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A note on nearly platonic graphs

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Abstract

We define a nearly platonic graph to be a finite k-regular simple planar graph in which all but a small number of the faces have the same degree. We show that it is impossible for such a graph to have exactly one disparate face, and offer some conjectures, including the conjecture that nearly platonic graphs with two disparate faces come in a small set of families.

1 Introduction

Several authors [4, 10, 11, 13] have been interested in planar embeddings of graphs in which almost all faces are of one type, with one or two exceptions. For the most part, these papers deal with *nearly regular* planar graphs: those in which most faces and vertices are of degrees that are a *multiple* of some m, and a small number of other faces have degrees that are not a multiple of m. The proof techniques involve transformations which may change the number of edges of one or more faces, preserving divisibility of their degrees by m. A typical theorem in the area is Lemma 2.2 of [13], which states that no 3-regular planar graph exists in which all but one face has degree a multiple of three.

These theorems thus leave open the full question with which this article is interested: is it possible to produce a vertex-regular planar graph in which almost all faces have one degree and a small number of faces have a different degree, regardless of whether the disparate face degrees are multiples of some m—e.g., can a 3-regular graph be drawn in which all faces are triangles except for a single 9-gon—and if so, what restrictions exist on the construction?[1]

For a single exceptional face, the answer is in the negative:

Theorem 1. *There is no finite, planar, regular graph that has all but one face of one degree and a single face of a different degree.*

Nearly platonic graphs with two exceptional faces do exist. All of our constructions at present are simple variants of the Platonic graphs; we conjecture that these are the only possibilities. For three exceptional faces, constructions become abundant.

For a question so easily stated, one suspects that the result is already folklore, perhaps demanding greater than usual diligence in checking the literature. If it is known, however, it is not widely known. Several commenters and one referee suggested checking the work of Michel Deza and collaborators on chemical graphs [5, 6] and related chemical graph software, CaGe and CPF [3]. Our main claim seems to appear in Deza's work, but in [5] without proof, and in [6] with a short paragraph directing the reader to examine another list. (We learned to our regret that M. Deza passed away during the preparation of this paper.) Certainly the result is apparently unknown to the combinatorialists who are our main audience. A search through standard graph theory textbooks [1, 7, 8], and others) yields no relevant theorem, preprint queries (such as [15]) attracted no firm answer, and citations of [4, 13] etc. remain interested in nearly regular graphs. We sought the advice of leading textbook authors in graph theory, who responded [2, 14] that our main theorem is "new to me, and of interest" and, for our conjecture, "I believe it but I don't know if it has actually been decided."

The only theorems our proof requires are basic theorems of graph theory, and some careful case by case vertex-counting. After the efforts described, we now have some confidence that publication of our elementary proof of this theorem and the description of the accompanying conjectures will be useful and hopefully stimulative of further investigation.

In Section 1.1 we recall the relevant theorems of graph theory and construct the basic properties we will make use of in the sequel. In Section 2 we establish the negative answer for the case with a single exceptional face; in the final section we discuss the cases of two and three exceptional faces, and offer some open questions.

[1]For the interested reader, the question arose in the context of teaching an introductory combinatorics course, in an attempt to construct a graph with exceptional outer face in anticipation of student error.

1.1 Basic theorems

A (v, e, f)-graph will denote a graph with that has v vertices and e edges that has a planar embedding with f faces. Consider such a graph in which the degree of each vertex is k, there are f_1 faces of degree d_1, and the remaining $f_2 = f - f_1$ have degree d_2. Every edge has two ends and abuts two faces, so twice the number of edges must equal both the sum of the degrees of all the vertices, and the sum of the degrees of the faces:

$$\begin{aligned} 2e &= kv \\ 2e &= f_1 d_1 + f_2 d_2. \end{aligned}$$

An important theorem in graph theory is Euler's formula, which holds that for all planar graphs,

$$v - e + f = 2.$$

Putting these pieces together and solving for various values we obtain:

$$f = \frac{kv - f_1(d_1 - d_2)}{d_2} \tag{1}$$

$$v(2d_2 - kd_2 + 2k) = 2f_1 d_1 + (4 - 2f_1)d_2 \tag{2}$$

$$\frac{e}{kd_2}\Big(4 - (k-2)(d_2 - 2)\Big) = \Phi(f_1, d_1, d_2), \tag{3}$$

where

$$\Phi(f_1, d_1, d_2) = 2 + \frac{f_1(d_1 - d_2)}{d_2} = 2 + f_1\left(\frac{d_1}{d_2} - 1\right).$$

If $k = 2$, our graph is just a polygon, which has two faces of equal degree (the inner and the outer). Ignoring those, we have $3 \leq k \leq 5$, since the minimum degree of a planar graph is at most 5, and $d_i \geq 3$, because faces must be at least triangles.

Now we can show that, regardless of k,

Lemma 2. *If $f_1 \leq 3$, then $\Phi(f_1, d_1, d_2) > 0$.*

Proof. If $d_1 \geq d_2$, then obviously $\Phi(f_1, d_1, d_2) > 0$, so we assume $d_1 < d_2$. Then

$$-1 < \frac{d_1 - d_2}{d_2} < 0$$

and so

$$2 - f_1 < \Phi(f_1, d_1, d_2) < 2.$$

Hence if $f_1 \leq 2$, then $\Phi(f_1, d_1, d_2) > 0$. We now consider $f_1 = 3$. If $d_2 \geq 6$, then

$$2e = d_2(f - 3) + 3d_1 \geq 6(f - 3) + 9 = 6f - 9 = 2(3f - 6) + 3 \geq 2e + 3$$

a contradiction. Hence $d_2 \leq 5$. Then

$$\Phi(3, d_1, d_2) \geq 2 + 3\left(\frac{3}{5} - 1\right) = \frac{4}{5} > 0.$$

□

Corollary 3. *If $f_1 \leq 3$, then*

$$(k, d_2) = (3,3), (3,4), (3,5), (4,3), \text{ or } (5,3). \tag{4}$$

Proof. The lemma shows that $\Phi(f_1, d_1, d_2)$ is positive, when $f_1 \leq 3$. Then Equation 3 forces

$$(k-2)(d_2-2) < 4.$$

There are only five integral solutions to this inequality when $k, d_2 \geq 3$. They are the solutions listed. □

If $f_1 = 0$ the five Platonic solids are obtained. One corresponds to each of the five possibilities enumerated in Corollary 3, and a little more work (see any relevant graph theory textbook, for instance [7]) shows that these are the only possible such graphs.

2 $f_1 = 1$: Nonexistence

If $f_1 = 1$, then a single face has degree different from all the others. We show that such a graph cannot exist.

We study the five possibilities for (k, d_2) above in turn. In each case, we will calculate the allowable number of vertices as a function of d_1: substitute k, d_2 and f_1 into Equation 2 and solve through for v.

$$v = \frac{2(d_1 + d_2)}{4 - (k-2)(d_2-2)} \tag{5}$$

Then we will consider how they might be adjacent to each other, eventually deriving a contradiction. In each case, what we essentially show, reformulated, is that the face regularity requirement has to be weakened further to gain any new graphs: the class of possible graphs for a given (k, d_2) with $f_1 \leq 1$ is still populated only by the Platonic graphs, $f_1 = 0$.

Without loss of generality we may assume that the graph has been drawn in the plane so that F, the unique face of degree d_1, is the outer face. Let $x_0 x_1 \ldots x_{d_1-1} x_0$ be ∂F, the cycle bounding F. All remaining vertices and edges are interior to ∂F. An edge that is not part of F's bounding cycle, but joins two vertices of the cycle, is called a chord.

Lemma 4. *For $f_1 = 1$, $(k, d_2) \in \{(3,3), (3,4), (3,5), (4,3), (5,3)\}$, the outer face has no chords.*

Proof. We proceed by contradiction. Assume there exists such a graph with outer face F and a chord.

Suppose without loss of generality that $x_0 x_j$ is the chord, and that j is minimal in that no edge $x_\ell x_k$ exists with $0 \leq \ell, k < j$. If $k = 3$, then obviously $3 \leq j \leq d_1 - 3$, for $j = 1$ would be a double edge (which we do not permit) and $j = 2$ would mean that x_1 would have a neighbor within the face that could not have a path other than through x_1 to x_0 or x_j; the result would be a face of degree greater than 5. Let y_i be the vertices within the region R_1 bounded by the cycle $x_0x_1 \dots x_jx_0$ and z_i the vertices within the region R_2 bounded by $x_jx_{j+1} \dots x_0x_j$.

$\boldsymbol{k = 3, d_2 = 3.}$ Because x_0 is already of degree 3, the path x_1, x_0, x_j must be on the boundary of a triangular face, forcing edge x_1x_j, which implies $\deg x_j \geq 4$, a contradiction.

$\boldsymbol{k = 3, d_2 = 4.}$ We observe that because both x_0 and x_j are already of degree 3, the path $x_1x_0x_jx_{j-1}$ must be on the boundary of a rectangular face. If $j \geq 4$, we would have edge x_1x_{j-1}, which contradicts minimality of j. If $j = 3$ and x_1 and x_2 have a common neighbor y_1, we either have a triangular face $x_1x_2y_1$ or a pentagonal one $x_0x_1y_1x_2x_3$ depending on placement of the third neighbor of y_1.

$\boldsymbol{k = 3, d_2 = 5.}$ We produce the contradiction illustrated in Figure 2.

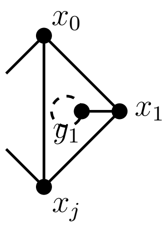

Figure 1: A basic inflorescence.

We have $x_1 \neq x_j$, $x_{j-1} \neq x_0$ to avoid a multigraph. Likewise $1 \neq j-1$ else x_1 must be adjacent to y_1 not on the boundary of F, and y_1 must in turn be adjacent to some other vertices within this face, since the boundary vertices are all of degree 3 already. But this makes y_1x_1 a bridge, and the face within which it lies is of degree strictly greater than 5. This is illustrated in Figure 1. Call such an instance an *inflorescence* for the remainder of this argument.

So there are at least two distinct vertices x_1 and x_{j-1}. Now x_1 must connect to some y_1 and x_{j-1} to some y_{j-1}. But then $y_1 = y_{j-1}$ to make the pentagon $y_1x_1x_0x_jx_{j-1}$. To give y_1 degree 3, it must be adjacent to some y_2; if it were adjacent to x_2 it would create a triangle, and to x_3 or higher a face of degree greater than 5, as x_2 would require an inflorescence.

Now x_1x_{j-1} is not an edge by minimality of j, nor is $x_2 = x_{j-2}$, else either x_2y_2 is an edge, creating at least one face of degree 4, or it is not an edge, in which case x_2 is adjacent to some y_3, which must be adjacent to y_2 to close two pentagonal faces, yet neither y_2 nor y_3 yet has degree 3, so inflorescences would increase the degree of one or both of the internal faces with x_2 on the boundary.

Now y_2 is not adjacent to x_2 or x_{j-2} (square, or greater with inflorescence), so it must be adjacent to two y_i, say y_3 and y_4. These must be adjacent to x_2 and

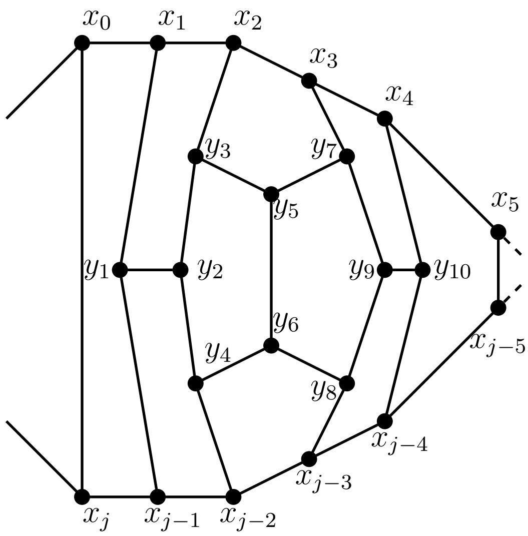

Figure 2: Contradiction for $k = 3$, $d_2 = 5$ boundary self-adjacency.

x_{j-2} to close the faces. We have x_2 not adjacent to x_{j-2}, else y_3 is connected by a path of length 2 to y_4, say via y_5, forming a face of degree 4 or, with an inflorescence from y_5, degree 6 or more.

Neither y_3 nor y_4 can be adjacent to each other (a triangle is formed, or a face of degree greater than 5 with an inflorescence), nor by a path via a y_5 of length 2 (a square is formed, or a face of degree 6 or more); thus y_3 is adjacent to y_4 by a path of length 3, say via y_5 and y_6. We cannot now have $x_3 = x_{j-3}$, since in such a case if x_3 is adjacent to y_5 or y_6, a face of degree 4 (or 6 or more) is formed, while if not adjacent to either, it must be adjacent to some y_7 which in turn is adjacent to both y_5 and y_6, forming a triangle. So x_3 and x_{j-3} exist and are distinct.

Now x_3 is not adjacent to y_5 (square), y_6 (y_5 would root an infloresence into a pentagon), so it is adjacent to some y_7, and y_7 must be adjacent to y_5 to close a face. Likewise x_{j-3} is adjacent to some y_8 in turn adjacent to y_6, with $y_7 \neq y_8$, else y_6 is on the boundary of a face of degree 6 or more.

We have y_7 not adjacent to y_8 (square), and so must be adjacent via a path of length 2; the intermediate vertex cannot be an x_i since this would increase the degree of the vertex to 4 or more, so say the intermediate vertex is y_9. We now have x_3 not adjacent to x_{j-3} by minimality of j, nor is $x_4 = x_{j-4}$, since if y_9 is adjacent to x_4 squares are created, and if not, the path from y_9 to x_4 would have at most one intermediate vertex which would root an inflorescence.

Now y_9 is not adjacent to x_4 or x_{j-4} (square), so it must be adjacent to a y_{10}, which to close faces must in turn be adjacent to x_4 and x_{j-4}. But now to make a face of degree 5, both x_5 and x_{j-5} must exist, but must be adjacent; but then any other path from x_5 to x_{j-5}, which must not include x_4, x_{j-4} or y_{10}, will be part of the boundary of a face of degree greater than 5, a contradiction.

The required vertices are illustrated in Figure 2.

$\boldsymbol{k = 4, d_2 = 3.}$ Suppose that the fourth neighbor of x_0 is in R_2, that is, it is either z_i or x_i for $j + 1 \leq d_1 - 2$. Then since x_0 is already of degree 4, the path $x_1x_0x_j$ must be on the boundary of a triangular face, forcing edge x_1x_j. Now since x_j is already of degree 4, the path x_1, x_j, x_{j-1} must be on the boundary of a triangular face, forcing edge x_1x_{j-1}. Once more, x_1 is now of degree 4, so the path x_2, x_1, x_{j-1} must be on the boundary of a triangular face, forcing edge x_2x_{j-1}. We continue until the forced edge reaches $x_{\lfloor \frac{j}{2} \rfloor}$ when j is odd or $x_{\frac{j}{2}+1}$ when j is even. Then there is only one vertex of degree 2 left on the boundary of R_1, namely $x_{\lceil \frac{j}{2} \rceil}$ when j is odd or $x_{\frac{j}{2}}$ when j is even, and the next forced edge would be a multiple edge, a contradiction. The argument works in the opposite direction if the fourth neighbor of x_0 is in R_1.

$\boldsymbol{k = 5, d_2 = 3.}$ Assume that x_0x_j is minimal in the sense that no chord x_ix_k exists with $0 \leq i < k \leq j$ other than x_0x_j itself. Since $k = 5$, x_0 and x_j both have two other neighbors. Clearly if both neighbors of x_0 (resp. x_j) are within R_2, then in order to make a triangular face, we must have a chord x_1x_j (resp. x_0x_{j-1}), a contradiction. If both neighbors of both x_0 and x_j are within R_1, then the path $x_{d-1}x_0x_jx_{j+1}$ must border a face of degree at least 4, also a contradiction.

Thus, either x_0 and x_j both have exactly one more neighbor in each of R_1 and R_2, or x_0 has both additional neighbors in R_1 and x_j has exactly one additional neighbor in each of R_1 and R_2, or x_0 has one neighbor in each R_i and x_j has both neighbors in R_1. The latter two are the same case after a relabeling, and so we deal with the former.

In both cases, the contradiction results from our conditions forcing the construction of the icosahedron; the minimality-contradicting edge is on the border of its planar embedding.

Case 1: Suppose both vertices have one neighbor in each region. We produce the contradiction to minimality illustrated in Figure 3.

The two neighbors of x_0 and x_j in R_1 must be the same, to produce a triangle bordering x_0x_j. Call this neighbor y_1. The paths $y_1x_0x_1$ and $y_1x_jx_{j-1}$ must close to create faces. We cannot have $x_1 = x_{j-1}$, else y_1 would root inflorescences in one or both of these triangles. Thus, y_1 has one additional neighbor, say y_2.

Now x_1 and x_{j-1} each have three additional neighbors, one of which must be y_2 as the faces bordered by their edge with y_1 must close. Now y_2 must have two addtional neighbors, one of which must be the neighbor of x_1 and the other of x_{j-1} along the edge incident to these vertices which is nearest to y_2 and on

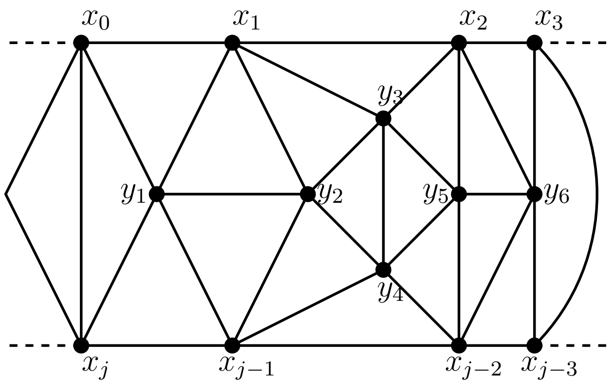

Figure 3: Contradiction for Case 1, $k = 5$, $d_2 = 3$.

the other side from y_1. These must be two distinct neighbors, else y_2 needs another neighbor (say z) inside one or the other of the two resulting faces; the putative z can then have only at most three neighbors on the boundary of the face and requires additional neighbors within the face, which lack sufficient boundary vertices to connect to and thus form boundaries of faces of degree greater than 3.

Let the new neighbors of y_2 be y_3 and y_4. They must be adjacent. Since x_1 and x_{j-1} need an additional neighbor outside the faces containing their edge with y_2, we must have an x_2 and x_{j-2} on the outer face; these cannot be equal, for if they were, x_2 would need to be adjacent to all four of x_1, x_{j-1}, y_3, and y_4, and would need an additional neighbor in one of its bounded faces, say z, which could be adjacent to at most three of its bounding neighbors; z would need additional neighbors which would form boundaries of faces of degree greater than 3.

Now x_2 must be adjacent to y_3 and x_{j-2} to y_4. Further, y_3 and y_4 require an additional neighbor each, not within any of their so-far closed faces (it would be unable to connect sufficiently). To form a triangle, it must be the same vertex, say y_5. Now x_2 and x_{j-2} must both be adjacent to y_5.

One more neighbor of y_5 is needed, as usual not in any of its so far closed nearby faces; call it y_6. We will have x_2 and x_{j-2} both adjacent to y_6, and requiring one more neighbor each, say x_3 and x_{j-3}, which cannot be the same neighbor: y_6 needs two more neighbors, and the extra neighbor would be create a face of degree too high.

But now y_6 already has five edges and thus x_3 and x_{j-3} must be adjacent to close the relevant face. But x_3 and x_{j-3} still need two more neighbors each to be of degree 5, which cannot appear within any of the so far completed faces, so this edge cannot be an edge of F. This contradicts our minimal choice of x_0x_j.

Case 2: The logic is extremely similar. Using x_0 as the vertex with its two additional neighbors in R_1 and x_j with one additional neighbor in each R_i, we illustrate the required vertices and eventual contradiction to minimality in Figure 4.

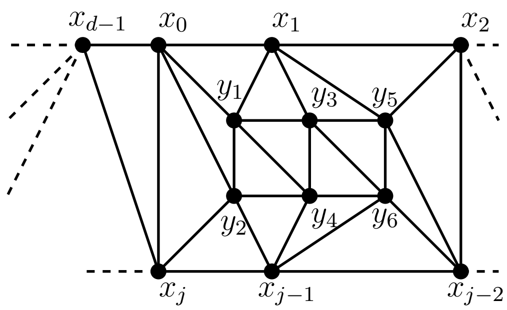

Figure 4: Contradiction for Case 2, $k = 5$, $d_2 = 3$.

We have exhausted all possibilities and the proof is now complete. □

Corollary 5. *For $k = 3, d_2 = 5$ there is no vertex y_t adjacent to two distinct vertices x_a, x_b.*

Proof. Suppose it is not the case, and let $a < b$. Then one of the paths $x_{a-1}x_ay_tx_bx_{b+1}$ or $x_{a+1}x_ay_tx_bx_{b-1}$ belongs to the boundary of a pentagonal face, forcing a chord $x_{a-1}x_{b+1}$ or $x_{a+1}x_{b-1}$, which is impossible by the previous Lemma 2. □

We now prove our main theorem.

Proof of Theorem 6

$\boldsymbol{k = 3, d_2 = 3.}$ In this situation after substitution in Equation 5 we obtain

$$v = 2\left(\frac{d_1 + 3}{3}\right). \tag{6}$$

Either the graph has vertices other than those that form the boundary of F the exceptional face, or it does not. If it does not, then $v = d_1 = 6$ and $e = 3v/2 = 9$. Hence there is a chord to F contrary to Lemma 2.

Thus the graph must have a vertex interior to F. Then

$$v = 2\left(\frac{d_1 + 3}{3}\right) \geq d_1 + 1.$$

But then $d_1 \leq 3$, a contradiction, because $d_1 \neq d_2$.

$\boldsymbol{k = 3, d_2 = 4.}$ In this situation after substitution in Equation 5 we obtain

$$v = d_1 + 4 \tag{7}$$

Hence there is a set Y of exactly 4 vertices interior to the face F. Also $d_1 \geq 6$, because $d_1 \neq d_2$ and v is even, because k is odd. Because there are no chords to F (Lemma 2) it follows that each x_i on the boundary of F is adjacent to some some vertex in Y.

Consider an edge xx' incident to F. Let y, y' be the vertices adjacent to x and x' respectively. Because $yxx'y'$ is a path of length 3, it follows that yy' is an edge. Hence every edge x_ix_{i+1} incident to F has a mate y_iy_{i+1} on Y. Thus because $d_1 \geq 6$, $d_2 = 4$ and $|Y| = 4$ at least two edges on Y are mated twice to edges on the boundary of F. Then because $k = 3$, there can be no edge with ends in Y incident to a doubly mated edge, contrary to the requirement that there be at least 6 mated edges.

$\boldsymbol{k = 3, d_2 = 5.}$ This part is longer, so we itemize briefly the statements we will prove:

- The y_i to which the x_i are adjacent are distinct.
- The y_i are also adjacent to a set $\{z_i\}$, none of which are y_i or x_i and all of which are distinct.
- The z_i are not adjacent to each other, and must be adjacent to w_i, which are not x_i, y_i or z_i.
- There must be exactly five w_i which form a face boundary, giving a contradiction.

Because $k = 3$, each vertex x_i is adjacent to exactly one y_i. The y_i are distinct by Lemma .

Each path $y_ix_ix_{i+1}y_{i+1}$ must be part of the boundary of a face of degree 5 with a fifth vertex z_i. The z_i cannot be any y_s: first, if $z_i = y_{i+2}$ or y_{i-1} a square is formed. Suppose instead that $z_i = y_{i+j}$, with j minimal in absolute value and either $j \geq 3$ or $j \leq -2$. The arguments are the same up to sign and a shift by 1, so suppose $j \geq 3$. Then the path $y_iz_ix_{i+j}x_{i+j-1}y_{i+j-1}$ must bound a pentagon with fifth vertex y_{i+1}. But this contradicts the minimality of j, for we now have a j one less is absolute value (which may be the previous case).

The z_i must be distinct. If $z_i = z_{i+1}$, then y_{i+1} either roots an inflorescence or the vertex other than z_i and x_{i+1} to which y_{i+1} is connected does so, while if $z_i = z_{i+j}$ with j minimal and at least 2, the vertex z_i is of degree at least 4.

None of the z_i are adjacent to each other: if z_i is adjacent to z_{i+1}, a triangle is formed; if to z_{i+2}, then z_{i+1} roots an inflorescence; if to z_{i+j} with j minimal, $j \geq 3$, a face of degree greater than 5 is formed.

So z_0 is adjacent to w_0, z_1 is adjacent to $w_1 \neq w_0$ (square), and w_1 is adjacent to w_0 to close a face. Next z_2 is adjacent to w_2, which is not w_1 (square) or w_0 (w_1 would root an inflorescence), and hence w_2 is adjacent to w_1. Next z_3 is adjacent to w_3, which is not w_2 (square), w_1 (already degree 3), or w_0 (hexagon or greater), and so w_3 is adjacent to w_2. Likewise z_4 must exist (with only three or four x_i, the w_i would all be adjacent and no inflorescence would be possible to increase the degree of the resulting triangle or square) and be adjacent to w_4, which is not w_3 (square), w_2 or w_1 (already degree 3), or w_0 (w_3 would root an inflorescence). Then w_4 is adjacent to w_3, and the cycle must close to form a face of degree 5 bounded by the w_i. But additional z_i would make a face abutting the edge w_4w_0 of too large a degree. Hence $d_1 = 5$, a contradiction.

$\boldsymbol{k = 4, d_2 = 3.}$ In this case $v = d_1 + 3$ and $e = 2v = 2d_1 + 6$. Hence there is a set $Y = \{y_1, y_2, y_3\}$ of exactly 3 vertices not incident to F. Each vertex x_i on F is adjacent to two vertices in Y, because $k = 4$ and F has no chords. This accounts for $3d_1$ edges. Thus $d_1 \leq 6$. Because $d_1 \neq d_2$, we have $4 \leq d_1 \leq 6$. Furthermore there are thus $e - 3d_1 = 6 - d_1$ edges on Y. But if y_i, y_j are incident to x_h, then $y_ix_hy_j$ is a path of length 3. Hence, because $d_2 = 3$, it follows that y_iy_j is an edge.

Suppose that x_1 is adjacent to y_1 and y_2. Then x_2 is also adjacent to, say, y_2. It must also be adjacent to another y_i. If x_2 is also adjacent to y_1, then y_3 is either within the regions bounded by the edges on x_1, x_2, y_1 and y_2, or not. If it is, then it may not be adjacent to x_1 or x_2, which are already of degree 4, and it is isolated from any other x_i, and hence has too few possible neighbors. If y_3 is external to this subgraph, then either y_1 or y_2 is internal to the cycle formed by the other three and is isolated from any possible fourth neighbors. Thus x_2 is adjacent to y_2 and thus also y_3, and hence y_2y_3 and then further y_3y_1 are edges. Now since x_0y_1 is an edge, the triangularity of faces requires that x_0y_3 be an edge, and now all y_i have four neighbors and no other external vertices are possible, i.e. we have constructed the octahedron.

$\boldsymbol{k = 5, d_2 = 3.}$ The leftmost non-boundary edge of x_i and the rightmost nonboundary edge of x_{i+1} must meet at vertex y_i to form a triangular face. We have the following: y_i cannot be any x_j since the bounding face has no chords; $y_i \neq y_{i+1}$ since x_i cannot be twice adjacent to the same y_i. Finally we have that $y_i \neq y_{i+j}$ for $j > 1, j \neq d$, for suppose j is a minimal contradiction to this claim. Then x_i and x_{i+1} both have neighbors along edges intermediate between those connecting them to y_i and, respectively, y_{i-1} and y_{i+1}; call these temporarily z_i and z_{i+1}. Now y_i must be adjacent to these z_i in order to close the triangular faces partially bounded by $y_ix_iz_i$ and $y_ix_{i+1}z_{i+1}$, since x_i and x_{i+1} are already of degree 5. But y_i is additionally a neighbor of x_i, x_{i+1}, and x_{i+j} and x_{i+j+1}, which requires too many edges. (The z_k cannot be any x_ℓ since this would be a chord, and the x_ℓ listed are distinct since $j > 1$.) Hence all y_i are distinct; ∂F consists of the base edges of a series of d triangles joined at their base vertices and otherwise distinct.

Each boundary vertex x_i has an additional neighbor which by definition is adjacent by an edge lying between y_i and y_{i-1}. Let such a vertex adjacent to x_i be called z_i. We again claim that all z_i are distinct and not equal to y_j or x_j for any j. The x_j clause is clear since this would be a chord of the boundary.

First, by definition, z_i cannot be y_i or y_{i-1}, as it is a separate neighbor of x_i. Now suppose that z_i is y_{i+j}, $j > 1$ and minimal among all such j, including with signs reversed and distances taken modulo d. In that case to close the triangular faces abutted by $z_i x_i y_{i-1}$ and $z_i x_i y_i$ we would require edges $z_i y_{i-1}$ and $z_i y_i$ respectively. This makes y_{i+j} of degree 5. Now since z_{i+j} is not y_{i+j}, to close the triangular face abutted by $y_i z_i x_{i+j}$ we would require $z_{i+j} = y_i$, contradicting the minimality of j once signs are reversed. Hence no z_i can be any y_k.

Next, in order to close triangular faces, each z_i must be adjacent to y_i, to close the face partially bounded by $z_i x_i y_i$, and y_{i-1}, to close the face partially bounded by $z_i x_i y_{i-1}$. If $z_i = z_{i+1}$, then y_i possesses two more neighbors, the edges for which will increase the degree of one of the faces that y_i abuts beyond 3. If $z_i = z_{i+j}$, $j > 1$, then z_i would have to be of degree at least 6 since z_i is adjacent to y_{i-1} and y_i, unless $y_{i+j} = y_{i-1}$, in which case we reverse the direction of labeling and argue as before for $j = 1$. Thus, all z_i are distinct and not equal to x_k or y_k for any k.

Each y_i requires another neighbor outside of the triangular faces it abuts so far; call these w_i. Since y_i is now of degree 5, each w_i is necessarily adjacent to z_{i-1} and z_i to close these faces, making the z_i of degree 5. But then the faces $w_i z_{i+1} w_i$ must close cyclically, and the resulting face must be triangular. Thus in the same manner as previous arguments we are led to the contradiction that $d_1 = d_2$, i.e. we have constructed the icosahedron.

By elimination of all cases, we have concluded the theorem:

Theorem 6. *There are no nearly platonic graphs with one disparate face.*

Remark: Our definition of nearly platonic graphs included finiteness. It is easy to construct infinite examples that satisfy all other criteria: surround a non-square with squares on its edges, close the corners with two sides, and repeat the construction for the resulting graph indefinitely to obtain a 4-regular graph with all but exactly one face squares. One wonders if the following method to prove Theorem 6 would be illuminating and more convenient: show that a regular, simple planar graph with a single disparate face must be infinite.

3 $f_1 = 2$ or 3

We will say that a k-regular simple plane graph is a $(k; d_1^{n_1} d_2^{n_2} \cdots d_t^{n_t})$-graph if it has n_i faces of degree d_i, $i = 1, 2, \ldots, t$, where $f = n_1 + n_2 + \cdots + n_t$.

In this section we primarily consider graphs of type $(k; d_1^2 d_2^{n_2})$, that is, nearly platonic graphs in which two faces are disparate from all others. This, it turns out, is probably the largest number of faces d_1 for which the term "nearly platonic" may be fairly applied. We find fifteen families of graphs of this type; interestingly, other than the cycle all seem to be related to platonic solids. The families are indexed by the equivalent possible pairs of distinct faces of platonic solids: the general idea is that one uses those faces as the two disparate faces, and repeats a fundamental unit around a long cycle. Prisms and antiprisms are common examples based on the cube and octahedron respectively. (Of course, the fundamental unit may be only a fraction of the related Platonic graph.)

One of the citations in the chemical graph literature previously mentioned, which thoroughly studies a similar concept, is [6]. In that article the authors study *polycycles*, which require that the two disparate faces not share vertices (a consideration motivated, we believe, by the physical context, which disallows several of our constructions) and allowing vertices on the boundary of a disparate face to have different degree (which a nearly platonic graph forbids). Our constructions are chains of such graphs.

The cycle is trivially the $(2; n^2 d_2^0)$ graph.

The tetrahedron has only one equivalent pair of faces, since any two faces share an edge. Cutting this edge and repeating the resulting graph results in a “thin cycle” which is not the skeleton of a polyhedron, because it is not connected; however, it is a $(3; (3d)^2 3^{2d})$-graph. Its fundamental unit is shown in Figure 5.

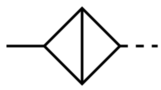

Figure 5: Tetrahedron thin cycle.

In the language of [6] this would be a chain of $\{3, 3\} - e$ polycycles.

There are two families related to the cube. The prisms are $(3; d^2 4^d)$-graphs isomorphic to $C_d \square P_2$. They exist for all $d \geq 3$; the $d = 4$ case is the cube. These are polyhedral.

The other family related to the cube is the related thin cycle, with fundamental unit shown in Figure 6.

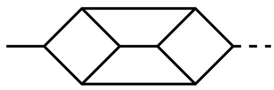

Figure 6: Cube thin cycle.

This as a fundamental unit does not seem to appear in [6]; of the remainder of our constructions, several can be found there and several are different due to the different problem being considered.

There are three families related to the octahedron. The antiprisms are $(4; d^2 3^{2d})$-graphs. They exist for all $d \geq 3$; the $d = 3$ case is the octahedron. They arise from choosing two opposite faces.

The thin cycle has fundamental unit given in Figure 7, from the choice of two faces that share an edge.

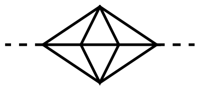

Figure 7: Octahedron thin cycle.

One may also choose two faces in the octahedron that share only one vertex, yielding an even less polyhedral $(4;(3d)^2 3^{6d})$-graph, since the two disparate faces share multiple isolated vertices, indicated in Figure 8.

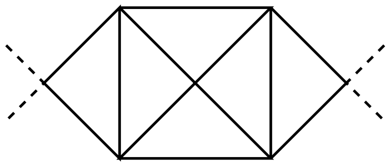

Figure 8: Octahedron vertex cycle.

There are three families related to the dodecahedron. One is prism-like, consisting of the skeleton of a truncated trapezohedron, formed by choosing two opposite faces in the dodecahedron. These are $(3;d^2 5^{2d})$-graphs. They exist for all $d \geq 3$; the $d = 5$ case is the dodecahedron. Its fundamental unit is illustrated in Figure 9. (Deza and Sikirić call these $snub\,APrism_m$, or in Deza's later work, $Barrel_m$.)

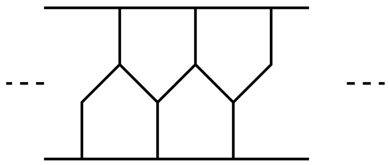

Figure 9: $Barrel_m$.

A thin cycle is formed from the dodecahedron by choosing two adjacent faces, with the fundamental unit shown in Figure 10.

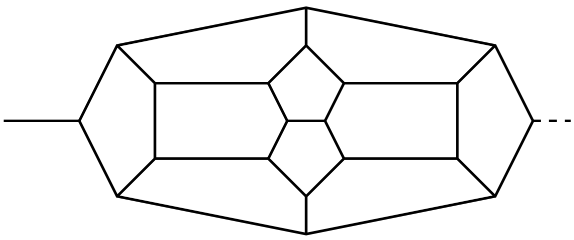

Figure 10: Dodecahedron thin cycle.

A "thick cycle" formed from the dodecahedron by choosing a face and a face neither adjacent nor opposite has the fundamental unit in Figure 11.

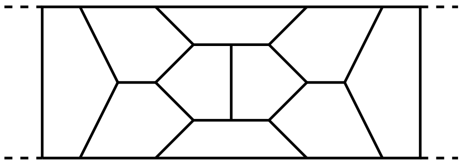

Figure 11: Dodecahedron thick cycle.

Finally, there are five families related to the icosahedron.

By choosing two faces sharing a side, we obtain the fundamental unit in Figure 12 for the related *icosahedron thin cycle*.

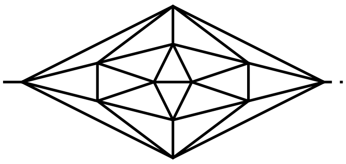

Figure 12: Icosahedron thin cycle.

By choosing two faces sharing exactly one vertex, we obtain the fundamental unit of the *icosahedron vertex cycle*, shown in Figure 13.

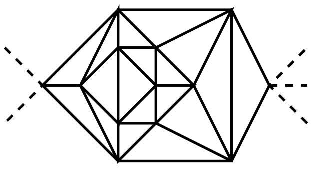

Figure 13: Icosahedron vertex cycle.

Choosing one face, and one of the three faces that shares a side with the face opposite the first, gives the *icosahedron first thick cycle*, yielding the $(5;(3d)^2 3^{18d})$-graph shown in Figure 14.

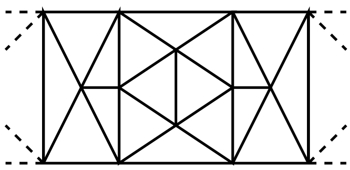

Figure 14: Icosahedron first thick cycle.

Choosing one face, and one of the six faces on the far side that shares a single vertex with the face opposite the first, yields a fundamental unit of the *icosahedron second thick cycle* illustrated in Figure 15.

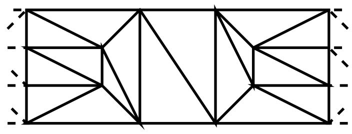

Figure 15: Icosahedron second thick cycle.

Finally, choosing two opposite faces yields the only fundamental unit where the two disparate faces of the resulting graph, the *icosahedron wide cycle*, are separated by a path of minimum length 2. Like the previous graphs of this type, it may be divided into a smaller repeatable fraction, in this case one third, as shown in Figure 16.

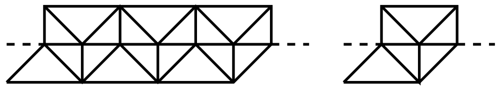

Figure 16: Icosahedron wide cycle.

In all fifteen of these families, one property is constant: both of the disparate faces have the same degree, since we produce the families by repeating a given fundamental unit around a cycle, and the units involved are axially symmetric. As of this writing, we have been unable to generate a counterexample to the following conjecture:

Conjecture 1. If a graph is vertex-regular and planar, and all but 2 faces are of one degree, then the remaining two faces must have the same degree as each other.

Another observation is that in all these families the longest path between the boundaries of the two disparate faces is at most two edges. Can the distance be increased indefinitely? Our suspicion is not. Both of these claims would follow if a much stronger possibility holds:

Question: Are the nearly platonic families listed above the only types of finite, regular planar graph with exactly two disparate faces?

When there are 3 or more disparate faces the disparate face degrees may be different. Indeed, it is possible to produce graphs with all three disparate faces having differing face degrees, as illustrated in Figure 17.

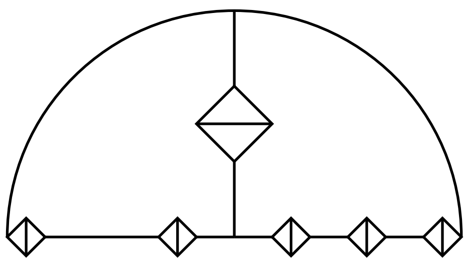

Figure 17: A 3-disparate graph with all disparate faces of differing degree.

Of course, there are also 3-disparate graphs which display symmetries, as Figure 18 shows.

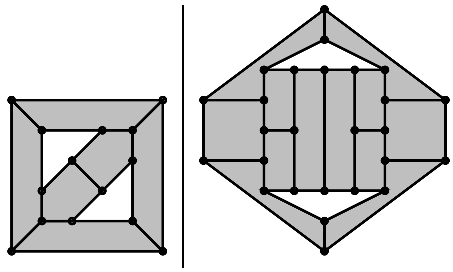

Figure 18: A pair of 3-disparate graphs with symmetries among the disparate faces.

Our concern in this paper is with the restricted cases, and so we do not delve into these graph types. It is intuitively obvious that as d_1 grows, construction of a $(k; f_1^{d_1} f_2^{d_2})$-graph becomes easier. It might be of interest to graph theorists to make this intuition more rigorous by means of some statistic on the set of planar graphs. Another direction of investigation into nearly platonic graphs could be an effort to catalog the regular planar graphs with three disparate faces, especially those with symmetries.

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