AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **70(1)** (2018), Pages 157–167

The integer-magic spectra and null sets of the Cartesian product of trees

Wai Chee Shiu*

Department of Mathematics
Hong Kong Baptist University
224 Waterloo Road, Hong Kong
P.R. China
wcshiu@math.hkbu.edu.hk

Richard M. Low

Department of Mathematics
San José State University
San José, CA 95192
U.S.A.
richard.low@sjsu.edu

Abstract

Let A be a non-trivial, finitely-generated abelian group and $A^* = A \setminus \{0\}$. A graph is A-magic if there exists an edge labeling f using elements of A^* which induces a constant vertex labeling of the graph. Such a labeling f is called an A-magic labeling and the constant value of the induced vertex labeling is called the A-magic value. The integer-magic spectrum of a graph G is the set

$$\mathrm{IM}(G) = \{k \in \mathbb{N} \mid G \text{ is } \mathbb{Z}_k\text{-magic}\},$$

where $\mathbb{N}$ is the set of natural numbers. The null set of G is the set of integers $k \in \mathbb{N}$ such that G has a $\mathbb{Z}_k$-magic labeling with magic value 0. In this paper, we determine the integer-magic spectra and null sets of the Cartesian product of two trees.

1 Introduction

All concepts and notation not explicitly defined in this paper can be found in [2]. Let $G = (V, E)$ be a connected simple graph. For any non-trivial, finitely generated

* This work was supported by Tianjin Research Program of Application Foundation and Advanced Technology (No. 14JCYBJC43100), National Natural Science Foundation of China (No. 11601391).

abelian group A (written additively), let $A^* = A \setminus \{0\}$. A mapping $f : E \to A^*$ is called an *edge labeling* of G. Any such edge labeling induces a *vertex labeling* $f^+ : V \to A$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$. If there exists an edge labeling f whose induced mapping on V is a constant map, we say that f is an A-*magic labeling* of G and that G is an A-*magic* graph. The corresponding constant is called an A-*magic value*. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k \in \mathbb{N} \mid G \text{ is } \mathbb{Z}_k\text{-magic}\}$, where $\mathbb{N}$ is the set of natural numbers. Here, $\mathbb{Z}_1$ is understood to be the set of integers. Generally speaking, it is quite difficult to determine the integer-magic spectrum of a graph. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values.

Group-magic graphs were studied in [7, 9, 15, 16, 26] and $\mathbb{Z}_k$-magic graphs were investigated in [4,6,8,10–14,17–22,27,28]. $\mathbb{Z}$-magic graphs were considered by Stanley [29, 30], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. They were also considered in [1, 23].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A-magic graph is due to J. Sedlacek [24, 25], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been considerable interest in graph labeling problems. The interested reader is directed to Wallis' [31] monograph on magic graphs and to Gallian's [3] excellent dynamic survey of graph labelings.

2 Cartesian product of a tree with a path

Some work on group-magic labelings of trees and their related graphs appear within the literature [11–14,17,21,22]. With regards to Cartesian products, Low and Lee [15] showed the following: If G and H are $\mathbb{Z}_k$-magic, then $G \times H$ is $\mathbb{Z}_k$-magic. In this section, we study the group-magicness of the Cartesian product of trees with paths.

With the purpose of constructing large classes of $\mathbb{Z}_k$-magic graphs, Salehi [19, 20] introduced the concept of a null set of a graph. The *null set* of a graph G, denoted by $N(G)$, is the set of integers $k \in \mathbb{N}$ such that G has a $\mathbb{Z}_k$-magic labeling with magic value 0. Hence, $N(G) \subseteq \mathrm{IM}(G)$.

It is easy to see that a graph G is $\mathbb{Z}_2$-magic if and only if the degrees of the vertices are of the same parity. Moreover, $2 \in N(G)$ if and only if the degree of each vertex of G is even.

Let G be a graph of order s and P_t be the path of order t. Let $V(G) = \{g_1, \dots, g_s\}$ and $V(P_t) = \{p_1, \dots, p_t\}$. Consider the Cartesian product graph $G \times P_t$. For a fixed i, the subgraph induced by $\{(g_i, p_j) \mid 1 \le j \le t\}$ is called a *vertical path* (or more precisely, the g_i-*path*). For a fixed j, the subgraph induced by $\{(g_i, p_j) \mid 1 \le i \le s\}$ is called a *horizontal graph* (or more precisely, the j-*th graph*).

Remark 2.1. For $P_2 \times P_2 \cong C_4$, we label the edges (clockwise) $1, -1, 1$ and -1. Thus, $N(P_2 \times P_2) = \mathbb{N} = \mathrm{IM}(P_2 \times P_2)$.

Lemma 2.1. *Let $s \geq 2$ and $t \geq 3$. Then, $N(P_s \times P_t) = \mathbb{N} \setminus \{2\} = \mathrm{IM}(P_s \times P_t)$.*

Proof: Since $P_s \times P_t$ contains vertices of even and odd degrees, it is not $\mathbb{Z}_2$-magic. Let $P_s = g_1 \cdots g_s$. Label the vertical g_1-path and g_s-path by 1 and the other vertical g_j-paths (if any) by 2, where $2 \leq j \leq s-1$; label the horizontal 1-st and t-th paths by -1 and the other horizontal paths by -2. This yields a $\mathbb{Z}_k$-magic labeling with magic value 0, for $k \in \mathbb{N} \setminus \{2\}$. □

For $s \geq 3$, $t \geq 2$ and $1 \leq r \leq s$, let $B(r; s, t)$ be the graph obtained from $P_s \times P_t$ by deleting all edges of the r-th vertical path. Note that $B(r; s, t) \cong B(s-r+1; s, t)$.

Remark 2.2. Observe that $B(2; 3, 2) \cong C_6$. In this case, we label the edges (clockwise) $1, -1, 1, -1, 1$ and -1. Thus, $N(B(2; 3, 2)) = \mathbb{N} = \mathrm{IM}(B(2; 3, 2))$.

Lemma 2.2. *Let $s \geq 3$, $t \geq 2$ and $2 \leq r \leq s-1$. If $(s, t) \neq (3, 2)$, then $N(B(r; s, t)) = \mathbb{N} \setminus \{2\} = \mathrm{IM}(B(r; s, t))$.*

Proof: Clearly, $B(r; s, t)$ is not $\mathbb{Z}_2$-magic. To obtain a $\mathbb{Z}_k$-magic labeling for $B(r; s, t)$ with magic value 0 (for $k \neq 2$), we perform the following steps:

1. Label $P_s \times P_t$ using the labeling found in the proof of Lemma 2.1.
2. Delete the edges of the r-th vertical path.
3. Multiply all edge labels that are to the left (or right) of the (former) r-th vertical path by -1. □

Example 2.1. Here are some labelings (see Figure 1) which illustrate the proofs of Lemmas 2.1 and 2.2 for $P_5 \times P_3$, $B(2; 5, 3)$ and $B(3; 5, 3)$, respectively:

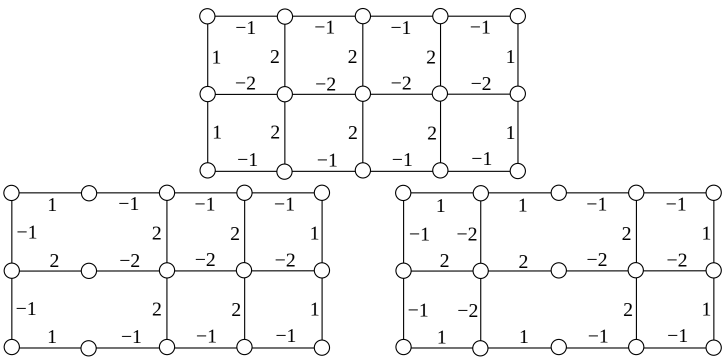

Figure 1

Definition 2.1. Let T be a tree, $u \in V(T)$ and $\deg(u) \geq 3$. We say that u has the 2-*pendant paths property* to mean the following:

- There exists two paths $uv_1v_2\cdots v_a$ and $uw_1w_2\cdots w_b$.
- T is the edge-disjoint union of $[T - (\{v_i \mid 1 \le i \le a\} \cup \{w_j \mid 1 \le j \le b\})]$ and path $w_b\cdots w_1uv_1\cdots v_a$, through identification of vertex u.

Lemma 2.3. *Let T be a tree which is not a path. Then, there exists a vertex $u \in V(T)$ which has the 2-pendant paths property.*

Proof: View T as a rooted tree. Since T is not a path, there is a vertex u furthest away from the root, where $\deg(u) \ge 3$. Then, there are at least two subtrees of u which are paths. Hence, u has the 2-pendant paths property. □

Lemma 2.4. *Let $s \ge 2$. If T_s is a tree of order s, then $\mathbb{N} \setminus \{2\} \subseteq N(T_s \times P_2) \subseteq \mathrm{IM}(T_s \times P_2)$.*

Proof: For $s = 2$, the claim holds by Remark 2.1. Now, let $s \ge 3$. Using mathematical induction, we assume that the claim holds for any tree of order less than s, where $s \ge 3$. Now consider T_s, a tree of order s. If $T_s = P_s$, then we are done by Lemma 2.1. Suppose that T_s is not a path. Then by Lemma 2.3, there exists a vertex u of T_s which has the 2-pendant paths property. Let $uv_1\cdots v_a$ and $uw_1\cdots w_b$ be two such pendant paths. Let $T = T_s - (\{v_i \mid 1 \le i \le a\} \cup \{w_j \mid 1 \le j \le b\})$ and $G = T \times P_2$. Let P be the path $w_b\cdots w_1uv_1\cdots v_a$, which is isomorphic to P_{a+b+1}. Let B be the graph obtained from $P \times P_2$ by deleting the edges of the $(b+1)$-st vertical path. Here, B is isomorphic to $B(b+1; a+b+1, 2)$. Now, G and B are edge-disjoint and $T_s \times P_2 = G \cup B$, (via identification of the copies of u in G with the vertices of the edge-deleted $(b+1)$-st vertical path in B). By the inductive hypothesis and Lemma 2.2 (or Remark 2.2, if $B \cong B(2; 3, 2) \cong C_6$), we know that G and B have $\mathbb{Z}_k$-magic labelings with magic value 0, for $k \ne 2$. Combining these two $\mathbb{Z}_k$-magic labelings, we get the required $\mathbb{Z}_k$-magic labeling of $T_s \times P_2$, for $k \ne 2$. Hence by mathematical induction, the claim is established. □

Theorem 2.5. *Let $s \ge 2$ and $t \ge 3$. If T_s is a tree of order s, then $N(T_s \times P_t) = \mathbb{N} \setminus \{2\} = \mathrm{IM}(T_s \times P_t)$.*

Proof: Since $T_s \times P_t$ contains vertices of even and odd degrees, it is not $\mathbb{Z}_2$-magic. From Lemma 2.1, the claim holds when $s = 2$ or $s = 3$. Using mathematical induction, we assume that the claim holds for any tree of order less than s, where $s \ge 4$. Now consider T_s, a tree of order s. If $T_s = P_s$, then we are done by Lemma 2.1. Suppose that T_s is not a path. Then by Lemma 2.3, there exists a vertex u of T_s which has the 2-pendant paths property. Let $uv_1\cdots v_a$ and $uw_1\cdots w_b$ be two such pendant paths. Let $T = T_s - (\{v_i \mid 1 \le i \le a\} \cup \{w_j \mid 1 \le j \le b\})$ and $G = T \times P_t$. Let P be the path $w_b\cdots w_1uv_1\cdots v_a$, which is isomorphic to P_{a+b+1}. Let B be the graph obtained from $P \times P_t$ by deleting the edges of the $(b+1)$-st vertical path. Here, B is isomorphic to $B(b+1; a+b+1, t)$. Now, G and B are edge-disjoint and $T_s \times P_t = G \cup B$, (via identification of the copies of u in G with the vertices of the edge-deleted $(b+1)$-st vertical path in B). By the inductive hypothesis and Lemma 2.2, we know that G and B have $\mathbb{Z}_k$-magic labelings with magic value 0, for $k \neq 2$. Combining these two $\mathbb{Z}_k$-magic labelings, we get the required $\mathbb{Z}_k$-magic labeling of $T_s \times P_t$, for $k \neq 2$. Hence by mathematical induction, the claim is established. □

Example 2.2. Here are $\mathbb{Z}_k$-magic labelings (see Figure 2), where $k \neq 2$ for $T_5 \times P_3$ and $T_7 \times P_3$, respectively:

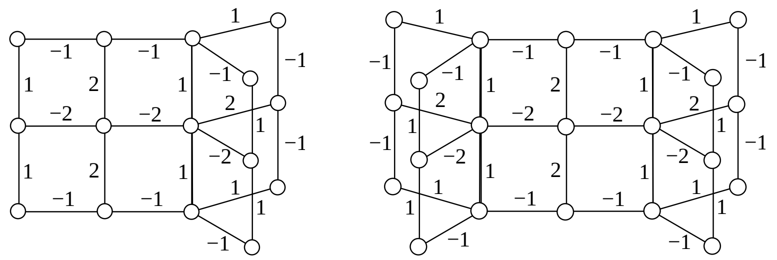

Figure 2

Example 2.3. Note that $K_{1,3} \times P_2$ is an Eulerian graph with an even number of edges. Traveling along an Eulerian circuit of $K_{1,3} \times P_2$, we can label the edges $1, -1, 1, -1, \ldots, 1, -1$. This is $\mathbb{Z}_k$-magic labeling with magic value 0, for $k \in \mathbb{N}$. See Figure 3.

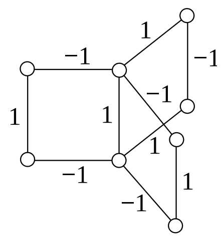

Figure 3

3 Cartesian product of two trees

Suppose T is a tree and $t \geq 3$. Let $B_T(r;t)$ be the graph obtained from $T \times P_t$ by deleting all the edges of the r-th horizontal tree, where $2 \leq r \leq t-1$.

Lemma 3.1. *Let T be a tree of order at least 3, $t \geq 4$ and $2 \leq r \leq t-1$. Then,* $N(B_T(r;t)) = \mathbb{N} \setminus \{2\} = \mathrm{IM}(B_T(r;t))$.

Proof: Since $B_T(r;t)$ contains vertices of even and odd degrees, it is not $\mathbb{Z}_2$-magic. To obtain a $\mathbb{Z}_k$-magic labeling for $B_T(r;t)$ with magic value 0 (for $k \neq 2$), we perform the following steps:

1. Label $T \times P_t$, as described in the proof of Theorem 2.5.

2. Delete the edges of the r-th horizontal tree.

3. Multiply all edge labels that are above (or below) the (former) r-th horizontal tree by -1.

This gives us a $\mathbb{Z}_k$-magic labeling of $B_T(r;t)$ with magic value 0, for $k \neq 2$. □

Remark 3.1. Suppose that T is a tree of order at least 3 and $t = 3$. Then the procedure described in the proof of Lemma 3.1 yields $\mathbb{N} \setminus \{2\} \subseteq N(B_T(2;3)) \subseteq \mathrm{IM}(B_T(2;3))$. If T has no vertex of even degree, $B_T(2;3)$ has no vertices of odd degree. In this case, labeling all of the edges of $B_T(2;3)$ with 1 gives a $\mathbb{Z}_2$-magic labeling with magic value 0. Thus, $N(B_T(2;3)) = \mathbb{N} = \mathrm{IM}(B_T(2;3))$. On the other hand, if T has a vertex of even degree, then $B_T(2;3)$ has vertices of even and odd degrees and hence, is not $\mathbb{Z}_2$-magic. In this case, $N(B_T(2;3)) = \mathbb{N} \setminus \{2\} = \mathrm{IM}(B_T(2;3))$.

Example 3.1. Here are some labelings which illustrate Remark 3.1. The integer-magic spectrum of $B_{T_5}(2;3)$ is $\mathbb{N} \setminus \{2\}$. See Figure 4. Now, let $T = K_{1,3}$. Then, the integer-magic spectrum of $B_T(2;3)$ is $\mathbb{N}$.

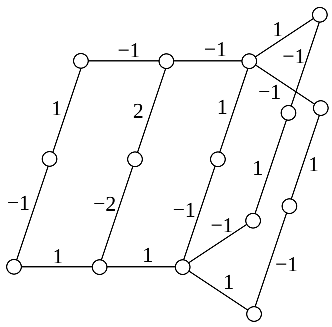

Figure 4

Theorem 3.2. *Let $s, t \geq 2$. If T_s and T_t are trees of order s and t, respectively, then $\mathbb{N} \setminus \{2\} \subseteq N(T_s \times T_t) \subseteq \mathrm{IM}(T_s \times T_t)$.*

Proof: Let $s \geq 2$. When $t = 2$, the claim holds by Lemma 2.4. When $t = 3$, the claim holds by Theorem 2.5. Using mathematical induction, we assume the claim holds for any tree of order less than t, where $t \geq 4$. Now consider T_t, a tree of order t. If $T_t = P_t$, then we are done by Theorem 2.5. Suppose that T_t is not a path. Then by Lemma 2.3, there exists a vertex u of T_t which has the 2-pendant paths property. Let $uv_1 \cdots v_a$ and $uw_1 \cdots w_b$ be two such pendant paths. Let $T = T_t - \big(\{v_i \mid 1 \leq i \leq a\} \cup \{w_j \mid 1 \leq j \leq b\}\big)$ and $G = T_s \times T$. Let P be the path $w_b \cdots w_1 uv_1 \cdots v_a$ which is isomorphic to P_{a+b+1}. Let B be the graph obtained from $T_s \times P$ by deleting the edges of the $(b+1)$-st horizontal tree. Here, B is isomorphic to $B_{T_s}(b+1;t)$. Now, G and B are edge-disjoint and $T_s \times T_t = G \cup B$. By the inductive hypothesis and Lemma 3.1, we know that G and B have $\mathbb{Z}_k$-magic labelings with magic-value 0, for $k \neq 2$. Combining these two $\mathbb{Z}_k$-magic labelings, we get the required $\mathbb{Z}_k$-magic of $T_s \times T_t$, for $k \neq 2$. Hence by mathematical induction, the claim is established. □

Remark 3.2. Theorem 3.2 establishes the entire integer-magic spectra and null sets of the Cartesian product of two trees, for all $k \neq 2$. To determine if 2 is contained in the integer-magic spectrum or null set of $T_s \times T_t$, one merely examines the parities of the degrees of the vertices in $T_s \times T_t$.

Example 3.2. Here is a construction of a $\mathbb{Z}_k$-magic labeling with magic value 0 of $K_{1,3} \times K_{1,3}$, using the ideas in the proofs of the above results.

(1) From the proof of Lemma 2.1, we obtain labelings of $P_2 \times P_3$ and $P_3 \times P_3$.

(2) Perform the steps described in the proof of Lemma 2.2 on $P_3 \times P_3$ to get a labeling of $B(3; 2, 3)$.

(3) From the proof of Theorem 2.5, we obtain a labeling of $K_{1,3} \times P_3$.

(4) From the proof of Lemma 3.1, we get a labeling of $B_{K_{1,3}}(2; 3)$.

(5) Combining the labeling of $K_{1,3} \times P_2$ obtained in Example 2.3, we get a labeling of $K_{1,3} \times K_{1,3}$.

All labelings obtained above are magic with magic value 0. Here are the resulting labelings (see Figure 5). Clearly, this is a $\mathbb{Z}_k$-magic labeling of $K_{1,3} \times K_{1,3}$ with magic value 0, for all $k \in \mathbb{N}$.

Theorem 3.3. *Let $s_i \geq 2$, for $1 \leq i \leq 2r$ and T_{s_i} be a tree of order s_i. Then, $\mathbb{N} \setminus \{2\} \subseteq \mathrm{IM}(T_{s_1} \times T_{s_2} \times T_{s_3} \times T_{s_4} \cdots \times T_{s_{2r-1}} \times T_{s_{2r}})$.*

Proof: In [15], it was shown that the Cartesian product of two $\mathbb{Z}_k$-magic graphs is $\mathbb{Z}_k$-magic. This, along with Theorem 3.2, establishes our claim. □

4 Miscellany

The main focus of this paper has been to determine the entire integer-magic spectra and null sets of $T_s \times T_t$. This section contains various miscellaneous results which the authors encountered along the way.

We first note that $\mathbb{Z}_k$-magic labelings can be obtained for $P_s \times P_t$ with any number of deleted vertical paths, excluding the 1-st and s-th vertical paths. This is accomplished by repeatedly using the procedure described in the proof of Lemma 2.2. Thus, we have the following theorem:

Theorem 4.1. *Let $s \geq 3$, $t \geq 2$ and $G = P_s \times P_t$ with some deleted vertical paths (excluding the 1-st and s-th vertical paths). Then, $\mathbb{N} \setminus \{2\} \subseteq N(G) \subseteq \mathrm{IM}(G)$.*

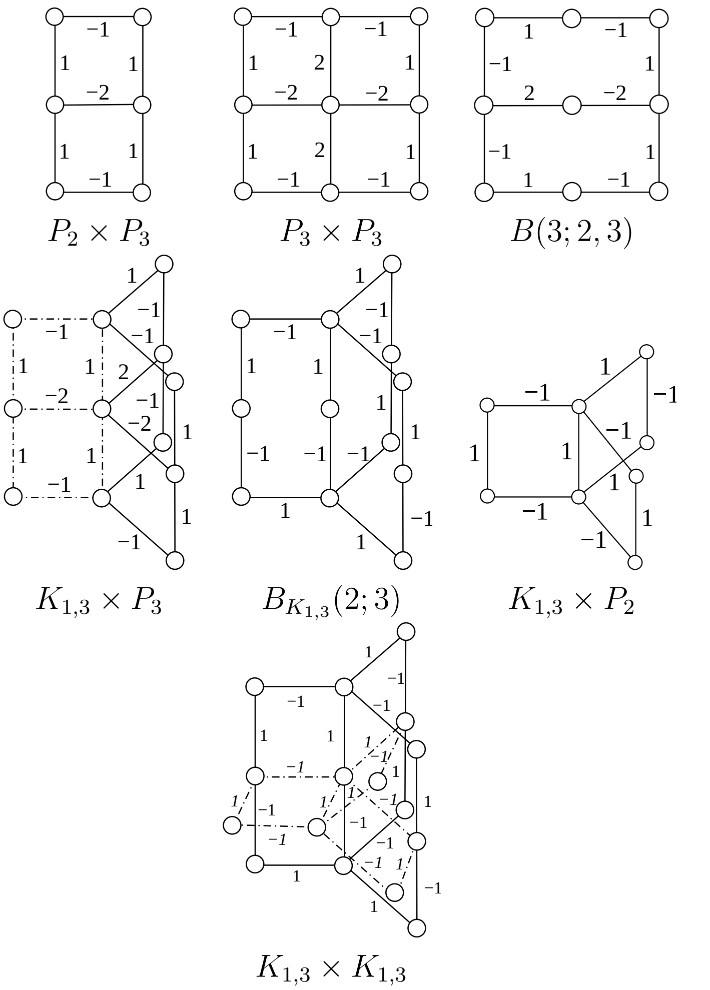

Figure 5

Example 4.1. Here is a $\mathbb{Z}_k$-magic labeling (see Figure 6) with magic value 0 ($k \neq 2$) of $P_5 \times P_3$ with its 2-nd and the 4-th vertical paths deleted. This was obtained by using the procedure described in the proof of Lemma 2.2 twice.

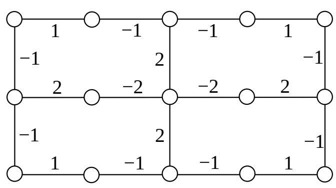

Figure 6

One can also obtain $\mathbb{Z}_k$-magic labelings for $T_s \times P_t$ (where T_s is a tree of order s) with any number of deleted horizontal trees, excluding the 1-st and t-th horizontal trees. This is accomplished by repeatedly using the procedure described in the proof of Lemma 3.1. Thus, we have the following theorem:

Theorem 4.2. *Let* $s \geq 3$, $t \geq 4$ *and* $G = T_s \times P_t$ *with some deleted horizontal trees (excluding the 1-st and* t*-th horizontal trees). Then,* $\mathbb{N} \setminus \{2\} \subseteq N(G) \subseteq \mathrm{IM}(G)$.

Theorem 4.3. *Suppose that* $2 \leq r \leq s-1$ *and* $t \geq 2$*. Let path* $P_s = u_1 \cdots u_s$ *and* $B(r; s, t)$ *be the graph obtained from* $P_s \times P_t$ *by deleting all edges of the* r*-th vertical path. Furthermore, suppose that* $G \times P_t$ *has a* $\mathbb{Z}_k$*-magic labeling with magic value* 0*, for* $k \neq 2$*. Let* H *be the one point union of* G *and* P_s *by identifying a vertex of* G *with the vertex* $u_r \in V(P_s)$*. Then,* $H \times P_t$ *has a* $\mathbb{Z}_k$*-magic labeling with magic value* 0.

Proof: Note that $H \times P_t \cong (G \times P_t) \cup B(r; s, t)$. The claim follows immediately from this. □

To determine if $H \times P_t$ (in Theorem 4.3) has a $\mathbb{Z}_2$-magic labeling, one merely examines the parities of the degrees of the vertices.

Acknowledgements

The authors wish to thank the referees for their valuable comments and suggestions.

References

[1] C. Berge, Regularisable graphs, *Annals of Discrete Math.* **3** (1978), 11–19.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, 1976.

[3] J. Gallian, A dynamic survey of graph labeling, *Elect. J. Combin.* **19** (2016), #DS6.

[4] M.C. Kong, S-M. Lee, and H. Sun, On magic strength of graphs, *Ars Combin.* **45** (1997), 193–200.

[5] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[6] S-M. Lee, Y-S. Ho and R.M. Low, On the integer-magic spectra of maximal planar and maximal outerplanar graphs, *Congr. Numer.* **168** (2004), 83–90.

[7] S-M Lee, A. Lee, H. Sun, and I. Wen, On group-magic graphs, *J. Combin. Math. Combin. Comput.* **38** (2001), 197–207.

[8] S-M. Lee and F. Saba, On the integer-magic spectra of two-vertex sum of paths, *Congr. Numer.* **170** (2004), 3–15.

[9] S-M. Lee, F. Saba, E. Salehi, and H. Sun, On the V_4-group magic graphs, *Congr. Numer.* **156** (2002), 59–67.

[10] S-M. Lee, F. Saba, and G.C. Sun, Magic strength of the k-th power of paths, *Congr. Numer.* **92** (1993), 177–184.

[11] S-M. Lee and E. Salehi, Integer-magic spectra of amalgamations of stars and cycles, *Ars Combin.* **67** (2003), 199–212.

[12] S-M. Lee, E. Salehi and H. Sun, Integer-magic spectra of trees with diameters at most four, *J. Combin. Math. Combin. Comput.* **50** (2004), 3–15.

[13] S-M. Lee, L. Valdes, and Y-S. Ho, On group-magic spectra of trees, double trees and abbreviated double trees, *J. Combin. Math. Combin. Comput.* **46** (2003), 85–95.

[14] R.M. Low and S-M. Lee, On the integer-magic spectra of tessellation graphs, *Australas. J. Combin.* **34** (2006), 195–210.

[15] R.M. Low and S-M Lee, On the products of group-magic graphs, *Australas. J. Combin.* **34** (2006), 41–48.

[16] R.M. Low and S-M. Lee, On group-magic eulerian graphs, *J. Combin. Math. Combin. Comput.* **50** (2004), 141–148.

[17] R.M. Low and W.C. Shiu, On the integer-magic spectra of graphs, *Congr. Numer.* **191** (2008), 193–203.

[18] R.M. Low and L. Sue, Some new results on the integer-magic spectra of tessellation graphs, *Australas. J. Combin.* **38** (2007), 255–266.

[19] E. Salehi, Zero-sum magic graphs and their null sets, *Ars Combin.* **82** (2007), 41–53.

[20] E. Salehi, On zero-sum magic graphs and their null sets, *Bull. Inst. Math. Acad. Sinica* **3** (2008), 255–264.

[21] E. Salehi and P. Bennett, On integer-magic spectra of caterpillars, *J. Combin. Math. Combin. Comput.* **61** (2007), 65–71.

[22] E. Salehi and P. Bennett, On integer-magic spectra of trees of diameter five, *J. Combin. Math. Combin. Comput.* **66** (2008), 105–111.

[23] L. Sandorova and M. Trenkler, On a generalization of magic graphs, *Combinatorics (Eger, 1987)*, 447–452, *Colloq. Math. Soc. Janos Bolyai* **52**, North-Holland, Amsterdam, 1988.

[24] J. Sedlacek, On magic graphs, *Math. Slov.* **26** (1976), 329–335.

[25] J. Sedlacek, Some properties of magic graphs, *Graphs, Hypergraph, and Bloc. Syst. 1976*, Proc. Symp. Comb. Anal., Zielona Gora (1976), 247–253.

[26] W.C. Shiu and R.M. Low, Group-magicness of complete N-partite graphs, *J. Combin. Math. Combin. Comput.*, **58** (2006), 129–134.

[27] W.C. Shiu and R.M. Low, Integer-magic spectra of sun graphs, *J. Combin. Optim.* **14** (2007), 309–321.

[28] W.C. Shiu and R.M. Low, $\mathbb{Z}_k$-magic labelings of fans and wheels with magic-value zero, *Australas. J. Combin.* **45** (2009), 309–316.

[29] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.* **40** (1973), 607–632.

[30] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen–Macaulay rings, *Duke Math. J.* **40** (1976), 511–531.

[31] W.D. Wallis, *Magic Graphs*, Birkhauser Boston, (2001).

(Received 22 May 2017; revised 3 Oct 2017)