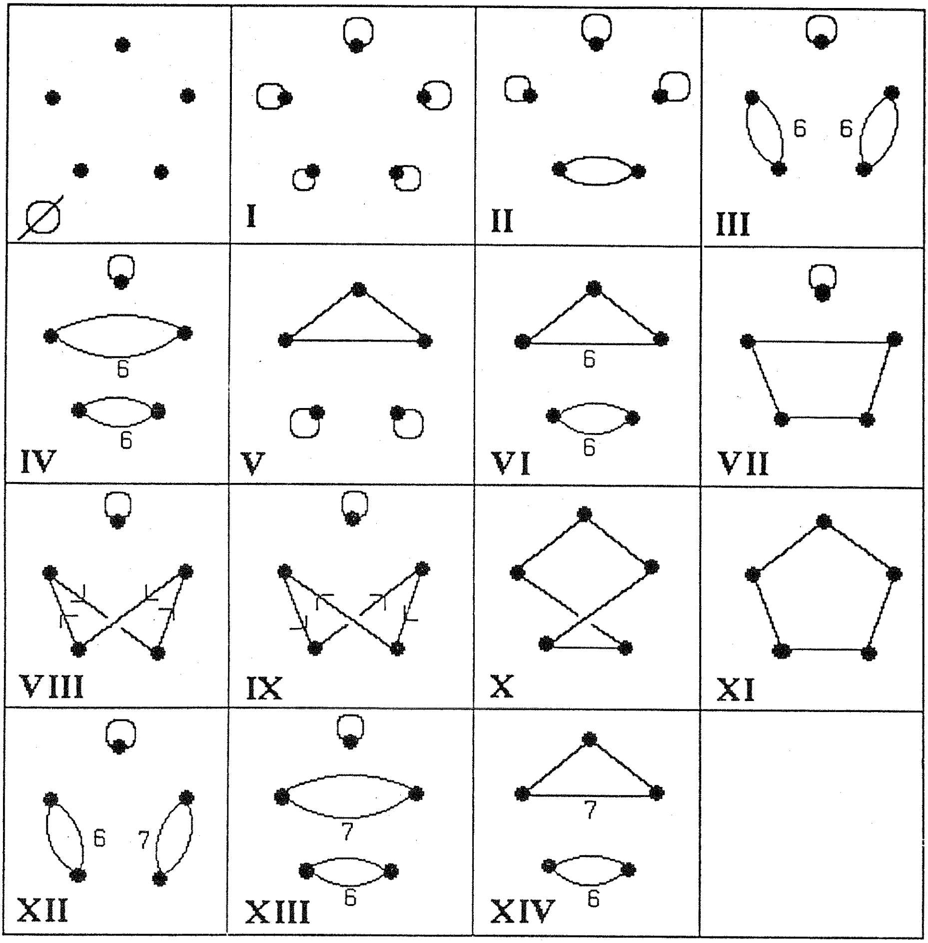

Table 3: Chain diagrams associated with section F. Note that these are directed graphs although, except for two cases, the direction of the circuits is immaterial. The vertices are to be labelled 1, 2, 3, 4, 5 as successive vertices of a regular pentagon. The numbers 6 and 7 refer to circuits on blocks containing 6 or 7 respectively.

The blocks [**5689ab**], [**5789ab**] are friendly with prongs **6** and **7**. In section B the blocks [**1234** . .], [**1234** . .] cannot intersect [**5689ab**] in one point, for if they did then they would have to contain 7, the prong of [**5789ab**], which is impossible. Thus the pair [**1234** . .], [**1234** . .] is marked "not **c**." Again, in section F, the block [**4689bc**] is friendly with the section E block [**46789c**] whose prong is **7**. Therefore the blocks [**1235** . .], [**1235** . .] are marked "not **a**"; and so on for each appropiate pair of section B (see Table 4). This process is called *not-gardening*.

By Lemma 8 of Part II, each pair from {**8, 9, a, b, c**} appears just once in section B. By Theorems 2 and 5 of Part II, no two blocks of section B intersect in five points. Therefore the blocks [**1234** . .], [**1234** . .] can be completed in just three ways, with disjoint pairs from {**8, 9, a, b**}. Once one of these ways is chosen, there are just two ways of completing the pair [**1235** . .], [**1235** . .]. Then the remaining pairs of section B can be completed in just one way. Hence there are six ways (i), (ii), ..., (vi), of completing section B, as given in Table 4. However, the largest subgroup of the group H fixing both section E and section F is generated by β^2. This interchanges (iii) with (vi), and (i) with (v), so the number of inequivalent ways of completing section B reduces to four; (i), (ii), (iii), (iv).

Section F	Section B							
168abc	**not c**	**1234 . .**	**89**	**8b**	**8a**	**8b**	**89**	**8a**
179abc		**1234 . .**	**ab**	**9a**	**9b**	**9a**	**ab**	**9b**
269abc	**not a**	**1235 . .**	**8b**	**8c**	**8b**	**89**	**8c**	**89**
278abc		**1235 . .**	**9c**	**9b**	**9c**	**bc**	**9b**	**bc**
3689ac	**not b**	**1245 . .**	**8c**	**89**	**89**	**8a**	**8a**	**8c**
3789bc		**1245 . .**	**9a**	**ac**	**ac**	**9c**	**9c**	**9a**
4689bc	**not 8**	**1345 . .**	**9b**	**9c**	**9a**	**9b**	**9a**	**9c**
4789ac		**1345 . .**	**ac**	**ab**	**bc**	**ac**	**bc**	**ab**
5689ab	**not 9**	**2345 . .**	**8a**	**8a**	**8c**	**8c**	**8b**	**8b**
5789ab		**2345 . .**	**bc**	**bc**	**ab**	**ab**	**ac**	**ac**
Class III			(i)	(ii)	(iii)	(iv)	(v)	(vi)

Table 4: The six completions of section B for a section F of Class III. The permutation β^2 leaves section F fixed and interchanges (ii) with (vi), and (i) with (v). In section F the prongs are underlined.

The process of not-gardening can be applied to all the classes of section F to produce the forty-six cases listed in Tables 5, 6, 7 and 8. We anticipate a little here by claiming that for the mixed chain classes XII, XIII and XIV, there are no continuations. The entries for section B for these classes are the same as those of classes III, IV and VI respectively. The tables also give the order of the automorphism group of the completed design arising from each class. For quick identification each case has been given a list number.

Class ∅	The 5-(12, 6, 1) design					
	List No		1			
	|G|		95,040			
Class I	F	B	(i)	(ii)		
	169abc	1234	8b	89		
Use α	179abc	1234	9a	ab		
	268abc	1235	89	8a		
	278abc	1235	ac	9c		
	3689bc	1245	8c	8c		
	3789bc	1245	9b	9b		
	4689ac	1345	8a	8b		
	4789ac	1345	bc	ac		
	5689ab	2345	9c	9a		
	5789ab	2345	ab	bc		
	List No		2	3		
	|G|		1440	8		
Class II	F	B	(i)	(ii)	(iii)	(iv)
	1689ab	1234	9c	9a	9a	9c
Use δ^2	179abc	1234	ab	bc	bc	ab
	268abc	1235	8c	89	8a	89
	278abc	1235	9a	ac	9c	ac
	3689bc	1245	89	8b	8c	8c
	3789bc	1245	bc	9c	9b	9b
	4689ac	1345	8b	8c	8b	8a
	4789ac	1345	ac	ab	ac	bc
	569abc	2345	8a	8a	89	8b
	5789ab	2345	9b	9b	ab	9a
	List No		4	5	6	7
	|G|		5	5	16	16
Class III	F	B	(i)	(ii)	(iii)	(iv)
	168abc	1234	89	8b	8a	8b
Use β^2	179abc	1234	ab	9a	9b	9a
	269abc	1235	8b	8c	8b	89
	278abc	1235	9c	9b	9c	bc
	3689ac	1245	8c	89	89	8a
	3789bc	1245	9a	ac	ac	9c
	4689bc	1345	9b	9c	9a	9b
	4789ac	1345	ac	ab	bc	ac
	5689ab	2345	8a	8a	8c	8c
	5789ab	2345	bc	bc	ab	ab
	List No		8	9	10	11
	8|G|		8	6	8	6

Table 5: Classes ∅, I, II and III.

Class IV	F	B	(i)	(ii)	(iii)			
Use β	1689ac	1234	8b	8a	8b			
	179abc	1234	9a	9b	9a			
	2689bc	1235	9c	9a	9b			
	278abc	1235	ab	bc	ac			
	368abc	1245	8a	8c	8c			
	3789bc	1245	bc	ab	ab			
	469abc	1345	8c	8b	89			
	4789ac	1345	9b	9c	bc			
	5689ab	2345	89	89	8a			
	5689ab	2345	ac	ac	9c			
	List No		12	13	14			
	|G|		8	8	144			
Class V	F	B	(i)	(ii)	(iii)	(iv)	(v)	(vi)
	1689bc	1234	8b	8a	8a	89	89	8b
	179abc	1234	9a	9b	9b	ab	ab	9a
	269abc	1235	8a	8c	89	8c	8a	89
	278abc	1235	9c	9a	ac	9a	9c	ac
	368abc	1245	8c	8b	8c	8a	8b	8a
	3789bc	1245	ab	ac	ab	bc	ac	bc
	4689ac	1345	9b	9c	9a	9b	9a	9c
	4789ac	1345	ac	ab	bc	ac	bc	ab
	5689ab	2345	89	89	8b	8b	8c	8c
	5789ab	2345	bc	bc	9c	9c	9b	9b
	List No		15	16	17	18	19	20
	|G|		6	55	6	6	6	6
Class VI	F	B	(i)	(ii)	(iii)	(iv)	(v)	(vi)
	1689bc	1234	8a	8c	89	8c	8a	89
	179abc	1234	9c	9a	ac	9a	9c	ac
	269abc	1235	8b	8a	8a	89	89	8b
	278abc	1235	9a	9b	9b	ab	ab	9a
	368abc	1245	8c	8b	8c	8a	8b	8a
	3789bc	1245	ab	ac	ab	bc	ac	bc
	4689ab	1345	9b	ab	9a	9b	9a	9c
	4789ac	1345	ac	9c	bc	ac	bc	ab
	5689ac	2345	89	89	8b	8b	8c	8c
	5789ab	2345	bc	bc	9c	9c	9b	9b
	List No		21	22	23	24	25	26
	|G|		5	5	5	24	24	5

Table 6: Classes IV, V and VI.

Class VII	F	B	(i)	(ii)	(iii)	(iv)	(v)	(vi)
	1689ac	1234	8a	8b	89	8a	8b	89
	179abc	1234	9b	9a	ab	9b	9a	ab
	269abc	1235	8b	8c	8b	89	89	8c
	278abc	1235	9c	9b	9c	bc	bc	9b
	368abc	1245	8c	8a	8a	8b	8c	8b
	3789bc	1245	ab	bc	bc	ac	ab	ac
	4689bc	1345	9a	9c	9b	9c	9b	9a
	4789ac	1345	bc	ab	ac	ab	ac	bc
	5689ab	2345	89	89	8c	8c	8a	8a
	5789ab	2345	ac	ac	9a	9a	9c	9c
	List No		27	28	29	30	31	32
	|G|		5	5	5	5	16	16

Class VIII	F	B	(i)	(ii)	(iii)
	168abc	1234	8b	89	8b
Use β	179abc	1234	9a	ab	9a
	2689ac	1235	8c	8b	89
	278abc	1235	9b	9c	bc
	369abc	1245	9c	9b	9b
	3789bc	1245	ab	ac	ac
	4689bc	1345	89	8c	8a
	4789ac	1345	ac	9a	9c
	5689ab	2345	8a	8a	8c
	5789ab	2345	bc	bc	ab
	List No		33	34	35
	|G|		72	5	72

Class IX	F	B	(i)	(ii)	(iii)
	1689bc	1234	8b	8a	8b
Use β	179abc	1234	9a	9b	9a
	269abc	1235	8c	8b	8a
	278abc	1235	ab	ac	bc
	3689ac	1245	8a	8c	89
	3789bc	1245	9c	9a	ac
	468abc	1345	9b	9c	9c
	4789ac	1345	ac	ab	ab
	5689ab	2345	89	89	8c
	5789ab	2345	bc	bc	9b
	List No		36	37	38
	|G|		72	5	24

Table 7: Classes VII, VIII and IX.

Class X	F	B	(i)	(ii)	(iii)	(iv)	(v)	(vi)
	1689ab	1234	8c	89	89	8b	8b	8c
	179abc	1234	9b	bc	bc	9c	9c	9b
	269abc	1235	8a	8c	8b	8c	8a	8b
	278abc	1235	bc	ab	ac	ab	bc	ac
	3689ac	1245	89	8a	8c	89	8c	8a
	3789bc	1245	ac	9c	9a	ac	9a	9c
	468abc	1345	9c	9b	9c	9a	9b	9a
	4789ac	1345	ab	ac	ab	bc	ac	bc
	5689bc	2345	8b	8b	8a	8a	89	89
	5789ab	2345	9a	9a	9b	9b	ab	ab
	List No		39	40	41	42	43	44
	|G|		8	8	6	8	6	8
Class XI	F	B	(i)	(ii)				
	168abc	1234	9c	9b				
Use α	179abc	1234	ab	ac				
	2689bc	1235	8b	89				
	278abc	1235	9a	ab				
	3689ac	1245	89	8c				
	3789bc	1245	ac	9a				
	4689ab	1345	8c	8b				
	4789ac	1345	9b	9c				
	569abc	2345	8a	8a				
	5789ab	2345	bc	bc				
	List No		45	46				
	|G|		55	6				
Class XII	None							
Class XIII	None							
Class XIV	None							

Table 8: Classes X, XI, XII, XIII and XIV.

6. Completing the Design; Prong Hunting

As a demonstration of a typical case we now complete Class III (i) to a 4–(12, 6, 4) design. Having completed sections F and B according to Table 5, let us now use some of the theorems from Part II to fill in further blocks. By Theorem 8 of Part II, the last block in each subsection of section C cannot have five points in common with any block of section B. Therefore, in section C, the block [**123** . . .] cannot contain any of the pairs **89**, **8b**, **9c**, **ab** and so has a unique completion to [**1238ac**]. The last blocks of other the subsections of section B likewise all have unique completions. In particular, we have the blocks [**1238ac**], [**1249bc**], [**125abc**].

Now, by Theorem 3 of Part II, the last blocks in the subsections of section B can never have five points in common with the last blocks of the subsections of section D. Therefore, in the first subsection of section D, the block [**12 . . . .**] has a unique completion to [**1289ab**]. In the same subsection, the block [**126 . . .**] must contain **9** by the RF applied to the blocks [**1678ab**] and [**168abc**]. Also, by the RF applied to [**2679bc**] and [**269abc**], [**126 . . .**] must contain **8**. Thus we have the partially completed block [**126 . 89**], and likewise the partially completed block [**127 . 89**]. Each subsection of section D can be similarly treated leading to the situation in table 9. The blocks marked with an asterisk are definitely known to be of Type I.

A
123456
123457

B
123489
1234ab
12358b
12359c
12458c*
12459a
13459b
1345ac
23458a
2345bc

C	
123679	14567c
1236 . .	1456 . .
1236 . .	1456 . .
1237 . .	1457 . .
1237 . .	1457 . .
1238ac	1458ab
12467b	23467a*
1246 . .	2346 . .
1246 . .	2346 . .
1247 . .	2347 . .
1247 . .	2347 . .
1249bc	2349ac
125678	23567c
1256 . .	2356 . .
1256 . .	2356 . .
1257 . .	2357 . .
1257 . .	2357 . .
125abc	2359ab
134678	245679
1346 . .	2456 . .
1346 . .	2456 . .
1347 . .	2457 . .
1347 . .	2457 . .
1348bc	24589b
13567a	34567b
1356 . .	3456 . .
1356 . .	3456 . .
1357 . .	3457 . .
1357 . .	3457 . .
13589a	34589c

D	
1267ac	24678c
126 . 89	246 . 8a
127 . 89	247 . 9b
1289ab	248abc
1367bc	2567ab
136 . 9b	256 . 8c
137 . 8a	257 . 9c
139abc	2589ac
14679a	34679c
146 . 9a	346 . ab
147 . 8b	347 . ab
1489ac	3489ab
15679b	356789
156 . 9c	356 . bc
157 . 8c	357 . ac
1589bc*	358abc
23678b	4567a8
236 . 8b	456 . ac
237 . 9a	457 . bc
2389bc	459abc

E
16789c
1678ab
26789a
2679bc
3678ac
3679ab*
46789b
467abc
5678bc
5679ac

F
168abc
179abc
269abc
278abc
3689ac
3789bc
4689bc
4789ac
5689ab
5789ab

Table 9: Class III (i). The situation at the end of the second paragraph of Section 6. Blocks definitely known to be of Type I are marked with an asterisk.

The completion to a 4–(12, 6, 4) design can be carried out in many ways and the details differ from case to case. Usually many, if not all, of the subsections of section D can be completed at this stage. For example, [**1589bc**] is a Type I block so the blocks [**156 . 9c**] and [**157 . 8c**] can only be completed with **a** if five point intersections with [**1589bc**] are to be avoided. Also, the blocks [**139abc**], [**1489ac**], [**2389bc**] and [**248abc**] all have friendly mates in section F, so by the RF, there must be blocks [**13689b**], [**1469ab**], [**2368ab**] and [**24689a**] in section D. Each triple from {**8, 9, a, b, c**} occurs just once with each of **6** and **7** in section D (Lemma 4, Part II) so [**126 . 89**] can only be completed with **c**, and so on.

123456
123457

B
123489
1234ab
12358b
12359c
12458c*
12459a
13459b
1345ac
23458a
2345bc

C	
123679	14567c
12369a	1456bc
1236bc	145689
1237ab	145789
12378c	1457ab
1238ac	1458ab
12467b	23467a*
12468b	23469b*
1246ac	23468c
12478a*	2347bc
12479c	234789
1249bc	2349ac
125678	23567c
12568a	2356ac
12569b	235689
12579a	23578a
1257bc	23579b
125abc	2359ab
134678	245679
13468a	24569c
13469c	2456ab
13479b	2457ac*
1347ac	24578b
1348bc	24589b
13567a	34567b
1356ab	34568b
13568c*	34569a
13578b	34579a
13579c	34578c
13589a	34589c

D	
1267ac	24678c
12689c	24689a
12789b	2479ab*
1289ab	248abc
1367bc	2567ab
13689b*	2568bc
13789a	25789c
139abc	2589ac
14679a	34679c
1469ab	346abc
1478bc	3478ab
1489ac	3489ab
15679b	356789
1569ac	3569bc*
1578ac*	357abc
1589bc*	358abc
23678b	4567a8
2368ab	4568ac
2379ac	4579bc
2389bc	459abc

E
16789c
1678ab
26789a
2679bc
3678ac
3679ab*
46789b
467abc
5678bc
5679ac

F
168abc
179abc
269abc
278abc
3689ac
3789bc
4689bc
4789ac
5689ab
5789ab

Table 10: Class III (i). The completed design. Type I blocks are marked with an asterisk.

To complete the subsections of section B the process of *prong hunting* will nearly always work on at least one of the blocks of section B. In the present case section B has the block [**12358b**] which has one-point intersections on **1, 3, 5, 8, b** with the blocks [**14679a**], [**34679c**] [**5679ac**], [**4789ac**], [**467abc**] respectively. The only point common to these five blocks is **7**, which, by the prong laws must be the prong of the mate of [**12358b**]. Therefore this mate is [**13578b**]. Then, in the **135**–subsection of section B the block [**1357 . .**] can only be completed with a pair from {**9, a, c**}. By the RF with the blocks [**13567a**], [**13589a**], [**357 . ac**], both **9a** and **ac** are forbidden. Therefore there is a block [**13579c**], which has [**12359c**] of section B as a friendly mate. A count of quadruples containing the triple **135** shows that the blocks [**1356 . .**], [**1356 . .**] between them must contain each of **8**, **a**, **b**, **c** just once. The RF forbids **8** to pair with either of **a** or **b** so the **135**–subsection is completed by the blocks [**13568c**] and [**1356ab**]. Now, in section D the only legimate way of completing [**137 . 8a**] is with a **9**, and so on. The completed design is given in Table 10.

7. Comments

It was found that each of the cases listed for classes I to XII can be completed to a 4–(12, 6, 4) design in a unique fashion, but none of the cases in classes XII, XIII and XIV have a completion. The completed designs always have Type II blocks in multiples of twelve. As is to be expected, there are isomorphisms occuring among the forty-six completed designs. The designs were checked using Cayley [5] and *nauty* [6]. These produced nine equivalence classes which, fortunately, are such that designs are in different classes if and only if the orders of their automorphism groups are different. To produce a set of standard designs a representative of each isomorphism class was chosen and given a new name. The 5–(12, 6, 1) design as modelled in Breach [2] we have called Design 1. Then the others, lexicographically ordered according to decreasing numbers of Type I blocks and group orders, have been called Design 2, Design 3, etc.

In Table 11 is given the complete list of standard designs together with their group orders and the numbers of blocks of each type. The blocks of all the ten standard designs are displayed in Breach, Elmes, Sharry and Street [3]. The design completed in this paper as a demonstration is Design 8.

As a matter of observation, in the forty-six completed models the blocks of section D that contain **7** but not **6** are always the same. If a short proof of this statement can be found beforehand then the completion of the forty-six models would be considerably simplified. But what is more interesting is that in any of the 4–(12, 6, 4) designs completed according to this paper the restriction on **7** is always a 4–(11, 5, 1) design; if all the blocks containing **7** are selected and **7** is then deleted from those blocks, then one would expect that the new blocks so formed are those of a 3–(11, 5, 4) design, not a 4–design. Again, if this could be predicted at the beginning then the classification of the 4–(12, 6, 4) designs would be much easier.

List No	STANDARD DESIGN	Group Order	No Blocks Type I	No Blocks Type II
List No 1	Design 1	95,040	132	0
List No 2	Design 2	1440	60	72
List No 14	Design 3	144	60	72
List No 31	Design 4	16	36	96
List No 33	Design 5	72	24	108
List No 24	Design 6	24	24	108
List No 9	Design 7	6	24	108
List No 8	Design 8	8	12	120
List No 4	Design 9	5	12	120
List No 45	Design 10	55	0	132

Table 11: Standard 4–(12, 6, 4) designs.

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