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Enforced hamiltonian cycles in circulant graphs

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Abstract

The H-force number of a hamiltonian graph G is the smallest number k with the property that there exists a set $W \subseteq V(G)$, $|W| = k$, such that each cycle passing through all vertices of W is hamiltonian. In this paper, we determine the H-force number of circulant graphs.

1 Introduction

Throughout this paper, we consider graphs without loops or multiple edges; for terminology not defined here, we refer to [5].

In research of hamiltonian graphs, there are several concepts setting a kind of stratification within this family of graphs, such as the number of different cycle lengths (and the related notion of pancyclicity), the number of edges that can be prescribed in a certain way such that it is possible to route a hamiltonian cycle through them (see [13], [10] or [7] for the case of 4-connected planar graphs). Another way of classifying hamiltonian graphs involves the notion of k-hamiltonicity: an n-vertex graph $G = (V, E)$ is called *k-hamiltonian* if, for all sets $U \subseteq V$, $0 \leq |U| \leq k$, the graph $G - U$ (obtained from G by deleting all vertices of U) is hamiltonian. In particular, a graph is 1-hamiltonian if it is hamiltonian and the graph that results from deletion of any vertex is also hamiltonian. There are several sufficient conditions for graphs to be 1-hamiltonian (see [3], [4] or [12]); in many cases, these conditions are similar to the classical conditions for hamiltonian connectivity.

Yet another concept of developing a hierarchy within hamiltonian graphs was defined first in [6] in the following way: let $G = (V, E)$ be a hamiltonian graph and let $W \subseteq V$, $W \neq \emptyset$. A cycle in G is a *W-cycle* if it contains all vertices of W. We say

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that W *enforces* a hamiltonian cycle in G (or, W is an *H-force set*) if each W-cycle of G is hamiltonian. The *H-force number* $h(G)$ is the cardinality of the smallest H-force set in G.

Note that if a graph $G = (V, E)$ is 1-hamiltonian, then $h(G) = |V|$, and vice versa. Thus, it is natural to consider graphs with H-force number less than their orders. The graphs with small H-force number were studied in [6], where there was presented, among other results, the complete characterization of graphs with H-force number two (or three in the case of 3-connected graphs, and four for 3-connected planar graphs, respectively).

In general, determining the H-force number of a hamiltonian graph is a difficult problem even for special graphs. In the papers [15] and [14], the H-force numbers of several special families of hamiltonian graphs were determined. In [9], an upper and a lower bound of H-force number were given using the cycle extendability property. Note also that the concepts of H-force set and H-force number were extended to hamiltonian digraphs and hypertournaments in [16] and [11].

In this paper, we deal with circulant graphs, defined as follows: for a finite set $\{a_1, a_2, \ldots, a_m\}$, $1 \le a_i \le n$ of positive integers (the set of parameters), the *circulant graph* $C_n(a_1, a_2, \ldots, a_m)$ has vertex set $[0, n-1] = \{0, 1, \ldots, n-1\}$ and two vertices u and v of $C_n(a_1, a_2, \ldots, a_m)$ are adjacent if $u - v \equiv \pm a_i (\text{mod } n)$.

2 Several properties of circulant graphs

In this section, we describe several properties which we will use in the sequel. Let $G = C_n(a_1, a_2, \ldots, a_m)$. The basic properties of G were studied in [1], [2], [8]. The following has been proved: G is bipartite if and only if every parameter is odd and n is even; G is connected if and only if $\gcd(n, a_1, \ldots, a_m) = 1$ and G is hamiltonian if and only if G is connected. Note that circulant graphs are vertex transitive which yields that if there is a cycle C^{-u} on all $n-1$ vertices of $V(G) - \{u\}$ (a cycle missing one vertex), then there is a cycle C^{-v} on all $n-1$ vertices of $V(G) - \{v\}$, for each $v \in V(G)$. In this paper, we will use the following isomorphisms several times:

1. The graph $C_n(a_1, \ldots, a_i, \ldots, a_m)$ is isomorphic to the graph $C_n(a_1, \ldots, n - a_i, \ldots, a_m)$.
2. Let $\gcd(n, a) = 1$. Then the graph $C_n(a)$ is isomorphic to the graph $C_n(1)$.
3. Let $\gcd(n, a) = 1$. Then the graph $C_n(a, b)$ is isomorphic to the graph $C_n(1, c)$, where $c \equiv a^{-1}b (\text{mod } n)$, a^{-1} being the multiplicative inverse of a modulo n.

Subsequently we assume, that $a_i \le n-2$ for each $i \in [1, m]$, $a_i \ne a_j$ and $a_i \ne n - a_j$ for all $i, j \in [1, m]$, $i \ne j$. Note that, in the following, all arithmetic on the vertices is assumed to be modulo n.

Subsequently we use one construction of a hamiltonian cycle in circulant graphs on two parameters, $\tilde{C}$. Let $G = C_n(a, b)$ be a hamiltonian circulant graph. If $\gcd(n, a) =$

1 or $\gcd(n,b)=1$, then $G \cong C_n(1,c)$ (isomorphism 3) and $\tilde{C}=(0,1,\dots,n-1,0)$ is a hamiltonian cycle of $C_n(1,c)$. Now let $\gcd(n,a)=g\geq 2$ and $\gcd(n,b)\geq 2$. A graph $C_n(a)$ has g isomorphic hamiltonian components G_i, $i\in[0,g-1]$ on $\frac{n}{g}$ vertices. Let $C_0=(0,a,2a,\dots,0)$ be a hamiltonian cycle in G_0. A cycle C_0 can be transformed to a hamiltonian cycle $C_i=(ib,ib+a,ib+2a,\dots,ib)$ in G_i, $i\in[1,g-1]$. Now

$$\tilde{C}=\bigcup_{i=0}^{g-1} C_i - \bigcup_{i=1}^{g-2}\{(ib,ib+a),(ib+a,ib+2a)\} -(0,a)-e$$
$$+\bigcup_{i=0}^{\lfloor\frac{g-2}{2}\rfloor}\{(2ib,(2i+1)b),((2i+1)b+2a,(2i+2)b+2a)\}+\bigcup_{i=0}^{g-2}(ib+a,(i+1)b+a),$$

where $e=((g-1)b+a,(g-1)b+2a)$ for odd g and $e=((g-1)b,(g-1)b+a)$ for even g, is a hamiltonian cycle of G (Fig. 1). In the next, the cycle $\tilde{C}$ is called *special hamiltonian cycle.*

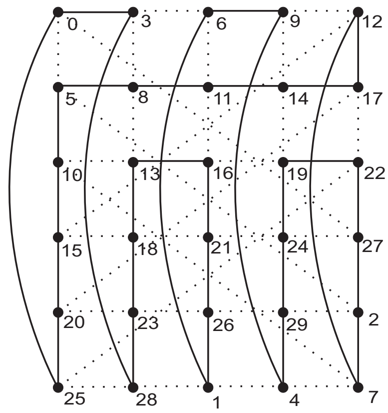

Fig. 1: Special hamiltonian cycle $\tilde{C}$ in $C_{30}(5,3)$

Lemma 1. *For a non-hamiltonian cycle C of G, every H-force set of G contains a vertex of $V(G)\setminus V(C)$.*

Lemma 2. *Let $G=C_n(a)$ be a hamiltonian circulant graph. Then $h(G)=1$.*

Proof. Let $G=C_n(a)$ be a hamiltonian circulant graph. Then G is isomorphic to a cycle (isomorphism 2) which implies $h(G)=1$. □

3 Bipartite circulant graphs on two parameters

In this section, we establish the H-force number for bipartite hamiltonian circulant graphs on two parameters.

Let $G=C_n(a,b)$ be a bipartite hamiltonian circulant graph, thus both its parameters are odd, n is even and $\gcd(n,a,b)=1$. Note that, if there is a cycle on $n-2$ vertices $V(G)-\{u,u+a\}$ (for such a cycle, we will use the notation $C^{-\{u,u+a\}}$ and we will say that a cycle misses two adjacent vertices), then due to vertex transitivity of circulant graphs, there is a cycle on $n-2$ vertices $V(G)-\{v,v+a\}$ (a cycle $C^{-\{v,v+a\}}$) for each v of $V(G)$.

Lemma 3. *Let $G = C_n(a,b)$ be a bipartite hamiltonian circulant graph and let $gcd(n,a)=1$ or $gcd(n,b)=1$. Then $h(G)=\frac{n}{2}$.*

Proof. Let $G=C_n(a,b)$ be a bipartite hamiltonian circulant graph and let $\gcd(n,a) = 1$ or $\gcd(n,b)=1$. Then $G \cong G' = C_n(1,c)$ (isomorphism 3). Obviously $h(G') \leq \frac{n}{2}$ (both bipartite sets of G' are H-force sets) and moreover, there is the cycle $C^{-\{0,c\}} = (1,2,\ldots,c-1,n-1,n-2,\ldots,c+1,1)$ of G' missing exactly two vertices 0 and c (Fig. 2). The cycle $C^{-\{0,c\}}$ can be transformed to a cycle $C^{-\{2i,2i+c\}}$ of G' missing exactly two vertices $2i$ and $2i+c$, where $i \in \left[0,\frac{n}{2}-1\right]$. Note that c is odd, so all $2i, 2i+c$ are distinct. By Lemma 1, at least one vertex of the pair of vertices $2i, 2i+c$ belongs to any H-force set, and thus $h(G)=\frac{n}{2}$. □

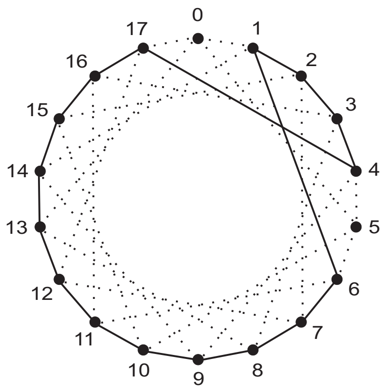

Fig. 2: $C^{-\{0,5\}}$ in $C_{18}(5,7)$

Lemma 4. *Let $G = C_n(a,b)$ be a bipartite circulant graph and let $gcd(n,a,b)=1$. Then $h(G)=\frac{n}{2}$.*

Proof. Let $G = C_n(a,b)$ be a bipartite hamiltonian circulant graph. Obviously $h(G) \leq \frac{n}{2}$ (both bipartite sets are H-force sets of G). If $\gcd(n,a)=1$ or $\gcd(n,b)=1$, then by previous Lemma $h(G)=\frac{n}{2}$. Let $\gcd(n,a)=g\geq 2$ and $\gcd(n,b)\geq 2$. Moreover, let $\tilde{C}$ be the special hamiltonian cycle of G described above. We use this cycle to construct a cycle $C^{-\{0,b\}}$, where $E(C^{-\{0,b\}}) = E(\tilde{C})$ $-(0,b)$ $-(0,n-a)$ $-(b,n-a+b)$ $+(n-a,n-a+b)$ (Fig. 3). The cycle $C^{-\{0,b\}}$ can be transformed to a cycle $C^{-\{2i,2i+b\}}$ of G missing exactly two vertices $2i$ and $2i+b$ where $i \in \left[0,\frac{n}{2}-1\right]$. By Lemma 1, at least one vertex of pair of vertices $2i, 2i+b$ belongs to any H-force set, thus $h(G)=\frac{n}{2}$. □

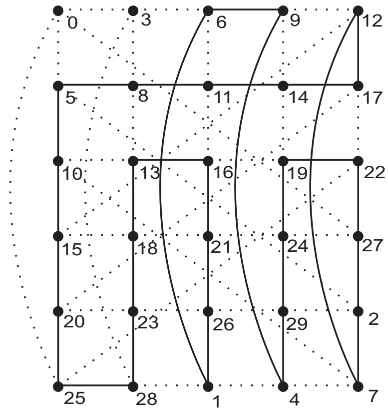

Fig. 3: $C^{-\{0,3\}}$ in $C_{30}(5,3)$

The next lemma will be used in the proof of Theorem 9.

Lemma 5. *Let $G = C_n(a,b)$ be a bipartite circulant graph with special hamiltonian cycle $\tilde{C}$. Then there is $k \in \{b, c \equiv ab^{-1}(\mathrm{mod}\ n)\}$ such that for every $i \in \left[0, \frac{n}{2}-1\right]$, a cycle $C^{-\{2i,2i+k\}}$ contains at least one edge of $\tilde{C}$.*

Proof. Let $C_n(a,b)$ be a bipartite circulant graph with special hamiltonian cycle $\tilde{C}$.

1. Let $\gcd(n,a)=1$ or $\gcd(n,b)=1$ and let $C^{-\{2i,2i+c\}}$ be the cycle of G described in Lemma 3. Then $\left|E(\tilde{C}) \cap E(C^{-\{2i,2i+c\}})\right| \geq n-4$, for every $i \in \left[0, \frac{n}{2}-1\right]$. Note that the smallest such graph is isomorphic to $C_6(1,3)$.
2. Let $\gcd(n,a) \geq 2$ and $\gcd(n,b) \geq 2$ and let $C^{-\{i,i+b\}}$ be the cycle of G described in Lemma 4. Then $\left|E(\tilde{C}) \cap E(C^{-\{2i,2i+b\}})\right| \geq n-4 \geq 2$, for every $i \in [0, n-1]$. Note that the smallest such graph is isomorphic to $C_{30}(3,5)$.

□

4 Non-bipartite circulant graphs on two parameters

In this section, we establish the H-force number for a non-bipartite hamiltonian circulant graph on two parameters.

Lemma 6. *Let $G = C_n(a,b)$ be a non-bipartite hamiltonian circulant graph and let gcd$(n,a)=1$ or gcd$(n,b)=1$. Then $h(G)=n$.*

Proof. Let $G = C_n(a,b)$ be a non-bipartite hamiltonian circulant graph and let $\gcd(n,a)=1$ or $\gcd(n,b)=1$. Then $G \cong G' = C_n(1,c)$ (isomorphism 3). Denote $d = n-c$. We assume that c is even (if c is odd, then n is also odd and $n-c$ is even, thus $C_n(1,c) \cong C_n(1,n-c)$ (isomorphism 1) and we denote $G' = C_n(1,n-c)$). We can see that $n \equiv d(\mathrm{mod}\ n)$. In the next we prove that there is a cycle C^{-0}.

1. Let $c \leq \frac{n+2}{2}$. Then $C^{-0} = P_1 \cup P_2$, where
$P_1 = (1+d, 1, 2, 2+d, 3+d, 3, 4, 4+d, 5+d, \ldots, n-2, n-1, c-1)$,
$P_2 = (c-1, c, c+1, \ldots, 1+d)$ with $V(P_1) = [1, c-1] \cup [d+1, n-1]$ and $V(P_2) = [c-1, d+1]$; note that all four vertex sets are non-empty (Fig. 4a).

2. Let $\frac{n+2}{2} < c \leq \frac{2}{3}n$. Then $d \leq \frac{n-2}{2} < \frac{n+2}{2}$. If d is even, then $h(G) = n$ (previous case). Now we assume that d is odd. Then $C^{-0} = P_1 \cup P_2 \cup P_3 \cup P_4$, where
$P_1 = (1, 2, \ldots, c-d)$,
$P_2 = (c-d, c, c+1, c+1-d, c+2-d, c+2, \ldots, d-1, d, 2d)$,
$P_3 = (2d, 2d+1, \ldots, n-1, c-1)$,
$P_4 = (c-1, c-2, \ldots, d+1, 1)$, with $V(P_1) = [1, c-d]$, $V(P_2) = [c-d, d] \cup [c, 2d]$, $V(P_3) = [2d, n-1] \cup \{c-1\}$ and $V(P_4) = [d+1, c-1]$; note that all four vertex sets are non-empty (Fig. 4b).

3. Let $\frac{2}{3}n < c$. Then $d \leq \frac{n}{3} < \frac{n+2}{2}$. If d is even, then $h(G) = n$ (first case). Now we assume that d is odd. Let $k = \left\lfloor \frac{2c-n-1}{n-c-1} \right\rfloor$ and note that $k \geq 1$ ($c > \frac{2}{3}n$). Then C^{-0} consists of two parts:

   - The first part of C^{-0} consists of k paths P_i, where
   $P_1 = (1, 2, \ldots, d)$ and
   $P_i = ((i-1)(d-1)+2, \ldots, (i-1)(d-1)+d = i(d-1)+1)$, $i \in [2, k]$.
   Now we merge these paths to the one path P by edges $(1, d+1)$ and $((i-1)(d-1)+2, (i-2)(d-1)+1)$, $i \in [3, k]$ (note that the first vertex is the initial vertex of P_i and the second vertex is the final vertex of P_{i-2}). Now P has end vertices $k(d-1)+1$ and $(k-1)(d-1)+1$; obviously $V(P) = [1, k(d-1)+1]$.
   - The second part of C^{-0} forms a path $P^* = C^* - (k(d-1)+1, k(d-1)+2)$, where C^* is a cycle of the graph $G^* = C_m(1, d)$, $m = n - k(d-1)$, which misses exactly one vertex 0 (a cycle from one of the previous cases). Denote every vertex j of P^* as $k(d-1)+j$. Now $V(P^*) = [k(d-1)+1, n-1]$.

   The cycle $C^{-0} = P \cup P^* + ((k-1)(d-1)+1, k(d-1)+2)$ (Fig. 4c).

The cycle C^{-0} can be transformed to a cycle C^{-i} of G' missing exactly one vertex i where $i \in [0, n-1]$. By Lemma 1, every vertex belongs to every H-force set; thus $h(G) = n$. □

Lemma 7. *Let $G = C_n(a, b)$ be a non-bipartite circulant graph and let $gcd(n,a,b) = 1$. Then $h(G) = n$.*

Proof. Let $G = C_n(a, b)$ be a non-bipartite circulant graph and let $\gcd(n, a, b) = 1$. If $\gcd(n, a) = 1$ or $\gcd(n, b) = 1$ then, by the previous lemma, $h(G) = n$. Let $\gcd(n, a) = g \geq 2$ and $\gcd(n, b) = f \geq 2$. A graph $C_n(a)$ has g isomorphic hamiltonian components G_i on $\frac{n}{g}$ vertices, $i \in [0, g-1]$. Let $C_0 = (0, a, 2a, \ldots, 0)$ be a hamiltonian cycle in G_0. The cycle C_0 can be transformed to a hamiltonian cycle

$C_i = (ib, ib+a, ib+2a, \ldots, ib)$ in G_i where $i \in [1, g-1]$. In what follows we prove that in G there exists a cycle C^{-b}.

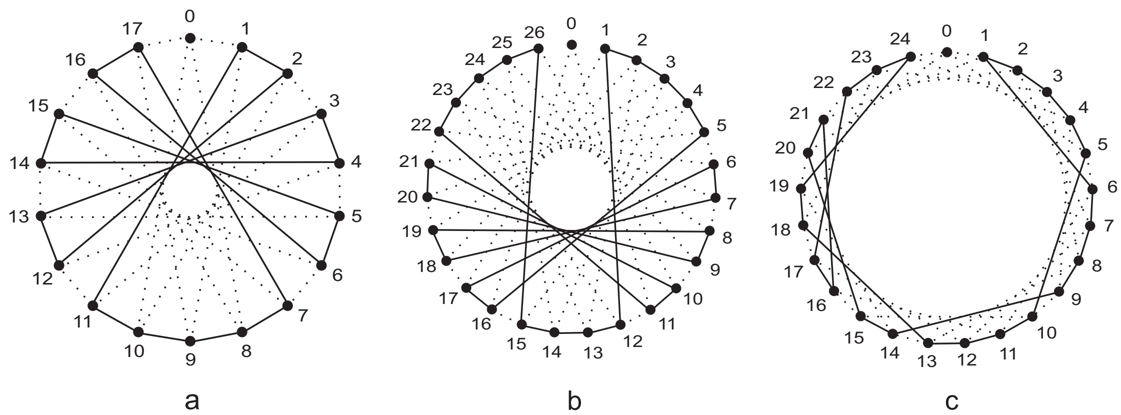

Fig. 4: C^{-0} in $C_{18}(1,8)$, $C_{27}(1,16)$ and $C_{25}(1,20)$

1. Let $\frac{n}{g}$ be odd. The cycle $C_{01}^{-b} = (0, a, a+b, 2a+b, 2a, 3a, 3a+b, 4a+b, \ldots, (\frac{n}{g}-1)a, 0)$ contains all vertices of C_0 and C_1, except one vertex b. Now $C^{-b} = (\tilde{C} - C_1 \cup C_2) \cup C_{01}^{-b} - \{(a+b, 2a+b)\} + \{(a+b, a+2b), (2a+b, 2a+2b)\}$ (Fig. 5a).

2. Let $\frac{n}{g}$ and $\frac{n}{f}$ be even. This means that n is even, one of a, b is odd (G is hamiltonian), and one is even (G is non-bipartite). Without lost of generality, assume that a is even and b is odd. Then g is even. Let k be the smallest integer such that $-b \equiv (g-1)b + ka (\bmod\ n)$. This is equal to $-b \equiv k\frac{a}{g} (\bmod\ \frac{n}{g})$ and it means that k is odd. The path $P_{01}^{-b} = (0, a, a+b, 2a+b, 2a, 3a, \ldots, ka, (k+1)a, \ldots, n-a, n-a+b, n-2a+b, \ldots, ka+b)$ contains all vertices of C_0 and C_1 except of one vertex b. For $i \in \left[2, \frac{n}{g}-1\right]$, we let $P_i = C_i - \{(ib+(k-1)a, ib+ka)\}$. Now $C^{-b} = P_{01}^{-b} \cup \bigcup_{i=2}^{\frac{n}{g}-1} P_i + \bigcup_{i=1}^{\frac{g}{2}-1} \{((2i-1)b+ka, 2ib+ka), (2ib+(k-1)a, (2i+1)b+(k-1)a)\}$ (Fig. 5b).

   The cycle C^{-b} can be transformed to a cycle C^{-i} of G missing exactly one vertex i where $i \in [0, n-1]$. By Lemma 1, every vertex belongs to every H-force set, thus $h(G) = n$.

$\square$

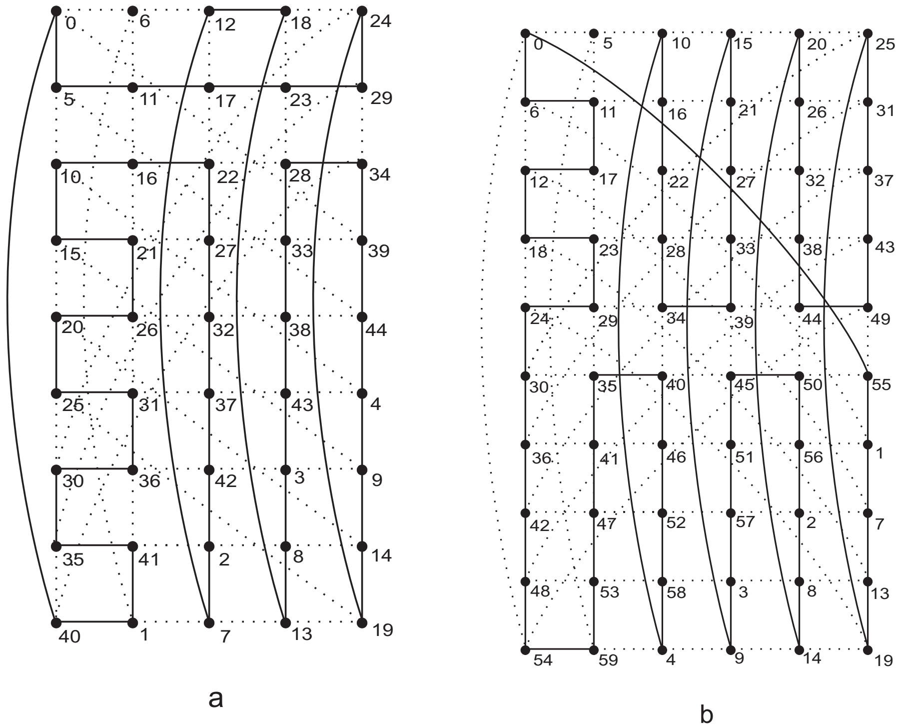

Fig. 5: C^{-b} in $C_{45}(5,6)$ and $C_{60}(6,5)$

The next lemma will be used in the proof of Theorem 9.

Lemma 8. *Let $G = C_n(a,b)$ be a non-bipartite circulant graph with special hamiltonian cycle $\tilde{C}$. Then for every $i \in [0, n-1]$, there exists a cycle C^{-i} containing at least one edge of $\tilde{C}$.*

Proof. Let $C_n(a,b)$ be a non-bipartite circulant graph with special hamiltonian cycle $\tilde{C}$.

1. Let $\gcd(n,a) = 1$ or $\gcd(n,b) = 1$ and let C^{-i} be a cycle of G described in Lemma 6. Then $\left|E(\tilde{C}) \cap E(C^{-i})\right| \geq \left\lfloor \frac{n}{2} \right\rfloor$, for every $i \in [0, n-1]$ (edges $(i, i+1) \in E(C^{-i})$). Note that the smallest such graph is isomorphic to $C_4(1,2)$.

2. Let $g = \gcd(n,a) \geq 2$ and $\gcd(n,b) \geq 2$ and let C^{-i} be a cycle of G described in Lemma 7. Then $\left|E(\tilde{C}) \cap E(C^{-i})\right| \geq n - \frac{n}{g}$, for every $i \in [0, n-1]$. Note that the smallest such graph is isomorphic to $C_6(2,3)$.

□

5 H-force number for circulant graphs

Theorem 9. *Let $G = C_n(a_1, a_2, \dots, a_m)$ be a hamiltonian circulant graph. Then*

$$h(G) = \begin{cases} 1, & \text{if } m = 1; \\ n, & \text{if } m \geq 2 \text{ and } G \text{ is non-bipartite}; \\ \frac{n}{2}, & \text{if } m \geq 2 \text{ and } G \text{ is bipartite}. \end{cases}$$

Proof. Let $G = C_n(a_1, a_2, \dots, a_m)$ be a hamiltonian circulant graph. If $m = 1$ or $m = 2$, then the assertion is true by Lemma 2 or by Lemmas 3, 4, 6 and 7, respectively. Let $m \geq 3$. Note that in the case when n is even and there are two parameters such that one is even and the second one is odd, we assume that a_1 is even and a_2 is odd. Let $G_t = C_n(a_1, \dots, a_t)$ and $g_t = \gcd(n, a_1, \dots, a_t)$, $t \in [1, m]$. Obviously, G_t has g_t hamiltonian components isomorphic to the graph $C_{\frac{n}{g_t}}(\frac{a_1}{g_t}, \dots, \frac{a_t}{g_t})$. We shall prove by induction that in the component of G_t which contains a vertex 0, there is a hamiltonian cycle $\tilde{C}_t$ and one of cycles from $S = \left\{ C_t^{-0}, C_t^{-a_2}, C_t^{-\{0,a_2\}}, C_t^{-\left\{0,a_1a_2^{-1}\right\}} \right\}$, which contains one edge from $\tilde{C}_t$. If $t \leq 3$, then the assertion is true by Lemmas 3,4,6,7 and by Lemmas 5 and 8. Now we assume that the assertion is true for every graph $G_3, \dots, G_t$. If $g_t = g_{t+1}$, there is nothing to show, since G_t and G_{t+1} have the same components. Now assume $g_{t+1} < g_t$. Let $k = \frac{g_t}{g_{t+1}}$. In G_t there are k components connected into one component of G_{t+1}. Denote these components by $Z_{u_1}, \dots, Z_{u_k}$, for $u_r = (r-1)a_{t+1}$, $r \in [1, k]$. By induction, the component Z_{u_1} $(0 \in V(Z_{u_1}))$ contains a cycle from S, which contains one edge from $\tilde{C}_t$. A hamiltonian cycle of Z_{u_1} can be transformed to a hamiltonian cycle of $Z_{u_r}, r \in [2, k]$ in the following way: if $(a_{t+1}, i_2, \dots, i_{\frac{n}{g_t}}, a_{t+1})$ is a hamiltonian cycle of Z_{u_1}, then $((r-1)a_{t+1}, i_2 + (r-1)a_{t+1}, \dots, i_{\frac{n}{g_t}} + (r-1)a_{t+1}, (r-1)a_{t+1})$ is a hamiltonian cycle of Z_{u_r}, $r \in [2, k]$. Now we will merge together a cycle of Z_{u_1} and hamiltonian cycles of Z_{u_r}, $r \in [2, k]$ into one cycle in a new component of G_{t+1}. This can be done by induction.

At first, we replace the edges $(i, i') \in E(Z_{u_1})$ and $(i + a_{t+1}, i' + a_{t+1}) \in E(Z_{u_2})$ by two new edges $(i, i + a_{t+1})$ and $(i', i' + a_{t+1})$ to obtain a new cycle. If we have constructed a cycle through all components $Z_{u_1}, \dots, Z_{u_r}$, $r \leq k$ (this cycle has at least one edge (j, j') of a hamiltonian cycle of Z_{u_r}), we can build a cycle through all vertices of $Z_{u_1}, \dots, Z_{u_{r+1}}$.

The cycle from S can be transformed to a cycle C^{-i} $(C^{-\{2i,2i+a_2\}})$ of G where $i \in [0, n-1]$ $(i \in \left[0, \frac{n}{2} - 1\right])$. By Lemma 1, every vertex (at least one vertex of every pair of vertices $2i, 2i+a_2$) belongs to every H-force set, and thus $h(G) = n$ $(h(G) \geq \frac{n}{2})$. $\square$

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