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Zero forcing and power domination for graph products

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Abstract

The power domination number arose from the monitoring of electrical networks, and methods for its determination have the associated application. The zero forcing number arose in the study of maximum nullity among symmetric matrices described by a graph (and also in control of quantum systems and in graph search algorithms). There has been considerable effort devoted to the determination of the power domination number, the zero forcing number, and maximum nullity for specific families of graphs. In this paper we exploit the natural relationship between power domination and zero forcing to obtain results for the power domination number of tensor products and the zero forcing number of lexicographic products of graphs. In addition, we establish a general lower bound for the power domination number of a graph based on the maximum nullity of the matrices described by the graph. We also establish results for the zero forcing number and maximum nullity of tensor products and Cartesian products of certain graphs.

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1 Introduction

Electric power companies need to monitor the state of their networks continuously to prevent system failure; a standard method is to place Phase Measurement Units (PMUs) at selected locations in the system, called electrical nodes or buses, where transmission lines, loads, and generators are connected. A PMU placed at an electrical node measures the voltage at the node and all current phasors at the node [3]; it also provides these measurements at other vertices or edges according to certain propagation rules. Due to the cost of a PMU, it is important to minimize the number of PMUs used while maintaining the ability to observe the entire system. This problem was first modeled using graphs by Haynes et al. in [17], where the vertices represent the electric nodes and the edges are associated with the transmission lines joining two electrical nodes (see Section 1.3 for the details and formal definitions). In this graph model, the power domination problem consists of finding a minimum set of vertices from where the entire graph can be observed according to certain rules; these vertices provide the locations where the PMUs should be placed in order to monitor the entire electrical system at minimum cost. Since its introduction in [17], the power domination number and its variations have generated considerable interest (see, for example, [5, 9, 10, 11, 20, 22]).

As was pointed out in [9], a careful examination of the definition of power domination leads naturally to the study of zero forcing. The zero forcing number was introduced in [2] as an upper bound for the maximum nullity of real symmetric matrices whose nonzero pattern of off-diagonal entries is described by a given graph, and independently by mathematical physicists studying control of quantum systems [6], and later by computer scientists studying graph search algorithms [23]. The study of maximum nullity, or equivalently, maximum multiplicity of an eigenvalue, was motivated by the inverse eigenvalue problem of a graph (see [13] and [12] for surveys of results on maximum nullity and zero forcing containing more than a hundred references). Since its introduction, zero forcing has attracted the attention of a large number of researchers who find the concept useful to model processes in a broad range of disciplines. There has been extensive work on determining the values of the power domination number and the zero forcing number for families of graphs. It is worth noting that the problem of deciding whether a graph admits a power dominating set of a given size is NP-complete [17], as is the analogous problem for zero forcing [1].

In Section 2 we establish results for the zero forcing number and maximum nullity of some families of tensor products, via a new upper bound on the zero forcing number of the tensor product of a complete graph with another graph (Theorem 2.1), and some Cartesian products of graphs. In Section 3 we use the connection between power domination and zero forcing established in [9] to obtain the only known general lower bound for the power domination number (Theorem 3.2). A zero forcing lower bound has not previously been applied to graphs other than the hypercube in [9]. Note that in [20] the author claimed to have obtained the first general lower bound for the power domination number, but a family of counterexamples to his claim was given in [16]. Here we use Theorem 3.2 to prove results for the power domination number of tensor products in Section 3.1 and the zero forcing number of lexicographic products of graphs in Section 3.2. The remainder of this introduction contains formal definitions of power domination and zero forcing, graph terminology, and matrix terminology.

1.1 Power domination and zero forcing definitions

A *graph* $G = (V, E)$ is an ordered pair formed by a finite nonempty set of *vertices* $V = V(G)$ and a set of *edges* $E = E(G)$ containing unordered pairs of distinct vertices (that is, all graphs are simple and undirected). The *order* of G is denoted by $|G| := |V(G)|$. We say the vertices u and v are *adjacent* or are *neighbors*, and write $u \sim v$, if $\{u, v\} \in E$. For any vertex $v \in V$, the *neighborhood* of v is the set $N(v) = \{u \in V : u \sim v\}$ (or $N_G(v)$ if G is not clear from context), and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set of vertices S, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$.

A vertex v in a graph G is said to *dominate* itself and all of its neighbors in G. A set of vertices S is a *dominating set* of G if every vertex of G is dominated by a vertex in S. The minimum cardinality of a dominating set is the *domination number* of G and is denoted by $\gamma(G)$.

In [17] the authors introduced the related concept of power domination by presenting propagation rules in terms of vertices and edges in a graph. In this paper we use a simplified version of the propagation rules that is equivalent to the original, as shown in [5]. For a set S of vertices in a graph G, define $PD(S) \subseteq V(G)$ recursively:

1. $PD(S) := N[S] = S \cup N(S)$.
2. While there exists $v \in PD(S)$ such that $|N(v) \cap (V(G) \setminus PD(S))| = 1$:
   $PD(S) := PD(S) \cup N(v)$.

We say that a set $S \subseteq V(G)$ is a *power dominating set* of a graph G if at the end of the process above $PD(S) = V(G)$. A *minimum power dominating set* is a power dominating set of minimum cardinality, and the *power domination number*, $\gamma_P(G)$, of G is the cardinality of a minimum power dominating set.

The concept of zero forcing can be explained via a coloring game on the vertices of G. The *color change rule* is: If u is a blue vertex and exactly one neighbor w of u is white, then change the color of w to blue. We say u forces w and denote this by $u \to w$. A *zero forcing set* for G is a subset of vertices B such that when the vertices in B are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. A *minimum zero forcing set* is a zero forcing set of minimum cardinality, and the *zero forcing number*, $\mathrm{Z}(G)$, of G is the cardinality of a minimum zero forcing set. The next observation is the key relationship between the two concepts.

Observation 1.1. [9] *The power domination process on a graph G can be described as choosing a set $S \subseteq V(G)$ and applying the zero forcing process to the closed neighborhood $N[S]$ of S. The set S is a power dominating set of G if and only if $N[S]$ is a zero forcing set for G.*

The *degree* of a vertex v, denoted by $\deg v$, is the cardinality of the set $N(v)$. The *maximum* and *minimum degree* of G are defined as $\Delta(G) = \max\{\deg v : v \in V\}$ and $\delta(G) = \min\{\deg v : v \in V\}$, respectively. A graph G is *regular* if $\delta(G) = \Delta(G)$.

The next observation is well known (and immediate since the color change rule cannot be applied in G without at least $\delta(G)$ blue vertices).

Observation 1.2. *For every graph G, $\delta(G) \le \mathrm{Z}(G)$.*

1.2 Graph definitions and notation

Let n be a positive integer. The *path* of order n is the graph P_n with $V(P_n) = \{x_i : 1 \le i \le n\}$ and $E(P_n) = \{\{x_i, x_{i+1}\} : 1 \le i \le n-1\}$. If $n \ge 3$, the *cycle* of order n is the graph C_n with $V(C_n) = \{x_i : 1 \le i \le n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} : 1 \le i \le n-1\} \cup \{\{x_n, x_1\}\}$. The *complete graph* of order n is the graph K_n with $V(K_n) = \{x_i : 1 \le i \le n\}$ and $E(K_n) = \{\{x_i, x_j\} : 1 \le i < j \le n\}$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be disjoint graphs. All of the following products of G and H have vertex set $V(G) \times V(H)$. The *tensor product* (also called the *direct product*) of G and H is denoted by $G \times H$; a vertex (g, h) is adjacent to a vertex (g', h') in $G \times H$ if $\{g, g'\} \in E(G)$ and $\{h, h'\} \in E(H)$. The *Cartesian product* of G and H is denoted by $G \,\square\, H$; two vertices (g, h) and (g', h') are adjacent in $G \,\square\, H$ if either (1) $g = g'$ and $\{h, h'\} \in E(H)$, or (2) $h = h'$ and $\{g, g'\} \in E(G)$. The *lexicographic product* of G and H is denoted by $G * H$; two vertices (g, h) and (g', h') are adjacent in $G * H$ if either (1) $\{g, g'\} \in E(G)$, or (2) $g = g'$ and $\{h, h'\} \in E(H)$. Note that $H \times G \cong G \times H$ and $H \,\square\, G \cong G \,\square\, H$, whereas $H * G$ need not be isomorphic to $G * H$.

For a graph G with no edges, $\mathrm{Z}(G) = \gamma_P(G) = \gamma(G) = |G|$, so we focus our attention on graphs that have at least one edge. In the case of the tensor product $G \times H$, this means we assume $|G|, |H| \ge 2$.

1.3 Matrix definitions and notation

Let $S_n(\mathbb{R})$ denote the set of all $n \times n$ real symmetric matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph* of A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \ne 0,\ 1 \le i < j \le n\}$. More generally, the graph of A is defined for any matrix that is *combinatorially symmetric*, i.e., $a_{ij} = 0$ if and only if $a_{ji} = 0$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The set of symmetric matrices described by a graph G of order n is defined as $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. The *maximum nullity* of G is $\mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$, and the *minimum rank* of G is $\operatorname{mr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{S}(G)\}$; clearly $\mathrm{M}(G) + \operatorname{mr}(G) = |G|$. The term 'zero forcing' comes from using the forcing process to force zeros in a null vector of a matrix $A \in \mathcal{S}(G)$, implying the following key relationship:

Proposition 1.3. [2, Proposition 2.4] *For a graph G, $\mathrm{M}(G) \le \mathrm{Z}(G)$.*

Although the relationship $\mathrm{M}(G) \le \mathrm{Z}(G)$ was originally viewed as an upper bound for the maximum nullity of a graph, we will repeatedly use this inequality to provide a lower bound for the zero forcing number.

A standard way to construct matrices of maximum nullity for a Cartesian product or a tensor product of graphs is to use the *Kronecker* or *tensor* product of matrices. Let A be an $n \times n$ real matrix and B be an $m \times m$ real matrix. Then $A \otimes B$ is the $n \times n$ block matrix whose ijth block is the $m \times m$ matrix $a_{ij}B$. It is known that $(A \otimes B)^T = A^T \otimes B^T$ and $\operatorname{rank}(A \otimes B) = (\operatorname{rank} A)(\operatorname{rank} B)$. If $A \in \mathcal{S}(G)$, $B \in \mathcal{S}(H)$, $|G| = n$, and $|H| = m$, then $A \otimes I_m - I_n \otimes B \in \mathcal{S}(G \,\square\, H)$. If $\mathbf{x}$ is an eigenvector of A for eigenvalue λ and $\mathbf{y}$ is an eigenvector of B for eigenvalue μ, then $\mathbf{x} \otimes \mathbf{y}$ is an eigenvector of $A \otimes I_m - I_n \otimes B$ for eigenvalue $\lambda - \mu$. Since a real symmetric matrix has an orthonormal basis of eigenvectors, the multiplicity of $\lambda - \mu$ is at least $\operatorname{mult}_A(\lambda) \operatorname{mult}_B(\mu)$ [2, Observation 3.5]. If $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$ and the diagonal entries of A and B are all zero, then $A \otimes B \in \mathcal{S}(G \times H)$. Define $\mathcal{M}_0(G) = \{A \in \mathbb{R}^{n \times n} : \mathcal{G}(A) = G \text{ and } a_{ii} = 0 \text{ for } i = 1, \ldots, n\}$; in contrast to a matrix in $\mathcal{S}(G)$, a matrix in $\mathcal{M}_0(G)$ need not be symmetric but must have a zero diagonal and be combinatorially symmetric. If $A \in \mathcal{M}_0(G)$ and $B \in \mathcal{M}_0(H)$, then $A \otimes B \in \mathcal{M}_0(G \times H)$.

2 Zero forcing for graph products

In this section we develop a tool for bounding the zero forcing number of tensor products of graphs and apply it to compute the zero forcing number and the maximum nullity of the tensor product of a complete graph with a path or a cycle. We also compute the zero forcing number and maximum nullity of the Cartesian product of two cycles.

2.1 Tensor products

For tensor products of graphs we use not only standard zero forcing but also skew zero forcing [19], defined by the *skew color change rule*: If a vertex v of G has exactly one white neighbor w, then v forces w to change color to blue; if v is white when it forces, the force $v \to w$ is called a *white vertex force*. The important distinction between skew zero forcing and standard zero forcing is that a white vertex can perform a force under the skew color change rule whereas only a blue vertex can perform a force under the standard rule; the requirement that the forcing vertex have only one white neighbor remains the same. The *skew zero forcing number*, $\mathrm{Z}^-(G)$, is the minimum cardinality of a *skew zero forcing set*, i.e., a (possibly empty) set of blue vertices that can color all vertices blue using the skew color change rule. Skew zero forcing was introduced in [19] for the study of *maximum skew nullity* of G, that is, the maximum nullity of all skew symmetric matrices having off-diagonal nonzero pattern described by the edges of G. Analogously to Proposition 1.3, $\mathrm{Z}^-(G)$ is bounded from below by the maximum skew nullity, or more generally, by the maximum nullity of matrices having zero diagonal and the off-diagonal nonzero pattern described by the edges of the graph. However, in this paper, we are using skew zero forcing simply as a tool to describe the (standard) zero forcing process in a tensor product graph and note that it is not directly connected to power domination.

For either a standard or skew zero forcing set B, color all the vertices of B blue and then perform zero forcing, listing the forces in the order in which they were performed. This list is a *chronological list of forces of* B and is denoted by $\mathcal{F}$.

For each vertex $g \in V(G)$, define the set $U_g = \{(g, h) : h \in V(H)\}$ in $G \times H$. We say vertices (g, h) and (g', h) are *associates* in $G \times H$ if $g \sim g'$ in G; associates are not adjacent in $G \times H$.

Theorem 2.1. *Let G be a graph and $n \geq 4$. Then*

$$\mathrm{Z}(G \times K_n) \leq (n-2)|G| + 2\,\mathrm{Z}^{-}(G).$$

Proof. For $g \sim g'$ in G, $(g, i) \sim (g', j)$ in $G \times K_n$ for all $j \neq i \in V(K_n)$. Choose a minimum skew zero forcing set B for G and a chronological list of forces $\mathcal{F}$ of B and denote the kth force by $g_k \to w_k$ (many vertices receive two labels, e.g., $g_k = w_\ell$). We describe how to choose $(n-2)|G| + 2|B|$ vertices to obtain a zero forcing set $\hat{B}$ for $G \times K_n$. For $g \in B$, let $\hat{B} \supset U_g$. For $v \notin B$, place $n-2$ vertices of U_v in $\hat{B}$; the selection of these vertices is determined when v is forced in G or when v performs a force in G, whichever comes first.

Consider the kth force in $\mathcal{F}$ ($k = 1$ is permitted). Suppose $g_k \to w_k$ is not a white vertex force. If no vertices of U_{w_k} are in $\hat{B}$ yet, then arbitrarily choose $n-2$ vertices in U_{w_k} to place in $\hat{B}$; otherwise, no additional vertices are placed in $\hat{B}$. Now suppose $g_k \to w_k$ is a white vertex force. Clearly, $w_k \notin B$ and w_k has not been forced previously in G. The only force w_k could have performed in G would be $w_k \to g_k$, in which case $g_k \to w_k$ would not be a white vertex force. Thus, no vertices in U_{w_k} have been previously placed in $\hat{B}$. Also, no vertices of U_{g_k} have been previously placed in $\hat{B}$. Since $n \geq 4$, it is possible to choose $n-2$ vertices in each of U_{g_k} and U_{w_k}, and place in $\hat{B}$, in such a way that for every pair of associated vertices in U_{g_k} and U_{w_k}, at least one member of the pair is placed in $\hat{B}$. By construction, $|\hat{B}| = |B|n + (|G| - |B|)(n-2)$.

The zero forcing process in $G \times K_n$ now follows the chronological list of forces $\mathcal{F}$. At stage k (before performing the kth force), we assume every vertex in U_v is blue for every v such that $v \in B$ or $v = w_\ell$ with $\ell < k$. Given the force $g_k \to w_k$ in $\mathcal{F}$, the two (blue) vertices in U_{g_k} that are associated with the two white vertices in U_{w_k} can each force the other's associate in U_{w_k}, so U_{w_k} is now entirely blue. Thus, all sets U_{w_k} will be turned entirely blue. □

The forcing process used in the proof of Theorem 2.1 is illustrated in Figure 1, where skew forcing is shown on P_6 and the corresponding standard zero forcing is shown on $P_6 \times K_5$; the number of each force in a chronological list of forces of P_6 is also shown. The first half of the skew forces in P_6 (forces 1, 2, and 3) are white vertex forces and the second half are not; all forces in the tensor product are standard forces.

Note that the bound in Theorem 2.1 need not be valid for $n = 3$, as shown in the next example.

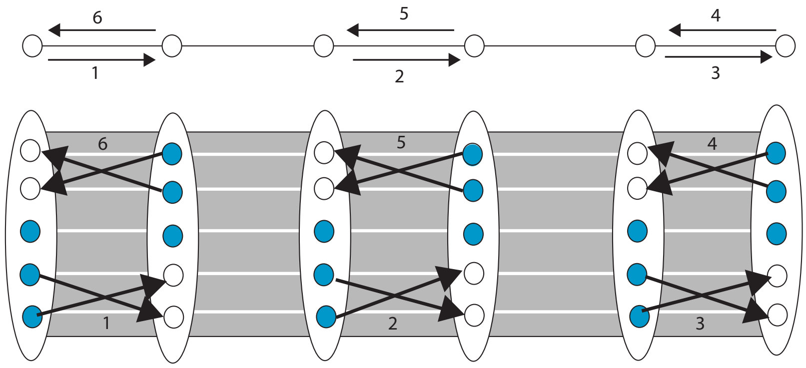

Figure 1: The skew zero forcing process on P_6 and the analogous zero forcing process on $P_6 \times K_5$ are illustrated. In the schematic diagram of $P_6 \times K_5$, the gray areas indicate that all possible edges are present except for the non-edges marked as white lines.

Example 2.2. Let H_3 denote the 3-sun shown in Figure 2 and consider $H_3 \times K_3$. Suppose $\hat{B}$ is a zero forcing set for $H_3 \times K_3$ of cardinality six. Observe that $\hat{B}$ must contain at least one vertex in U_g for each $g \in V(H_3)$, or no vertices in U_g could ever be colored blue, so necessarily $\hat{B}$ contains exactly one vertex of each U_g. In $H_3 \times K_3$ there are three sets U_g that contain vertices of degree 2 (corresponding to the three vertices of degree 1 in H_3) and three sets of vertices of degree 6. With appropriately staggered choices of which vertex of U_g is in $\hat{B}$, each of the three blue vertices of degree 2 can force one degree 6 vertex blue. Now the only blue vertices that have not yet forced all have degree 6. Each has (at least) one white neighbor of degree 2. In order for such a vertex v to perform a force, it must have no other white neighbors. That is, all four of its degree 6 neighbors must be blue. This implies two degree 6 white vertices must be associates, preventing any further forcing after v forces its degree 2 neighbor. Thus $\mathrm{Z}(H_3 \times K_3) \geq 7 > (3-2)6+0$; note $\mathrm{Z}^-(H_3) = 0$. (The zero forcing number of this example was originally found by use of the *Sage* zero forcing software [21], which provided a zero forcing set of order 7 and determined no smaller ones exist.)

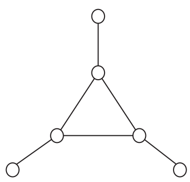

Figure 2: The 3-sun H_3

We apply Theorem 2.1 to the tensor product of a path and a complete graph. The case of odd paths has already been done:

Theorem 2.3. [18, Theorem 15] *If* $t \geq 1$ *is odd and* $n \geq 2$, *then* $\mathrm{M}(P_t \times K_n) = \mathrm{Z}(P_t \times K_n) = (n-2)t + 2$.

The method used to prove [18, Theorem 15] is the standard one of exhibiting a matrix and a zero forcing set with cardinality equal to the nullity of the matrix. Specifically, if $A \in \mathcal{S}(G)$ and S is a zero forcing set for G with $|S| = \operatorname{null} A$, then

$$|S| = \operatorname{null} A \leq \mathrm{M}(G) \leq \mathrm{Z}(G) \leq |S|$$

by Proposition 1.3. For even paths, we use the same matrix as [18, Theorem 15] to establish a lower bound on maximum nullity. For $n \geq 4$, Theorem 2.1 gives an equal upper bound on the zero forcing number; a specific zero forcing set is exhibited for $n = 3$.

Theorem 2.4. *If $t \geq 2$ is even and $n \geq 3$, then*

$$\mathrm{M}(P_t \times K_n) = \mathrm{Z}(P_t \times K_n) = (n-2)t.$$

Proof. A symmetric $nt \times nt$ matrix with rank $2t$ described by the tensor product graph $P_t \times K_n$ can be constructed as follows. Define the n-vectors $\mathbb{1} = [1, 1, \ldots, 1]^T$ and $\mathbf{y} = [1, 2, \ldots, n]^T$, and define $A_n = [\mathbb{1}\ \mathbf{y}] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{1}^T \\ \mathbf{y}^T \end{bmatrix}$. Then $\operatorname{rank} A_n = 2$, $A_n^T = -A_n$, and $A_n \in \mathcal{M}_0(K_n)$. Define B_t to be a $t \times t$ skew adjacency matrix (a tridiagonal matrix with 1s on the first superdiagonal, 0s on the main diagonal, and -1s on the first subdiagonal); note that since t is even, $\operatorname{rank} B_t = t$ because $\det(B_t) = (-1)^2 \det(B_{t-2})$. Then $B_t \otimes A_n \in \mathcal{S}(P_t \times K_n)$ and $\operatorname{rank}(B_t \otimes A_n) = 2t$. Thus $(n-2)t = tn - 2t \leq \mathrm{M}(P_t \times K_n) \leq \mathrm{Z}(P_t \times K_n)$. For $n \geq 4$, $\mathrm{Z}(P_t \times K_n) \leq (n-2)t$ by Theorem 2.1, since $\mathrm{Z}^-(P_t) = 0$ for t even [19].

Now assume $n = 3$. The forcing order in $P_t \times K_3$ is slightly more complicated than the one in Theorem 2.1, since initially only a single vertex can force at a time. Label the vertices of $P_t \times K_3$ as ordered pairs (r, s) with $1 \leq r \leq t$ and $1 \leq s \leq 3$. Define $B = \{(2i-1, 3), (2i, 1) : i = 1, \ldots, \frac{t}{2}\}$. First, $(1, 3)$ forces $(2, 2)$. Continue in increasing order of sets, so $(2i-1, 3)$ forces $(2i, 2)$ for $i = 1, \ldots, \frac{t}{2}$. Then the process is repeated in reverse order, starting at $2i = t$. Now, $(t, 1)$ and $(t, 2)$ are both blue with one white neighbor each, so $(t, 1)$ forces $(t-1, 2)$ and $(t, 2)$ forces $(t-1, 1)$. Continue in decreasing order, so $(2i, 1)$ forces $(2i-1, 2)$ and $(2i, 2)$ forces $(2i-1, 1)$ for $i = \frac{t}{2}$ down to $i = 1$, turning all odd-numbered sets all blue. Finally, in increasing order again, $(2i-1, 2)$ forces $(2i, 3)$ for $i = 1, \ldots, \frac{t}{2}$. □

Observe that the formula in Theorem 2.4 fails for $n = 2$: $P_t \times K_2$ is the disjoint union of two copies of P_t, so $\mathrm{M}(P_t \times K_2) = \mathrm{Z}(P_t \times K_2) = 2 \neq (2-2)t$.

Theorem 2.5. *If $n, t \geq 3$, then*

$$\mathrm{M}(C_t \times K_n) = \mathrm{Z}(C_t \times K_n) = \begin{cases} (n-2)t + 2 & \text{if } t \text{ is odd,} \\ (n-2)t + 4 & \text{if } t \text{ is even.} \end{cases}$$

Proof. Let the matrix A_n be as defined in the proof of Theorem 2.4, so $\operatorname{rank} A_n = 2$, $A_n^T = -A_n$, and $A_n \in \mathcal{M}_0(K_n)$. Define B_t to be the $t \times t$ skew adjacency matrix (with 1s in one cyclic direction and -1s in the other). Since $B_t \mathbb{1} = \mathbf{0}$, $\operatorname{rank} B_t \leq t - 1$. For odd t (respectively, even t) deleting the last one (resp., two) row(s) and column(s) of B_t gives the skew adjacency matrix of P_{t-1} (resp., P_{t-2}), which is nonsingular because $t-1$ (resp., $t-2$) is even. So $\operatorname{rank} B_t \geq t-1$ for odd t and $\operatorname{rank} B_t \geq t-2$ for even t. Since the rank of a skew symmetric matrix must be even, $\operatorname{rank} B_t = t-2$ for even t. Then $B_t \otimes A_n \in \mathcal{S}(C_t \times K_n)$ and

$$\operatorname{rank}(B_t \otimes A_n) = \begin{cases} 2t-2 & \text{if } t \text{ is odd,} \\ 2t-4 & \text{if } t \text{ is even.} \end{cases}$$

Since $\mathrm{M}(C_t \times K_n) \leq \mathrm{Z}(C_t \times K_n)$, it suffices to exhibit a zero forcing set of cardinality $(n-2)t+2$ for t odd and $(n-2)t+4$ for t even. Color all the vertices in U_t blue for t odd, or all the vertices in U_{t-1} and in U_t blue for t even. We now consider the graph obtained by deleting these blue vertices, i.e., $P_{t-1} \times K_n$ or $P_{t-2} \times K_n$, respectively, both having the form of a tensor product of an even path with a complete graph. We construct a minimum zero forcing set $\hat{B}'$ (of cardinality $(n-2)(t-1)$ or $(n-2)(t-2)$, respectively) and perform zero forcing as in Theorem 2.4. The zero forcing set for $C_t \times K_n$ is $\hat{B}' \cup U_t$ or $\hat{B}' \cup U_t \cup U_{t-1}$, respectively. $\square$

2.2 Cartesian products

Next we determine the zero forcing number and maximum nullity of the Cartesian product of two cycles.

Theorem 2.6. *For $m \geq n \geq 3$,*

$$\mathrm{M}(C_n \,\square\, C_m) = \mathrm{Z}(C_n \,\square\, C_m) = \begin{cases} 2n-1 & \text{if } m = n \text{ and } n \text{ is odd,} \\ 2n & \text{otherwise.} \end{cases}$$

Proof. For $m = n \geq 3$, by [8, Theorem 2.18] $\mathrm{M}(C_n \,\square\, C_n) = \mathrm{Z}(C_n \,\square\, C_n) = n + 2\left\lfloor \frac{n}{2} \right\rfloor$, so $\mathrm{M}(C_n \,\square\, C_n) = \mathrm{Z}(C_n \,\square\, C_n) = 2n-1$ for n odd and $\mathrm{M}(C_n \,\square\, C_n) = \mathrm{Z}(C_n \,\square\, C_n) = 2n$ for n even.

So assume $m > n \geq 3$. It is shown in [2, Corollary 2.8] that the vertices of two consecutive cycles C_n form a zero forcing set, so $\mathrm{Z}(C_n \,\square\, C_m) \leq 2n$. To complete the proof we construct a matrix in $\mathcal{S}(C_n \,\square\, C_m)$ with nullity $2n$, so $2n \leq \mathrm{M}(C_n \,\square\, C_m) \leq \mathrm{Z}(C_n \,\square\, C_m) \leq 2n$.

Let $k = \left\lceil \frac{n}{2} \right\rceil$. Let A be the matrix obtained from the adjacency matrix of C_n by changing one pair of symmetrically placed entries from 1 to -1. Then as discussed in the proof of [2, Theorem 3.8], the distinct eigenvalues of A are $\mu_i = 2\cos\frac{\pi(2i-1)}{n}$, $i = 1, \ldots, k$, each with multiplicity 2 except μ_k, which has multiplicity 1 when n is odd. Assuming that there exists a matrix $B \in \mathcal{S}(C_m)$ such that μ_i is an eigenvalue of B with multiplicity 2 for $i = 1, \ldots, k$, it follows that $A \otimes I_m - I_n \otimes B$ has eigenvalue zero with multiplicity at least $2n$, because every eigenvalue of A has a corresponding eigenvalue of B with multiplicity 2.

It remains to establish the existence of a matrix $B \in \mathcal{S}(C_m)$ such that μ_i is an eigenvalue of B with multiplicity 2 for $i = 1, \ldots, k$. In [14, Theorem 4.3] Ferguson showed that that for any set of $\ell+1$ distinct real numbers $\lambda_1 > \cdots > \lambda_{\ell+1}$, there is a matrix $B \in \mathcal{S}(C_{2\ell+1})$ having eigenvalues $\lambda_1, \ldots, \lambda_{\ell+1}$ with multiplicities $1, 2, \ldots, 2$, respectively. In [15, Theorem 3.3] Fernandes and da Fonseca extended Ferguson's method to show that for any set of ℓ distinct real numbers $\lambda_1 > \cdots > \lambda_\ell$, there is a matrix $B \in \mathcal{S}(C_{2\ell})$ having eigenvalues $\lambda_1, \ldots, \lambda_\ell$ with multiplicities $2, \ldots, 2$. For even m, choose $\lambda_i = \mu_i$ for $i = 1, \ldots, k$. For odd m, since $m > n$ we can choose $\lambda_{i+1} = \mu_i$ for $i = 1, \ldots, k$. □

3 Zero forcing lower bound for power domination number

The power domination number of several families of graphs has been determined using a two-step process: finding an upper bound and a lower bound. The upper bound is usually obtained by providing a pattern to construct a set, together with a proof that the constructed set is a power dominating set. The lower bound is usually found by exploiting structural properties of the particular family of graphs, and it often consists of a very technical and lengthy process (see, for example, [11]). Therefore, finding good general lower bounds for the power domination number is an important problem.

An effort in that direction is the work by Stephen et al. [22, Theorem 3.1] in which a lower bound is presented and successfully applied to finding the power domination number of some graphs modeling chemical structures. However, their lower bound depends heavily on the choice of a family of subgraphs satisfying certain properties. While in some graphs it is possible to find families of subgraphs that yield good lower bounds, in others it is not, and the bound depends on the family of subgraphs chosen rather than on the graph itself.

The lower bound for the power domination number of a hypercube presented in Dean et al. [9] is based on the following result:

Theorem 3.1. [9, Lemma 2] *Let G be a graph with no isolated vertices, and let $S = \{u_1, \ldots, u_t\}$ be a power dominating set for G. Then $\mathrm{Z}(G) \le \sum_{i=1}^{t} \deg u_i$.*

Although $\mathrm{Z}(G) \le \sum_{i=1}^{t}(\deg u_i + 1)$ follows trivially from Observation 1.1, the improved bound in the previous result yields a tight bound in the next theorem by removing the '+1' from the denominator of the equally immediate lower bound $\frac{\mathrm{Z}(G)}{\Delta(G)+1} \le \gamma_P(G)$. The next theorem, which follows from Theorem 3.1, can be used to map zero forcing results to power dominating results and vice versa.

Theorem 3.2. *Let G be a graph that has an edge. Then $\left\lceil \frac{\mathrm{Z}(G)}{\Delta(G)} \right\rceil \le \gamma_P(G)$, and this bound is tight.*

Proof. Choose a minimum power dominating set $\{u_1, \ldots, u_t\}$, so $t = \gamma_P(G)$, and observe that $\sum_{i=1}^{t} \deg u_i \le t\Delta(G)$. If G has no isolated vertices, the result follows from Theorem 3.1. Each isolated vertex of G contributes one to both the zero forcing number and the power domination number, hence the result still holds. Since $\mathrm{Z}(K_n) = \Delta(K_n) = n-1$ and $\gamma_P(K_n) = 1$, the bound is tight. □

The next corollary is immediate from Proposition 1.3. Although weaker than Theorem 3.2, Corollary 3.3 can sometimes be applied using a well known matrix such as the adjacency or Laplacian matrix of the graph, even if $\mathrm{M}(G)$ and $\mathrm{Z}(G)$ are not known. In addition, Corollary 3.3 permits the incorporation of a new set of tools based on linear algebra into the study of power domination.

Corollary 3.3. *For a graph G that has an edge and any matrix $A \in \mathcal{S}(G)$, $\left\lceil \frac{\operatorname{null} A}{\Delta(G)} \right\rceil \leq \gamma_P(G)$.*

3.1 Applications to computation of power domination number

Dorbec et al. studied the power domination problem for the tensor product of two paths [10]. We study the tensor product of a path and a complete graph and of a cycle and a complete graph.

Proposition 3.4. *Let $n \geq 3$. If $G = P_t$ with $t \geq 2$ or $G = C_t$ with $t \geq 3$, then*

$$\gamma_P(G \times K_n) \leq \begin{cases} \left\lceil \frac{t}{2} \right\rceil & \text{if } t \not\equiv 2 \mod 4, \\ \frac{t}{2} + 1 & \text{if } t \equiv 2 \mod 4. \end{cases}$$

Proof. Denote the vertices of $G \times K_n$ as ordered pairs (r, s) for $1 \leq r \leq t, 1 \leq s \leq n$. Define a set S in the following way (throughout, k is a positive integer):

If $t = 4k$, let $S = \{(4i-2, 1), (4i-1, 1) : 1 \leq i \leq k\}$.

If $t = 4k+1$, let $S = \{(4i-2, 1), (4i-1, 1) : 1 \leq i \leq k\} \cup \{(4k, 1)\}$.

If $t = 4k+2$, let $S = \{(4i-2, 1), (4i-1, 1) : 1 \leq i \leq k\} \cup \{(4k+1, 1), (4k+2, 1)\}$.

If $t = 4k+3$, let $S = \{(4i-2, 1), (4i-1, 1) : 1 \leq i \leq k\} \cup \{(4k+2, 1), (4k+3, 1)\}$.

It is easy to verify that S is a power dominating set for $G \times K_n$ and thus, $\gamma_P(G \times K_n) \leq |S|$. $\square$

Observation 3.5. *The degree of an arbitrary vertex (g, h) in $G \times H$ is the product $deg_G(g) deg_H(h)$. Therefore, $\Delta(G \times H) = \Delta(G)\Delta(H)$.*

Theorem 3.6. *Let $t \geq 3$ and $G = P_t$ or $G = C_t$. Suppose t is odd and $n \geq t$, or suppose t is even and either (1) $G = P_t$ and $n \geq \frac{t}{2} + 2$, or (2) $G = C_t$ and $n \geq \frac{t}{2}$. Then*

$$\gamma_P(G \times K_n) = \begin{cases} \left\lceil \frac{t}{2} \right\rceil & \text{if } t \not\equiv 2 \mod 4, \\ \frac{t}{2} \text{ or } \frac{t}{2} + 1 & \text{if } t \equiv 2 \mod 4. \end{cases}$$

Proof. Proposition 3.4 provides an upper bound on $\gamma_P(G \times K_n)$. We obtain a lower bound on $\gamma_P(G \times K_n)$ from Theorem 3.2 and results in Section 2, by considering two cases depending on the parity of t. Observation 3.5 yields $\Delta(G \times K_n) = \Delta(G)\Delta(K_n) = 2(n-1)$.

If $t = 2k+1$ for some positive integer k, then by Theorems 2.3, 2.5, and 3.2 we know that

$$\begin{aligned} \gamma_P(G \times K_n) &\geq \left\lceil \frac{(n-2)(2k+1)+2}{2(n-1)} \right\rceil = \left\lceil \frac{2k(n-1)+n-2k}{2(n-1)} \right\rceil = \left\lceil k + \frac{n-2k}{2(n-1)} \right\rceil \\ &\geq k+1 \text{ if } n-2k > 0. \end{aligned}$$

That is, $\lceil \frac{t}{2} \rceil \le \gamma_P(G \times K_n)$ if t is odd and $n \ge t$.

Let $t = 2k$. Define $c = 0$ for $G = P_t$ and $c = 2$ for $G = C_t$. Then by Theorems 2.4, 2.5, and 3.2 we know that

$$\begin{aligned}\gamma_P(G \times K_n) &\ge \left\lceil \frac{(n-2)(2k)+2c}{2(n-1)} \right\rceil = \left\lceil \frac{k(n-1)-k+c}{(n-1)} \right\rceil = \left\lceil k - \frac{k-c}{n-1} \right\rceil \\ &= k \text{ if } n-1 > k-c.\end{aligned}$$

That is, $\frac{t}{2} \le \gamma_P(G \times K_n)$ if $G = P_t$ and $n \ge \frac{t}{2}+2$, or if $G = C_t$ and $n \ge \frac{t}{2}$. □

Remark 3.7. For $t = 2$ and $n \ge 4$, $2 = \left\lceil \frac{2(n-2)}{n-1} \right\rceil \le \gamma_P(P_2 \times K_n) \le 2 = \frac{t}{2}+1$, with the lower bound by Theorems 2.4 and 3.2 and the upper bound by Proposition 3.4. Computations using *Sage* power domination software [21] also suggest that for $n \ge 4$ and larger $t \equiv 2 \mod 4$, the correct value is $\gamma_P(G \times K_n) = \frac{t}{2}+1$ if (1) $G = P_t$ and $n \ge \frac{t}{2}+2$ or (2) $G = C_t$ and $n \ge \frac{t}{2}$. As noted earlier, $n = 3$ can behave differently. It is easy to see that $\gamma_P(P_2 \times K_3) = 1 = \frac{t}{2}$, and Sage computations show $\gamma_P(C_6 \times K_3) = 3 = \frac{t}{2}$ (these are the only cases where the bounds $n \ge \frac{t}{2}+2$ for $G = P_t$ and $n \ge \frac{t}{2}$ for $G = C_t$ permit $n = 3$).

Remark 3.8. It is shown in [4, Theorem 4.2][1] that for $m \ge n \ge 3$,

$$\gamma_P(C_n \,\square\, C_m) \le \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 2 \mod 4, \\ \frac{n}{2}+1 & \text{if } n \equiv 2 \mod 4. \end{cases}$$

It follows immediately from Theorems 2.6 and 3.2 that this inequality is an equality whenever $n \not\equiv 2 \mod 4$, and $\frac{n}{2} \le \gamma_P(C_n \,\square\, C_m) \le \frac{n}{2}+1$ for $n \equiv 2 \mod 4$. There is an unpublished proof in [7] that

$$\gamma_P(C_n \,\square\, C_m) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 2 \mod 4, \\ \frac{n}{2}+1 & \text{if } n \equiv 2 \mod 4. \end{cases}$$

3.2 Applications to computation of zero forcing number

In the preceding section, we obtained the power domination numbers of certain graphs from the corresponding zero forcing numbers. We take the opposite approach in this section, using Theorem 3.2 and known power domination numbers to obtain the corresponding zero forcing numbers.

In [10, Theorem 4.1] it was proved that:

$$\gamma_P(G * H) = \begin{cases} \gamma(G) & \text{if } \gamma_P(H) = 1, \\ \gamma_t(G) & \text{otherwise,} \end{cases} \tag{1}$$

where $\gamma_t(G)$ denotes the *total domination number of* G, defined as the minimum cardinality of a dominating set S in G such that $N(S) = V(G)$.

[1] There is a typographical error in the statement of this theorem, but it is clear from the proof that this is what is intended.

Now, from Theorem 3.2 we know $\mathrm{Z}(G*H) \leq \gamma_P(G*H)\Delta(G*H)$. It follows easily from the definition of lexicographic product that $\deg_{G*H}(g,h) = (\deg_G g)|V(H)| + \deg_H h$ for any vertex $(g,h) \in V(G*H)$, and therefore $\Delta(G*H) = \Delta(G)|V(H)| + \Delta(H)$. Then from (1) above, we obtain

$$\mathrm{Z}(G*H) \leq \begin{cases} \gamma(G)\left(\Delta(G)|V(H)| + \Delta(H)\right) & \text{if } \gamma_P(H) = 1, \\ \gamma_t(G)\left(\Delta(G)|V(H)| + \Delta(H)\right) & \text{otherwise.} \end{cases} \tag{2}$$

In particular, we obtain the following result for lexicographic products of regular graphs with low domination and power domination numbers.

Theorem 3.9. *Let G and H be regular graphs with degree d_G and d_H, respectively. If $\gamma_P(H) = 1$ and $\gamma(G) = 1$, then $\mathrm{Z}(G*H) = d_G|V(H)| + d_H$.*

Proof. Since G is d_G-regular, H is d_H-regular, and $\gamma_P(H) = \gamma(G) = 1$, the bound in (2) gives $\mathrm{Z}(G*H) \leq d_G|V(H)| + d_H$. Moreover, since $G*H$ is $(d_G|V(H)| + d_H)$-regular, Observation 1.2 implies $d_G|V(H)| + d_H = \delta(G*H) \leq \mathrm{Z}(G*H)$. □

Corollary 3.10. *For $n \geq 2$ and $m \geq 3$, $\mathrm{Z}(K_n * C_m) = (n-1)m + 2$.*

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