AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **70(1)** (2018), Pages 75–85

On local matching property in groups and vector spaces

Mohsen Aliabadi Mano Vikash Janardhanan

Department of Mathematics, Statistics, and Computer Science
University of Illinois
851 S. Morgan St, Chicago, IL 60607
U.S.A.
maliab2@uic.edu mjana2@uic.edu

Abstract

In this paper, we define locally matchable subsets of a group which are derived from the concept of matchings in groups and used as a tool to give alternative proofs for existing results in matching theory. We also give the linear analogue of a local matching property for subspaces in a field extension. Our tools mix additive number theory, combinatorics and algebra.

1 Introduction

The notion of *matchings* in groups was used to study an old problem of Wakeford concerning canonical forms for symmetric tensors [8]. Losonczy in [6] introduced matchings in order to generalize a geometric property of lattices in Euclidean space. A matching in an abelian group $(G, +)$ is a bijection $f : A \to B$, where A, B are finite subsets of G such that $0 \notin B$, fulfilling $a + f(a) \notin A$, for all $a \in A$, and G is said to have the *matching property*, if a matching always exists, as long as A and B are finite of the same cardinality. This topic has found some interest in the literature: for example, in 2006 Eliahou-Lecouvey generalized Losonczy's results to arbitrary groups [3], and in 2010 Eliahou-Lecouvey went over to subspaces in field extensions [4].

The subject of the present paper is to consider *local matchings*: given a proper subgroup $H < G$ such that $H \cap B \neq \emptyset$ and $a + H \subseteq A$, for some $a \in A$, there is a bijection $f : A' \to H \cap B$, for some $A' \subseteq A$, such that $a + f(a) \notin A$, for all $a \in A'$. In this case, A is said to be *locally matched* to B. Any matching being a local matching, it is natural to ask whether, conversely, a local matching property implies the matching property. The answer to this question is "Yes" and we will use this result to give an alternative proof for Losonczy's main result in [6]. Moreover, these questions are also discussed in the context of subspaces in field extensions.

The purpose of this paper is to find the relations between the local matching property and the matching property in groups and vector spaces, to give alternative proofs for existing results on the matching property for groups and also the linear analogue. Section 2 is devoted to the results proved on matchings in groups and vector spaces and also some tools of additive number theory required to prove our main results. In Section 3, we will show the equivalence between matchings and local matchings for subsets of a group. Section 4 is concerned with the linear analogue of one of Losonzy's results on matchings for cyclic groups. Finally, in Section 5, we show that for vector spaces in a field extension whose algebraic elements are separable, the linear local matching property implies the matching property.

2 Preliminaries

First, we define the matching property for subspaces in a field extension. Let $K \subset L$ be a field extension and let A and B be n-dimensional K-subspaces of the field extension L. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be bases of A and B, respectively. It is said that $\mathcal{A}$ is *matched* to $\mathcal{B}$ if

$$a_i b \in A \ \Rightarrow \ b \in \langle b_1, \ldots, \hat{b}_i, \ldots, b_n \rangle,$$

for all $b \in B$ and $i = 1, \ldots, n$, where $\langle b_1, \ldots, \hat{b}_i, \ldots, b_n \rangle$ is the hyperplane of B spanned by the set $\mathcal{B} \setminus \{b_i\}$; moreover, it is said that A is *matched* to B if every basis $\mathcal{A}$ of A can be matched to a basis $\mathcal{B}$ of B. As is seen, the matchable bases are defined in a natural way based on the definition of a matching in a group. Indeed, we can consider $\mathcal{A}$ and $\mathcal{B}$ as subsets of the multiplicative group L^* and so the bijection $a_i \mapsto b_i$ is a matching in the group setting sense. It is said that L has the *linear matching property* if, for every $n \geq 1$ and all n-dimensional subspaces A and B of L with $1 \notin B$, the subspace A is matched with B. A *strong matching* from A to B is a linear isomorphism $\varphi : A \to B$ such that any basis $\mathcal{A}$ of A is matched to the basis $\varphi(\mathcal{A})$ of B. It is proved that there is a strong matching from A to B if and only if $AB \cap A = \{0\}$. In this case, any isomorphism $\varphi : A \to B$ is a strong matching [4].

Now, we give our definition for matchable subsets of two matchable bases:

Definition 2.1 Let $\tilde{A}$ and $\tilde{B}$ be two non-zero m-dimensional K-subspaces of A and B, respectively. We say that $\tilde{A}$ is *A-matched* to $\tilde{B}$, if for any basis $\tilde{\mathcal{A}} = \{a_1, \ldots, a_m\}$ of $\tilde{A}$, there exists a basis $\tilde{\mathcal{B}} = \{b_1, \ldots, b_m\}$ of $\tilde{B}$ for which $a_i b_i \notin A$, for $i = 1, \ldots, m$. In this case, it is also said that $\tilde{\mathcal{A}}$ is *A-matched* to $\tilde{\mathcal{B}}$.

The following is the linear analogue of locally matchable subsets for the vector spaces in a field extension.

Definition 2.2 Let $K \subset L$ be a field extension and let A, B be two n-dimensional K-subspaces of L. We say that A is *locally matched* to B if for any intermediate subfield $K \subset H \subsetneq L$ with $H \cap B \neq \{0\}$ and $aH \subseteq A$, for some $a \in A$, one can find a subspace $\tilde{A}$ of A such that $\tilde{A}$ is A-matched to $H \cap B$.

Definition 2.3 We say that $K \subset L$ has the *linear local matching property* if, for every $n \geq 1$ and all n-dimensional subspaces A and B of L with $1 \notin B$, the subspace A is locally matched to B.

The following theorem is a dimension criterion for matchable bases [4, Proposition 3.1]. This will be used as a tool to prove Theorem 4.2. For more results on linear versions of matchings, see [1].

Theorem 2.4 *Let $K \subset L$ be a field extension and let A and B be two n-dimensional K-subspaces of L. Suppose that $\mathcal{A} = \{a_1, \ldots, a_n\}$ is a basis of A. Then $\mathcal{A}$ can be matched to a basis of B if and only if, for all $J \subseteq \{1, \ldots, n\}$, we have:*

$$\dim_K \bigcap_{i \in J} \left(a_i^{-1} A \cap B\right) \leq n - \#J.$$

The following theorem gives a necessary and sufficient condition for a field extension to have the linear matching property.

Theorem 2.5 *[4, Theorem 5.2] Let $K \subset L$ be a field extension. Then L has the linear matching property if and only if L contains no proper finite dimensional extension over K.*

For proving our main results, we shall need the following theorem from [7, Theorem 4.3, p. 116].

Theorem 2.6 (Kneser) *If $C = A + B$, where A and B are finite subsets of an abelian group G, then*

$$\#C \geq \#A + \#B - \#H,$$

where H is the subgroup $H = \{g \in G : \ C + g = C\}$.

See [2] for more details regarding the following theorem which is the linear analogue of Kneser's theorem.

Theorem 2.7 *Let $K \subset L$ be a field extension in which every algebraic element of L is separable over K. Let $A, B \subset L$ be non-zero finite-dimensional K-subspaces of L and H be the stabilizer of $\langle AB \rangle$, i.e. $H = \{x \in L; \, x\langle AB \rangle \subseteq \langle AB \rangle\}$. Then*

$$\dim_K \langle AB \rangle \geq \dim_K A + \dim_K B - \dim_K H.$$

We remark that in the above theorem, we denote by $\langle AB \rangle$ the K-subspace of L generated by the Minkowski product AB which is defined as

$$AB := \{ab; \, a \in A, b \in B\}.$$

3 Local matching property for groups

The following theorem shows that the local matching property is equivalent to the matching property in abelian groups. The main idea of our proof is obtained from the Loconczy paper [6, Theorem 3.1] and Eliahou-Lecouvey paper [3, Theorem 3.3].

Theorem 3.1 *Let G be an additive abelian group and A, B be non-empty finite subsets of G satisfying the conditions $\#A = \#B$ and $0 \notin B$. If A is locally matched to B, then A is matched to B.*

We shall need the following lemma to prove Theorem 3.1.

Lemma 3.2 *Let G, A and B be as Theorem 3.1. For any non-empty subset S of A, assume that $\#S \leq \#(B \setminus U)$, where $U = \{b \in B; s + b \in A$, for any $s \in S\}$. Then, there is a matching from A to B.*

Proof. Assume that $A = \{a_1, \ldots, a_n\}$ and define $S_J = \{a_i;\, i \in J\}$, for any $J \subseteq \{1, \ldots, n\}$. Set $U_J = \{b \in B;\, s + b \in A$, for any $s \in S_J\}$. Clearly, $U_J = \bigcap\limits_{i \in J} U_{\{i\}}$. Consider the collection $\mathcal{E} = \{B \setminus U_{\{1\}}, \ldots, B \setminus U_{\{n\}}\}$. We have

$$\#\bigcup_{i \in J} \left(B \setminus U_{\{i\}}\right) = \# \left(B \setminus \bigcap_{i \in J} U_{\{i\}} \right) = \# \left(B \setminus U_J\right) \geq \#S_J = \#J,$$

for any $J \subseteq \{1, \ldots, n\}$. Then by Hall's Marriage Theorem [5, Theorem 2], one can find a transversal $(b_1, \ldots, b_n) \in \mathcal{E}$. The mapping $a_i \mapsto b_i$ is a matching from A to B. $\square$

Proof of Theorem 3.1. We remark that for the case that G is a finite group and $A = G \setminus \{0\}$, we have the matching $\begin{array}{rl} f : A & \to B \\ a & \mapsto -a \end{array}$. Thus, we may assume that $A \neq G \setminus \{0\}$. Suppose there is no matching from A to B. We are going to reach a contradiction. Using Lemma 3.2, there exists a non-empty finite subset S of A such that $\#(B \setminus U) < \#S$, where $U = \{b \in B :\ s + b \in A$, for any $s \in S\}$. Let $\#A = \#B = n$, then $\#U + \#S > n$. Set $U_0 = U \cup \{0\}$. Using Kneser's Theorem one can find the subgroup H of G such that

$$\#(U_0 + S) \geq \#U_0 + \#S - \#H, \tag{1}$$

where $H = \{g \in G :\ g + U_0 + S = U_0 + S\}$. Applying Kneser's Theorem for $U' = H \cup U$ and S, we can find the subgroup H' of G for which

$$\#(U' + S) \geq \#U' + \#S - \#H', \tag{2}$$

where $H' = \{g \in G : \ g + U' + S = U' + S\}$. We claim that $H = H'$ and to prove this, it suffices to show that $U' + S = U_0 + S$. We have

$$\begin{aligned} U' + S &= (H \cup U) + S = (H + S) \cup (U_0 + S) \\ &= (H + S) \cup (U_0 + S + H) \\ &= H + (S \cup (U_0 + S)) \\ &= H + (U_0 + S) = U_0 + S. \end{aligned} \tag{3}$$

Then $H = H'$ and it follows from (2) that

$$\#(U_0 + S) \geq \#U' + \#S - \#H. \tag{4}$$

Using (3), (4) we obtain

$$\begin{aligned} \#(U_0 + S) &= \#(U' + S) \\ &= \#U' + \#S - \#H \\ &= \#(H \cup U) + \#S - \#H \\ &= \#H + \#U - \#(H \cap U) + \#S - \#H \\ &= \#U + \#S - \#(H \cap U). \end{aligned} \tag{5}$$

As $U_0 + S = S \cup (S + U)$, (5) implies

$$\#(S \cup (S + U)) \geq \#U + \#S - \#(H \cap U). \tag{6}$$

Now, we have two cases for $H \cap U$.

1. If $H \cap U$ is empty, then by (6) we conclude that $\#(S \cup (S + U)) \geq n$. On the other hand $S \cup (S + U)$ is a subset of A. We would have $\#A > n$, which contradicts $\#A = n$ above.

2. If $H \cap U$ is non-empty, so is $H \cap B$. Also, if $s \in S \subseteq A$, then according to the definition of H, $s + H \subseteq U_0 + S + H = U_0 + S \subseteq A$. As A is locally matched to B, then there is a subset $\tilde{A}$ of A and a bijection $f : \tilde{A} \to H \cap B$ such that $a + f(a) \notin A$, for any $a \in \tilde{A}$. We claim that $f^{-1}(H \cap U) \cap (U_0 + S)$ is empty. If not and $a \in f^{-1}(H \cap U) \cap (U_0 + S)$, then $a + f(a) \in (U_0 + S) + H$ as $a \in U_0 + S$ and $f(a) \in H \cap U \subseteq H$. Since $U_0 + S \subseteq A$, then $a + f(a) \in A$ which is a contradiction. Therefore $f^{-1}(H \cap U) \cap (U_0 + S)$ is empty. As the sets $f^{-1}(H \cap U)$ and $U_0 + S$ are both subsets of A and have nothing in common, then $\#f^{-1}(H \cap U) + \#(U_0 + S) \leq n$. Thus $\#(H \cap U) + \#(U_0 + S) \leq n$ and this tells us $\#(H \cap U) + \#(S \cup (S + U)) \leq n$. Next, using (6) yields that $\#U + \#S \leq n$ which is a contradiction.

Therefore in both cases we extract contradictions. Then there is a matching from A to B. □

Remark 3.3 Note that in the second case above, $H \neq G$. We argue this in two cases:

1. Suppose that $0 \in A$. Assume to the contrary $G = H$. Then we have

$$\#G = \#H \leq \#(H + U_0 + S) = \#(U_0 + S) \leq \#A.$$

Then $\#G = \#A = \#B$. This contradicts $0 \notin B$.

2. Suppose that $0 \notin A$. Assume to the contrary $G = H$. Then we have

$$\#G = \#H \leq \#(H + U_0 + S) = \#(U_0 + S) \leq \#(A \cup \{0\}).$$

Then $G = A \cup \{0\}$ and this contradicts $A \neq G \setminus \{0\}$.

Example 3.4 Assume that $G = \mathbb{Z}/8\mathbb{Z}$, $A, B \subseteq G$ with $A = \{0, 2, 6\}$ and $B = \{1, 3, 4\}$. The only non-trivial subgroup of G which satisfies the condition $a + H \subseteq A$ is $H = \{0, 4\}$. Note that here $a = 2$. If $A' = \{0\} \subseteq A$, then for the bijection $f : A' \to H \cap B$ defined as $f(0) = 4$ we have $0 + f(0) = 4 \notin A$. Then A is locally matched to B and using Theorem 3.1, A is matched to B.

Using Theorem 3.1, we give an alternative proof to the following result of Losonzcy [6, Theorem 3.1].

Theorem 3.5 *An abelian group G has the matching property if and only if it is torsion-free or cyclic of prime order.*

Proof. Assume that G is either torsion-free or cyclic of prime order. Then G has no non-trivial subgroup of finite order. This means if $A, B \subset G$ with $\#A = \#B$ and $0 \notin B$, then A is locally matched to B (because in this case $H = \{0\}$ is the only proper subgroup of G. But $H \cap B = \emptyset$.) Using Theorem 3.1 yields that A is matched to B and so G has matching property.

Conversely, assume that G is neither torsion-free nor cyclic of prime order. Then it has a non-trivial finite subgroup H. Choose $g \in G \setminus H$, set $A = H$ and $B = H \cup \{g\} \setminus \{0\}$. Clearly, $H \cap B \neq \emptyset$ and $a + H \subseteq A$ for some $a \in A$ (Indeed for any $a \in A$). If A is locally matched to B, then one can find an A-matching f from a subset A_0 of A to $H \cap B$. But if $a \in A_0$, then $a + f(a) \in H + (H \cap B) = H + (H \setminus \{0\}) = H = A$, which is a contradiction. Then A is not locally matched to B and so by Theorem 3.1, A is not matched to B. Therefore G has no matching property. □

Corollary 3.6 *Let G, A and B be as Theorem 3.1 and $\#A = \#B = n > 1$. Denote by $n(G)$ the smallest cardinality of a non-zero subgroup of G. If $n < n(G)$, then A is matched to B.*

Proof. Since $n < n(G)$, then it is clear that A is locally matched to B. Using Theorem 3.1 yields A is matched to B. □

4 The linear analogue of Losonzcy's result on matchable subsets

In this section, we formulate and prove the linear analogue of the following theorem of Losonzcy proven in [6] which basically investigates the matchable subspaces in a simple field extension.

Theorem 4.1 *Let G be a non-trivial finite cyclic group such that $\#A = \#B$ and every element of B is a generator of G. Then there exists at least one matching from A to B.*

We say that $K \subset L$ is a simple field extension if $L = K(\alpha)$, for some $\alpha \in L$. Also, if B is a K-subspace of L such that $K(b) = L$, for any $b \in B \setminus \{0\}$, we say that B is a primitive K-subspace of L. The main ingredient in our proof is the linear version of Kneser's theorem.

Theorem 4.2 *Let $K \subset L$ be separable field extension and A and B be two n-dimensional K-subspaces in L with $n \geq 1$ and B is a primitive K-subspace of L. Then A is matched with B.*

Proof. Assume that A is not matched to B. Using Theorem 2.4, one can find $J \subseteq \{1, \ldots, n\}$ and a basis $\mathcal{A} = \{a_1, \ldots, a_n\}$ of A such that

$$\bigcap_{i \in J} \left(a_i^{-1} A \cap B\right) > n - \#J. \tag{7}$$

Set $S = \langle a_i : \ i \in J \rangle$ the K-subspace of A spanned by a_i's, $i \in J$, $U = \bigcap_{i \in J} \left(a_i^{-1} A \cap B\right)$ and $U_0 = U \cup \{1\}$. Now, by Theorem 2.7 one can find a subfield H of L such that

$$\dim_K \langle U_0 S \rangle \quad \geq \quad \dim_K U_0 + \dim_K S - \dim_K H,$$

where H is the stabilizer of $\langle U_0 S \rangle$, i.e. $H = \{x \in L : \ x \langle U_0 S \rangle \subseteq \langle U_0 S \rangle\}$. Define $U' = \langle H \cup U \rangle$. Using Theorem 2.7 again, we have

$$\dim_K \langle U' S \rangle \geq \dim_K \langle U' \rangle + \dim_K \langle S \rangle - \dim_K H',$$

where H' is the stabilizer of $\langle U' S \rangle$. Next, we have

$$\begin{aligned} \langle U' S \rangle &= \langle (H \cup U) S \rangle = \langle HS \cup U_0 S \rangle \\ &= \langle HS \cup HU_0 S \rangle = H \langle S \cup U_0 S \rangle \\ &= H \langle U_0 S \rangle = \langle U_0 S \rangle. \end{aligned} \tag{8}$$

From this it follows that $H = H'$ and then

$$\dim_K \langle U' S \rangle \geq \dim_K \langle U' \rangle + \dim_K S - \dim_K H. \tag{9}$$

Using (8) and (9), we have

$$\begin{aligned}\dim_K\langle U_0S\rangle &\geq \dim_K U' + \dim_K S - \dim_K H \\ &= \dim_K\langle H\cup U\rangle + \dim_K S - \dim_K H. \end{aligned} \tag{10}$$

Using (10), the fact that $\langle U_0S\rangle = \langle S\cup SU\rangle$ and the inclusion-exclusion principle for vector spaces we have:

$$\begin{aligned}\dim_K\langle S\cup SU\rangle &\geq \dim_K\langle H\cup U\rangle + \dim_K S - \dim_K H \\ &= \dim_K H + \dim_K U - \dim_K(H\cap U) + \dim_K S - \dim_K H \\ &= \dim_K U + \dim_K S - \dim_K(H\cap U). \end{aligned} \tag{11}$$

Now, we have two cases for the subspace $H \cap U$.

1. If $H\cap U = \{0\}$, then (7) and (11) imply $\dim_K\langle S\cup SU\rangle \geq n$ and this is impossible as $S\cup SU \subseteq A$ and $\dim_K A = n$.
2. If $H\cap U \neq \{0\}$, then $H\cap B \neq \{0\}$ and since B is a primitive subspace of L, then $H = L$. By the definition of U and S, $HUS \subseteq A$ and this follows $LUS \subseteq A$ and so $A = L$. Then $B = L$ as $\dim_K A = \dim_K B$ and this means $K \subseteq B$. Therefore if $a \in K \setminus \{0\}$, then $K = K(a) = L$, which is impossible.

In both cases, we extract contradictions. Then A is matched to B. □

5 Local matching property for subspaces in a field extension

The following theorem shows that the linear local matching property implies the linear matching property for subspaces of a field extension whose algebraic elements are separable. Note that this result can probably be reformulated for any field extension $K \subset L$ without any condition on separability.

Theorem 5.1 *Let $K \subset L$ be a field extension in which every algebraic element of L is separable over K. Let $A, B \subset L$ be two non-zero n-dimensional K-subspaces with $1 \notin B$. If A is locally matched to B, then A is matched to B.*

Proof. Assume to the contrary A is not matched to B. Then, by Theorem 2.3 there exist a basis $\mathcal{A} = \{a_1, \dots, a_n\}$ of A and $J \subseteq \{1, \dots, n\}$ such that

$$\dim_K \bigcap_{i\in J} \left(a_i^{-1}A\cap B\right) > n - \#J.$$

Set $S = \langle a_i :\ i \in J\rangle$ as a K-subspace of A, $U = \bigcap_{i\in J}(a_i^{-1}A\cap B)$ and $U_0 = \langle U\cup\{1\}\rangle$. Using Theorem 2.7 there exists an intermediate subfield H of $K \subset L$ such that

$$\dim_K\langle U_0S\rangle \geq \dim_K U_0 + \dim_K S - \dim_K H, \tag{12}$$

where H is the stabilizer of $\langle U_0S\rangle$. Define $U' = H \cup U$. Reusing Theorem 2.7, one can find an intermediate subfield H' of $K \subset L$ for which

$$\dim_K\langle U'S\rangle \geq \dim_K\langle U'\rangle + \dim_K S - \dim_K H', \tag{13}$$

where H' is the stabilizer of $\langle U'S\rangle$. The following computations show that $\langle U'S\rangle = \langle U_0S\rangle$;

$$\begin{aligned}\langle U'S\rangle &= \langle (H\cup U)S\rangle = \langle HS\rangle \cup \langle U_0S\rangle \\ &= \langle HS\rangle \cup \langle U_0SH\rangle = H\langle S\cup U_0S\rangle \\ &= H\langle U_0S\rangle = \langle U_0S\rangle. \end{aligned}\tag{14}$$

Then, the stabilizers of these two subspaces must be the same, i.e. $H = H'$. Then we would have

$$\dim_K\langle U'S\rangle \geq \dim_K\langle U'\rangle + \dim_K S - \dim_K H. \tag{15}$$

Bearing (13) and (14) in mind and using the inclusion-exclusion principle for vector spaces we obtain:

$$\begin{aligned}\dim_K\langle U_0S\rangle &= \dim_K\langle U'S\rangle \\ &\geq \dim_K\langle U'\rangle + \dim_K S - \dim_K H \\ &= \dim_K\langle H\cup U\rangle + \dim_K S - \dim_K H \\ &= \dim_K H + \dim_K U - \dim_K(H\cap U) + \dim_K S - \dim_K H \\ &= \dim_K U + \dim_K S - \dim_K(H\cap U). \end{aligned}\tag{16}$$

Now, we have two cases for $H \cap U$.

1. If $H\cap U = \{0\}$, then $\dim_K\langle S\cup SU\rangle > n$. On the other hand since $S\cup SU \subseteq A$, we would have $\dim_K A > n$, contradicting our assumption $\dim_K A = n$.
2. If $H \cap U$ is a non-zero vector space, then $H \cap B$ is non-zero. It is clear that $aH \subseteq A$, for some $a \in A$ (Indeed, $USH \subseteq A$). Since A is locally matched to B, one can find a subspace $\tilde{A}$ of A such that $\tilde{A}$ is A-matched to $H \cap B$. Let $\tilde{A} \cap \langle U_0S\rangle \neq \{0\}$ and choose a non-zero element a of it. We extend $\{a\}$ to a basis $\{a, a_2, \ldots, a_m\}$ for $\tilde{A}$. Then, there exists a basis $\{b, b_2, \ldots, b_m\}$ of $H \cap B$ such that $ab \notin A$ and $a_ib_i \notin A$, where $2 \leq i \leq m$, as A is locally matched to B. But, we have $ab \in \langle U_0S\rangle H = \langle U_0S\rangle \subseteq A$, which contradicts the case $\tilde{A}$ is A-matched to $H \cap B$. So $\tilde{A} \cap U_0S = \{0\}$. Then, $\dim_K \tilde{A} + \dim_K\langle U_0S\rangle \leq n$. This yields $\dim_K\langle H \cap U\rangle + \dim_K\langle (U \cup \{1\})S\rangle \leq n$. This follows $\dim_K\langle H \cap U\rangle + \dim_K\langle S \cup SU\rangle \leq n$. So, by (16) we have $\dim_K U + \dim_K S \leq n$, which is impossible.

Then in both cases, we extract contradictions and so A is matched to B. □

Remark 5.2 Note that in the second case above, $H \neq L$. We justify this as follows; assume to the contrary $H = L$. Then we have

$$\begin{aligned}[L : K] = [H : K] &\leq \dim_K H\langle U_0S\rangle \\ &= \dim_K\langle U_0S\rangle = \dim_K\langle US\cup S\rangle \\ &\leq \dim_K A = \dim_K B.\end{aligned}$$

Then $B = L$ and this contradicts $1 \notin B$.

Example 5.3 Assume that $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt[35]{2})$. Let A be the subspace of those elements of $\mathbb{Q}(\sqrt[5]{2})$ whose trace over $\mathbb{Q}$ is zero. Then $\dim_k A = 4$. Let V be the subspace of those elements of $\mathbb{Q}(\sqrt[7]{2})$ whose trace over $\mathbb{Q}$ is zero and take a four-dimensional subspace B of V. Clearly, for every non-trivial intermediate subfield H of $K \subseteq L$, $[H : K] \geq 5$ and then $aH \nsubseteq A$, for any $a \in A$. Thus A is locally matched to B. Using Theorem 5.1, A is matched to B.

Corollary 5.4 *Let $K \subset L$ be a field extension in which every algebraic element of L is separable over K. If $K \subset L$ has the local linear matching property, then it possesses the linear matching property.*

As we mentioned, in Theorem 5.1 the extension $K \subset L$ is assumed to have all its algebraic elements separable. Are these results valid without this hypothesis? We conjecture that this is the case.

Conjecture 5.5 *Let $K \subset L$ be a field extension. Then $K \subset L$ possesses the linear matching property if it possesses the local linear matching property.*

Using Theorems 5.1, we give a short proof of a special case of Theorem 2.5.

Let $K \subset L$ be a field extension whose algebraic elements are separable and has no proper intermediate field with a finite degree. If A and B are two n-dimensional K-subspaces of L with $n \geq 1$ and $1 \notin B$, clearly A is locally matched to B and then A is matched to B. This means $K \subset L$ has the linear matching property.

Remark 5.6 That the linear matching property implies the local linear matching property is immediate from Theorem 2.5. However, If A is matched to B, in the field extension setting sense, whether A is locally matched to B is still unsolved. This is valid in some specific cases. For example, if there is a strong matching from A to B, one can prove that A is locally matched to B. Further investigations along those lines could prove to be worthwhile.

Acknowledgements

We are grateful to Professor Shmuel Friedland for his useful suggestions and comments and detecting several mistakes in the earlier version of the proof of Theorem 3.1. We would also like to thank the referees for making several useful comments.

References

[1] M. Aliabadi, M. Hadian and A. Jafari, On matching property for groups and field extensions, *J. Algebra Appl.* **15** no. 1 (2016), 1650011, 13 pp.

[2] S. Eliahou, M. Kervaire and C. Lecouvey, On the product of vector spaces in a commutative field extension, *J. Number Theory* **129** (2009) no. 2, 339–348.

[3] S. Eliahou and C. Lecouvey, Matching in arbitrary groups, *Adv. Appl. Math.* **40** (2008), 219–224.

[4] S. Eliahou and C. Lecouvey, Matching subspaces in a field extension, preprint. `arXiv`: 1208.2792v1.

[5] P. Hall, On representatives of subsets, *J. London Math. Soc.* **10** (1935), 26–30.

[6] J. Losonzcy, On matchings in groups, *Adv. Appl, Math.* **20** (1998), 385–391.

[7] M. B. Nathanson, *Additive Number Theory; Inverse problems on geometry of subsets*, *Springer-Verlag*, New York, Berlin, Heidelberg (1996).

[8] E. K. Wakeford, On canonical forms, *Proc. London Math, Soc.* **18** (1918-1919), 403–410.

(Received 17 Oct 2016; revised 19 Aug 2017, 12 Oct 2017)