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Covering contractible edges in 2-connected graphs

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Abstract

In this paper we prove that, for any 2-connected graph G nonisomorphic to K_3, the set of contractible edges $E_C(G)$ cannot be covered by one vertex. All 2-connected graphs whose contractible edges can be covered by exactly two vertices are characterized. We also prove that if a vertex subset S covers $E_C(G)$ such that $|V(G)| \geq 2|S|+1$, then $G-S$ is not connected. Finally, we provide tight upper bounds for the order, size and number of non-trivial components of $G-S$ (components having more than one vertex) in terms of $|S|$, and characterize all the extremal graphs.

1 Introduction

Covers for contractible edges in 3-connected graphs were first studied by Ota and Saito [4] who proved that the set of contractible edges $E_C(G)$ in a 3-connected graph G of order at least six cannot be covered by two vertices (see also Saito [5]). Later, Hemminger and Yu [3] characterized all 3-connected graphs of order at least ten whose contractible edges can be covered by three vertices. Yu [6] showed that for any 3-connected graph G nonisomorphic to K_4, if S covers $E_C(G)$ such that $|V(G)| \geq 3|S|-1$, then $G-S$ is not connected. Hemminger and Yu [2] provided upper bounds for the order, size and number of non-c-components of $G-S$ (refer to the paper for the definition) in terms of $|S|$. Inspired by the above work, we prove the corresponding results for 2-connected graphs.

All graphs considered in this paper are finite and simple. Standard graph-theoretical terminology can be found in Diestel [1]. Consider any 2-connected graph G. An edge is *contractible* if its contraction results in a 2-connected graph. Denote the set of contractible edges of G by $E_C(G)$. Let S be a subset of $V(G)$. A component of $G-S$ is *trivial* if its order is one. A *fragment* F of S is a union of at least one but not all components of $G-S$. Define $\overline{F} := G-S-F$ which is also a fragment of S. Denote the vertex set, edge set and component set of all non-trivial components of $G - S$ by $VN(G, S)$, $EN(G, S)$ and $CN(G, S)$ respectively. We say S is a *cover* of $E_C(G)$ if every contractible edge in G is incident to a vertex in S. For any two disjoint subsets A and B of $V(G)$, denote $E_G(A, B)$ to be the set of all edges between A and B in G. Consider the complete bipartite graph $K_{2,k}$ and let $\{x, y\}$ be the partition class of the two vertices. Define $K_{2,k}^+ := K_{2,k} + xy$. Also, the following construction of a new 2-connected graph based on G will be useful later. For each edge e in a subset D of $E(G)$, add a vertex x_e together with two edges from x_e to $V(e)$. Denote the resulting graph by $G\#D$.

The paper is organized as follows. In Section 2, we will show that for any 2-connected graph G nonisomorphic to K_3, the set of contractible edges cannot be covered by one vertex. All 2-connected graphs whose contractible edges can be covered by exactly two vertices are characterized. We also prove that if a vertex subset S covers $E_C(G)$ such that $|V(G)| \geq 2|S| + 1$, then $G - S$ is not connected. In Section 3, we provide tight upper bounds for the order, size and number of non-trivial components of $G - S$ in terms of $|S|$, and characterize all the extremal graphs.

2 Small vertex cover of contractible edges

We begin with two basic results concerning contractible and non-contractible edges in any 2-connected graph G nonisomorphic to K_3. Note that e is a non-contractible edge in G if and only if $G - V(e)$ is not connected.

Lemma 2.1. *Let G be any 2-connected graph nonisomorphic to K_3 and e be an edge of G. Then $G - e$ or G/e is 2-connected.*

Proof. Let $e = xy$. Suppose $G - e$ is not 2-connected. Let z be a cutvertex of $G - e$. Then $G - e - z$ has exactly two components, say C and D such that e joins C and D. Obviously, $z \notin V(e)$ and every x-y path other than e passes through z. Suppose G/e is not 2-connected. Then $G - V(e)$ is not connected. Let B be the component of $G - V(e)$ containing z. Now, there exists an x-y path in $G - B - e$ not passing through z, a contradiction. □

Lemma 2.2. *Let G be any 2-connected graph nonisomorphic to K_3, and e and f be two non-contractible edges of G. Then f is a non-contractible edge of $G - e$.*

Proof. By Lemma 2.1, $G - e$ is 2-connected. Since f is non-contractible in G, $G - V(f)$ is not connected. Therefore, $G - e - V(f)$ is not connected and f is a non-contractible edge of $G - e$. □

By the above two fundamental lemmas, every vertex of G is incident to at least two contractible edges and hence $|V(G)| \leq |E_C(G)|$. Also, the subgraph induced by all the contractible edges $(V(G), E_C(G))$ is 2-connected.

Lemma 2.3. *Consider any 2-connected graph G nonisomorphic to K_3. Let x, y be any two vertices of G and C be a component of $G - x - y$. Then $E_G(x, C)$ contains a contractible edge and so does $E_G(y, C)$. Moreover, if $|C| > 1$, then there exist two independent contractible edges in $E_G(\{x, y\}, C)$.*

Proof. Suppose all edges in $E_G(x, C)$ are non-contractible. By Lemma 2.1 and 2.2, we can delete all edges in $E_G(x, C)$ and the resulting graph $H := G - E_G(x, C)$ is 2-connected. However, either x is an isolated vertex of H or y is a cutvertex of H, a contradiction.

Now, assume $|C| > 1$. Suppose there are no two independent contractible edges in $E_G(\{x, y\}, C)$. Then there exists a vertex z in C that covers $E_G(\{x, y\}, C) \cap E_C(G)$, and xz and yz are the only contractible edges in $E_G(\{x, y\}, C)$. By the 2-connectedness of G, there exists an edge joining $\{x, y\}$ to a vertex w of C other than z. Without loss of generality, assume w is adjacent to y. Obviously, yw is non-contractible. Let D be a component of $G - y - w$ not containing z. Then $D \subsetneq C$ and from above, $E_G(y, D)$ contains a contractible edge not covered by z, a contradiction. □

Lemma 2.4. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Suppose $G - S$ contains two vertices x and y. Let C be any component of $G - x - y$. Then the following statements hold.*

(a) $C \cap S \neq \emptyset$.

(b) *If* $|C \cap S| = 1$, *then* $|C| = 1$.

(c) *If* $|C \cap S| > 1$, *then there exist two independent contractible edges in* $E_G(\{x, y\}, C)$.

Proof. (a) follows from the first part of Lemma 2.3 while (b) and (c) follow directly from the second part of Lemma 2.3. □

We now prove that for any 2-connected graph nonisomorphic to K_3, a vertex cover of the set of all contractible edges contains at least two vertices, and characterize all graphs whose contractible edges can be covered by exactly two vertices.

Theorem 2.1. *For any 2-connected graph G nonisomorphic to K_3, $E_C(G)$ cannot be covered by one vertex.*

Proof. Suppose x is a vertex in G that covers $E_C(G)$. Obviously, there exists an edge yz that is not incident to x. Therefore, yz is non-contractible. But this contradicts Lemma 2.4(a) by considering a component of $G - y - z$ not containing x. □

Theorem 2.2. *Let G be any 2-connected graph nonisomorphic to K_3. Then $E_C(G)$ can be covered by two vertices if and only if G is isomorphic to $K_{2,k}$ or $K_{2,k}^+$ where $k \geq 2$.*

Proof. ($\Leftarrow$) Easy.

($\Rightarrow$) Let $S := \{u, v\}$ be a cover of $E_C(G)$. Consider any component C of $G - S$. If $|C| > 1$, then C contains a non-contractible edge, say xy. By Lemma 2.4, $G - x - y$ has exactly two components both of order one, namely u and v. We have $G = K_{2,2}^+$.

Now, assume that every component of $G - S$ consists of exactly one vertex. Then G is isomorphic to $K_{2,k}$ or $K_{2,k}^+$ where $k \geq 2$. □

Next, we show that if a vertex cover S of the set of all contractible edges is small enough, then $G - S$ is not connected.

Lemma 2.5. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Let xy be an edge in $G - S$ and F be a fragment of $G - x - y$. Consider $G' := (V(F) \cup \{x, y, z\}, E(G[F \cup xy]) \cup \{xz, yz\})$ and $S' := (S \cap V(F)) \cup z$. Then G' is 2-connected, $E_C(G') = (E_C(G) \cap E(G[F \cup xy]) \cup \{xz, yz\}$ and S' covers $E_C(G')$.*

Proof. Suppose G' contains a cutvertex w. Then $w \in F$. But then w is a cutvertex of G, a contradiction. Hence, G' is 2-connected.

To prove S' covers $E_C(G')$, we will show that $E_C(G') = (E_C(G) \cap E(G[F \cup xy]) \cup \{xz, yz\}$. Since both $G' - x - z$ and $G' - y - z$ are connected, xz and yz are contractible edges in G'. Let $uv \in E_C(G') \setminus \{xz, yz\}$. Note that $uv \neq xy$. Then $G' - u - v$ is connected and so is $G' - u - v - z$. Hence, $G - u - v$ is connected and $uv \in E_C(G) \cap E(G[F \cup xy])$.

Suppose $st \in E_C(G) \cap E(G[F \cup xy])$. Then $G - s - t$ is connected and so is $G - s - t - \overline{F}$. Therefore, $G' - s - t$ is connected and $st \in E_C(G')$. □

Theorem 2.3. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. If $|V(G)| \geq 2|S| + 1$, then $G - S$ is not connected.*

Proof. The proof is by induction on $|V(G)|$. The result is trivially true for $|V(G)| = 4$ by Theorem 2.1. Suppose the theorem is true for all 2-connected graphs with less than k vertices. Consider any 2-connected graph G with k vertices. Let S be a cover of $E_C(G)$ such that $|S| \leq \frac{k-1}{2}$. Suppose $G - S$ is connected. Note that all edges in $G - S$ are non-contractible. Let xy be any edge in $G - S$ and $C_1, C_2, \ldots, C_m$ be the components of $G - x - y$. For each C_i, define $G_i := (V(C_i) \cup \{x, y, x_i\}, E(G[C_i \cup xy]) \cup \{x_i x, x_i y\})$.

Suppose $m \geq 3$, or $m = 2$ and both C_1 and C_2 contain at least two vertices. Then $|V(G_i)| < |V(G)|$. By Lemma 2.5, $S_i := (S \cap C_i) \cup x_i$ is a vertex cover of all contractible edges of G_i. Since $G - S$ is connected, $G_i - S_i$ is also connected. By induction, $|V(G_i)| \leq 2|S_i| = 2|S \cap C_i| + 2$. Now, $|V(G)| = 2 + \sum_i |V(C_i)| = 2 + \sum_i (|V(G_i)| - 3) \leq 2 + \sum_i (2|S \cap C_i| - 1) = 2 - m + 2|S| \leq 2|S|$, a contradiction. Therefore, $m = 2$, and one of C_1 and C_2 contains exactly one vertex.

For each edge e in $G - S$, define x_e to be the single vertex component of $G - V(e)$. Note that $x_e \in S$, $N_G(x_e) = V(e)$, and for any two distinct edges e, f in $G - S$, $x_e \neq x_f$. Since $G - S$ is connected, $|S| \geq |E(G - S)| \geq |V(G - S)| - 1 = |V(G)| - |S| - 1$ implying $|V(G)| \leq 2|S| + 1$. Consequently, $|V(G)| = 2|S| + 1$, $|S| = |E(G - S)|$ and $G - S$ is a tree. But then G is not 2-connected, a contradiction. □

The bound $2|S| + 1$ is best possible as demonstrated by K_4^- (K_4 minus an edge) for $|S| = 2$ and $K_3 \# E(K_3)$ for $|S| = 3$. For $|S| = k \geq 4$, let H be any 2-connected outerplanar graph of order k. Note that $|E_C(H)| = |V(H)|$. Consider $H \# E_C(H)$ and take S to be the set of verices not in H.

3 Order, size and number of non-trivial components

In this section, we derive tight upper bounds for the order, size and number of non-trivial components of $G - S$ in terms of $|S|$ where S is a vertex cover of $E_C(G)$, and characterize all the extremal graphs. The first two theorems investigate the situation when S has order three or four, and are needed for induction arguments later.

Theorem 3.1. *Let G be any 2-connected graph nonisomorphic to K_3. Suppose S is a cover of $E_C(G)$ of order three. Then $|VN(G,S)| \leq 3$, $|EN(G,S)| \leq 3$ and $|CN(G,S)| \leq 1$.*

Proof. Let $S := \{x, y, z\}$. If $G - S$ is independent, then $|VN(G,S)| = |EN(G,S)| = |CN(G,S)| = 0$. Suppose $G - S$ contains an edge uv. Obviously, uv is non-contractible. By Lemma 2.4(a), $G - u - v$ contains exactly two or three components. Suppose $G - u - v$ consists of three components. By Lemma 2.4(b), the components are precisely x, y and z, and $G[u, v]$ is the only non-trivial component of $G - S$. Otherwise, let C and D be the two components of $G - u - v$. Without loss of generality, by Lemma 2.4(a) and (b), assume $C = z$ and $x, y \in D$. Then uz and vz are contractible edges. By Lemma 2.4(c), we can assume ux and vy are contractible edges. Denote $T := S \cup \{u, v\}$. Note that $G[T]$ is connected. Suppose $G - T$ contains an edge e. Obviously, e is non-contractible. By Lemma 2.3, there exists a contractible edge not covered by S, a contradiction. Therefore, $E(G - T) = \emptyset$.

Suppose $V(G) = T$. Then xy is an edge and $G[u, v]$ is the only non-trivial component of $G - S$. Now, let $V(G) - T := \{a_1, a_2, \ldots, a_k\}$ where $k \geq 1$. Then the neighbors of a_i belong to $\{u, v, x, y\}$. Since $a_i u$ and $a_i v$, if exist, are non-contractible edges, $a_i x$ and $a_i y$ are contractible edges in G. Suppose $k \geq 2$. Since $G - a_i - u$ and $G - a_i - v$ are connected, none of $a_i u$ and $a_i v$ exist, and $G[u, v]$ is the only non-trivial component of $G - S$. Suppose $k = 1$. If both $a_1 u$ and $a_1 v$ are absent, then $G[u, v]$ is the only non-trivial component of $G - S$. Otherwise, $G[u, v, a_1]$ is the only non-trivial component of $G - S$ and $|VN(G,S)| = 3$. Now, $|EN(G,S)| = 3$ if and only if both $a_1 u$ and $a_1 v$ are present. □

Theorem 3.2. *Let G be any 2-connected graph nonisomorphic to K_3. Suppose S is a cover of $E_C(G)$ of order four. Then $|VN(G,S)| \leq 4$, $|EN(G,S)| \leq 5$ and $|CN(G,S)| \leq 2$.*

Proof. Let $S := \{w, x, y, z\}$. If $G - S$ is independent, then $|VN(G,S)| = |EN(G,S)| = |CN(G,S)| = 0$. Suppose $G - S$ contains an edge uv. Obviously, uv is non-contractible. By Lemma 2.4(a), $G - u - v$ contains exactly two, three or four components.

Suppose $G - u - v$ consists of four components. By Lemma 2.4, each component is precisely one vertex of S. We have $|VN(G,S)| = 2$, $|EN(G,S)| = 1$ and $|CN(G,S)| = 1$.

Suppose $G - u - v$ consists of three components. Then by Lemma 2.4, two components consist of one vertex of S while the third contains two vertices of S. By arguing as in the proof of Theorem 3.1, we have $|VN(G,S)| \leq 3$, $|EN(G,S)| \leq 3$ and $|CN(G,S)| = 1$.

Suppose $G-u-v$ consists of two components, namely C and D. If $|C\cap S| = 2$ and $|D \cap S| = 2$, by arguing as in the proof of Theorem 3.1, we have $|VN(G,S)| \leq 4$, $|EN(G,S)| \leq 5$ and $|CN(G,S)| = 1$. By Lemma 2.4(c), without loss of generality, suppose uw, vx, uy and vz are contractible edges where $w, x \in C$ and $y, z \in D$. If $|VN(G,S)| = 4$ or $|EN(G,S)| = 5$, then both C and D contain exactly three vertices. Let c be the vertex of C other than w and x, and d be the vertex of D other than y and z. Note that cw, cx, dy, dz are contractible edges in G. Now, $|VN(G,S)| = 4$ if and only if $N_G(c) \cap \{u,v\} \neq \emptyset$ and $N_G(d) \cap \{u,v\} \neq \emptyset$. Whereas $|EN(G,S)| = 5$ if and only if c is adjacent to both u and v, and d is adjacent to both u and v.

Suppose $|C\cap S| = 1$ and $|D\cap S| = 3$. By Lemma 2.4(b), $|C| = 1$ and let $C := w$. Also, from now on, we may assume that:

($*$) For each non-contractible edge $u'v'$ in $G-S$, $G-u'-v'$ consists of exactly two components, one of which is comprised of a single vertex from $\{w,x,y,z\}$.

By Lemma 2.4(c),there exist two independent contractible edges in $E_G(\{u,v\}, D)$, say ux and vy. Let $T := \{u,v,w,x,y\}$. Note that $G[T]$ is connected and $z \in G-T$. Let $V(G) - T := \{a_1, a_2, \ldots, a_m\}$ where $a_1 = z$. If $m = 1$, then $|VN(G,S)| = 2$, $|EN(G,S)| = 1$ and $|CN(G,S)| = 1$. Therefore, assume $m \geq 2$. Since every vertex is incident to at least two contractible edges, $N_G(a_i) \cap \{x,y\} \neq \emptyset$ for all $1 < i \leq m$. Suppose $G-T$ is independent. Every vertex a_i other than z is adjacent to both x and y, and a_ix and a_iy are contractible edges in G. Since D is connected, $N_G(z) \cap \{x,y\} \neq \emptyset$. If $m = 2$, then by ($*$), $|VN(G,S)| \leq 3$, $|EN(G,S)| \leq 2$ and $|CN(G,S)| = 1$. If $m > 2$, then both a_iu and a_iv are absent for all $i > 1$ as $G-a_i-u$ and $G-a_i-v$ are connected. We have $|VN(G,S)| = 2$, $|EN(G,S)| = 1$ and $|CN(G,S)| = 1$.

Now, assume that $G-T$ is not independent. Suppose $G-T$ contains a non-contractible edge ab. By Lemma 2.3, $z \cap \{a,b\} = \emptyset$. By ($*$), $G-a-b$ consists of exactly two components, one of which is z. Without loss of generality, by Lemma 2.4(c), assume ax and by are contractible edges. Note that by Lemma 2.3, every non-contractible edge of G lies in $G[u,v,x,y,a,b]$. Consequently, every vertex in $H := G-S-u-v-a-b$, if exists, is adjacent to x and y only. By ($*$), ub and va are absent. Therefore, $|VN(G,S)| = 4$ and $|EN(G,S)| \leq 4$. We also have $|CN(G,S)| \leq 2$ with equality holds if and only if ua and vb are both absent.

Suppose all edges in $G-T$ are contractible, and hence incident to z. In particular, $a_2, \ldots, a_m$ are independent in G. Let $a_2, \ldots, a_l$ be all the neighbors of z in $V(G)-T$. Note that $l \geq 2$ since $G-T$ is not independent. Suppose there exists a vertex in $G-T-z$ that is not adjacent to z (i.e. $l < m$). For every $l+1 \leq i \leq m$, a_ix and a_iy are contractible edges. Consider the case $l+1 < m$. By ($*$), a_iu and a_iv are absent for all $l+1 \leq i \leq m$. If none of a_iu and a_iv exist for all $1 < i \leq l$, then $|VN(G,S)| = 2$, $|EN(G,S)| = 1$ and $|CN(G,S)| = 1$. Suppose a_2u exists. By ($*$), $N_G(z) = \{a_2, u\}$

and $l = 2$. Also, a_2v is absent by $(*)$. We have $|VN(G,S)| = 3$, $|EN(G,S)| = 2$ and $|CN(G,S)| = 1$. Consider the case $l+1 = m$. If none of a_iu and a_iv exist for all $1 < i \leq l$, then by the connectedness of D and $(*)$, $|VN(G,S)| \leq 3$, $|EN(G,S)| \leq 2$ and $|CN(G,S)| = 1$. Suppose a_2u exists. By $(*)$, $N_G(z) = \{a_2, u\}$ and $l = 2$. Now, a_2v is absent by $(*)$. Since D is connected, $N_G(a_2) \cap \{x, y\} \neq \emptyset$. Without loss of generality, assume a_2x exists. As ux is contractible, $G - u - x$ is connected and hence, a_2y exists. By $(*)$, a_3u and a_3v are both absent. Hence, $|VN(G,S)| \leq 3$, $|EN(G,S)| \leq 2$ and $|CN(G,S)| = 1$.

Suppose every vertex in $G-T-z$ is adjacent to z. If none of a_iu and a_iv exist for all $i > 1$, then $|VN(G,S)| = 2$, $|EN(G,S)| = 1$ and $|CN(G,S)| = 1$. Suppose a_2u exists. By $(*)$, either $N_G(x) = \{a_2, u\}$ or $N_G(z) = \{a_2, u\}$. If $N_G(z) = \{a_2, u\}$, then $m = 2$, $|VN(G,S)| = 3$, $|EN(G,S)| \leq 3$ and $|CN(G,S)| = 1$. Suppose $N_G(x) = \{a_2, u\}$. If $m = 2$, then $|VN(G,S)| = 3$, $|EN(G,S)| \leq 3$ and $|CN(G,S)| = 1$. Assume $m \geq 3$. Then a_iy is a contractible edge for all $i \geq 3$. By $(*)$, a_iu is absent for all $i \geq 3$ and a_2v is absent as well. Therefore, $\{y, z\} \subseteq N_G(a_i) \subseteq \{v, y, z\}$ for all $i \geq 3$. If a_iv exists for some $i \geq 3$, then by $(*)$, $N_G(y) = \{a_i, v\}$ and $m = 3$. We have $|VN(G,S)| = 4$, $|EN(G,S)| \leq 3$ and $|CN(G,S)| = 1$. Otherwise, a_iv are absent for all $i \geq 3$, and $|VN(G,S)| = 3$, $|EN(G,S)| \leq 2$ and $|CN(G,S)| = 1$. □

Now, we are ready for the main results concerning the order, size and number of non-trivial components of $G - S$.

Theorem 3.3. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Then $|VN(G,S)| \leq 2|S| - 4$ for $|S| \geq 4$.*

Proof. The statement is true for $|S| = 4$ by Theorem 3.2. All the extremal graphs together with their corresponding S are given in the proof of Theorem 3.2. Suppose the theorem holds for all $|S| < k$ where $k \geq 5$. Consider a 2-connected graph G and a cover S of $E_C(G)$ such that $|S| = k$. If $G - S$ is independent, then $|VN(G,S)| = 0$ and the theorem is trivially true. Let xy be any edge in $G - S$. Consider any fragment F of $\{x, y\}$. Define $F_1 := F$ and $F_2 := \overline{F}$. By Lemma 2.4, $|F_i \cap S| \geq 1$ for $i = 1, 2$. Note that $|F_1 \cap S| + |F_2 \cap S| = k \geq 5$. For each F_i, define $G_i := (V(F_i) \cup \{x, y, x_i\}, E(G[F_i \cup xy]) \cup \{x_ix, x_iy\})$ and $S_i := x_i \cup (S \cap F_i)$. By Lemma 2.5, G_i is 2-connected and S_i covers $E_C(G_i)$.

(I) Suppose $|F_1 \cap S| \geq 3$ and $|F_2 \cap S| \geq 3$. Then $|VN(G_1, S_1)| \leq 2|S_1| - 4$ and $|VN(G_2, S_2)| \leq 2|S_2| - 4$. By Lemma 2.5, we have $|VN(G,S)| = |VN(G_1, S_1)| + |VN(G_2, S_2)| - 2 \leq 2|S_1| - 4 + 2|S_2| - 4 - 2 = 2(|S_1| + |S_2| - 2) - 6 = 2|S| - 6 < 2|S| - 4$.

(II) Suppose $|F_1 \cap S| = 2$ and $|F_2 \cap S| \geq 3$. Then $|VN(G_1, S_1)| \leq 3$ by Theorem 3.1 and $|VN(G_2, S_2)| \leq 2|S_2| - 4$. By Lemma 2.5, we have $|VN(G,S)| = |VN(G_1, S_1)| + |VN(G_2, S_2)| - 2 \leq 3 + 2|S_2| - 4 - 2 = 2(3 + |S_2| - 2) - 5 = 2|S| - 5 < 2|S| - 4$.

Suppose $G - x - y$ has at least four components. By choosing F to be the union of any two components or its complement, we have either (I) or (II). Suppose $G - x - y$ has exactly three components. If there exists a component C such that $|C \cap S| \geq 3$, then by choosing F to be the union of the two components other than C, we have either (I) or (II). Assume for each component C, $|C \cap S| \leq 2$. Then there are at least two components such that the equality holds. By taking F to be one such component, we have (II). Suppose $G-x-y$ has exactly two components C and D. If $|C\cap S|\geq 2$ and $|D\cap S|\geq 2$, then by choosing F to be C or D, we have either (I) or (II).

From now on, we can assume that for every edge e in $G-S$, $G-V(e)$ has exactly two components, one of which consists of exactly one vertex denoted by x_e. Note that $x_e\in S$ and $x_e\neq x_f$ for any two distinct edges in $G-S$. Therefore, $|S|\geq |E(G-S)|=|EN(G,S)|$ and $|VN(G,S)|\leq 2|EN(G,S)|$.

Suppose $|CN(G,S)|=1$. Let B be the non-trivial component of $G-S$. If B is a tree, then $|VN(G,S)|=|EN(G,S)|+1$. Since $G[B\cup\bigcup_{e\in E(B)}x_e]$ is connected but not 2-connected, $|S|\geq|EN(G,S)|+1$. We have $|VN(G,S)|\leq|S|<2|S|-4$. If B is not a tree, then $|VN(G,S)|\leq|EN(G,S)|$. We have $|VN(G,S)|\leq|S|<2|S|-4$.

Suppose $|CN(G,S)|>1$. Let B be the union of all non-trivial components of $G-S$. Obviously, B is not connected and so is $G[B\cup\bigcup_{e\in E(B)}x_e]$. Since G is 2-connected, we need at least two vertices of S outside $\bigcup_{e\in E(B)}x_e$ to connect B together. Therefore, $|S|\geq|EN(G,S)|+2$. We have $|VN(G,S)|\leq 2|EN(G,S)|\leq 2|S|-4$. Equality holds if and only if all edges in $G-S$ are independent and $|S\setminus\bigcup_{e\in E(B)}x_e|=2$. Equivalently, $V(G):=\{x,y\}\cup\bigcup_{i=1}^{k-2}\{x_i,y_i,z_i\}\cup\bigcup_{j=1}^{l}\{a_j\}$, $E(G):=\bigcup_{i=1}^{k-2}\{z_ix_i,z_iy_i,x_iy_i,x_ix,y_iy\}\cup\bigcup_{j=1}^{l}\{a_jx,a_jy\}\cup F$ where $F\subseteq xy\cup\bigcup_{i=1}^{k-2}\{x_iy,y_ix\}$, and $S:=\{x,y\}\cup\bigcup_{i=1}^{k-2}\{z_i\}$. □

Theorem 3.4. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Then $|EN(G,S)|\leq 2|S|-3$ for $|S|\geq 2$. Equality holds if and only if $G=K_2\#E(K_2)$ for $|S|=2$, $G=K_3\#E(K_3)$ for $|S|=3$, and $G=H\#E_C(H)$ for $|S|\geq 4$ where H is any 2-connected maximally outerplanar graph of order $|S|$ with S being the set of all degree two vertices.*

Proof. The statement is true for $|S|=2$ and $|S|=3$ by Theorem 2.2 and Theorem 3.1. For $|S|=2$, the extremal graph is $K_2\#E(K_2)$ with S being the set of all degree two vertices. For $|S|=3$, the extremal graph is $K_3\#E(K_3)$ with S being the set of all degree two vertices. Suppose the theorem holds for all $|S|<k$ where $k\geq 4$. Consider a 2-connected graph G and a cover S of $E_C(G)$ such that $|S|=k$. If $G-S$ is independent, then $|EN(G,S)|=0$ and the theorem is trivially true. Let xy be any edge in $G-S$. Consider any fragment F of $\{x,y\}$. Define $F_1:=F$ and $F_2:=\overline{F}$. By Lemma 2.4, $|F_i\cap S|\geq 1$ for $i=1,2$. Note that $|F_1\cap S|+|F_2\cap S|=k\geq 4$. For each F_i, define $G_i:=(V(F_i)\cup\{x,y,x_i\},E(G[F_i\cup xy])\cup\{x_ix,x_iy\})$ and $S_i:=x_i\cup(S\cap F_i)$. By Lemma 2.5, G_i is 2-connected and S_i covers $E_C(G_i)$.

For induction to proceed, we are interested in the condition (I) $|F_1\cap S|\geq 2$ and $|F_2\cap S|\geq 2$. Suppose $G-x-y$ has at least four components. By choosing F to be the union of any two components, (I) holds. Suppose $G-x-y$ has exactly three components. Then there exists a component C such that $|C\cap S|\geq 2$. By choosing F to be C, (I) holds. Suppose $G-x-y$ has exactly two components C and D. If $|C\cap S|\geq 2$ and $|D\cap S|\geq 2$, then by choosing F to be C, (I) holds.

Now, suppose for every edge e in $G-S$, $G-V(e)$ has exactly two components, one of which consists of exactly one vertex denoted by x_e. Note that $x_e \in S$ and $x_e \neq x_f$ for any two distinct edges in $G-S$. Therefore, $|EN(G,S)| \leq |S| < 2|S|-3$.

Finally, if $|F_1 \cap S| \geq 2$ and $|F_2 \cap S| \geq 2$, then $|EN(G_1,S_1)| \leq 2|S_1|-3$ and $|EN(G_2,S_2)| \leq 2|S_2|-3$. By Lemma 2.5, we have $|EN(G,S)| = |EN(G_1,S_1)| + |EN(G_2,S_2)| - 1 \leq 2|S_1| - 3 + 2|S_2| - 3 - 1 = 2(|S_1| + |S_2| - 2) - 3 = 2|S| - 3$. Equality holds if and only if for $i = 1,2$, $G_i = H_i \# E_C(H_i)$ where H_i is any 2-connected maximally outerplanar graph of order $|S_i|$ with S_i being the set of all degree two vertices. Equivalently, $G = H \# E_C(H)$ where H is any 2-connected maximally outerplanar graph of order $|S|$ with S being the set of all degree two vertices. □

Theorem 3.5. *Let G be any 2-connected graph nonisomorphic to K_3 and S be a cover of $E_C(G)$. Then $|CN(G,S)| \leq |S|-2$ for $|S| \geq 3$.*

Proof. The statement is true for $|S| = 3$ by Theorem 3.1. All the extremal graphs together with their corresponding S are given in the proof of Theorem 3.1. Suppose the theorem holds for all $|S| < k$ where $k \geq 4$. Consider a 2-connected graph G and a cover S of $E_C(G)$ such that $|S| = k$. If $G - S$ is independent, then $|CN(G,S)| = 0$ and the theorem is trivially true. Let xy be any edge in $G-S$. Consider any fragment F of $\{x,y\}$. Define $F_1 := F$ and $F_2 := \overline{F}$. By Lemma 2.4, $|F_i \cap S| \geq 1$ for $i = 1,2$. Note that $|F_1 \cap S| + |F_2 \cap S| = k \geq 4$. For each F_i, define $G_i := (V(F_i) \cup \{x,y,x_i\}, E(G[F_i \cup xy]) \cup \{x_i x, x_i y\})$ and $S_i := x_i \cup (S \cap F_i)$. By Lemma 2.5, G_i is 2-connected and S_i covers $E_C(G_i)$.

For induction to proceed, we are interested in the condition (I) $|F_1 \cap S| \geq 2$ and $|F_2 \cap S| \geq 2$. Then $|CN(G_i,S_i)| \leq |S_i| - 2$ for $i = 1,2$. By Lemma 2.5, $|CN(G,S)| = |CN(G_1,S_1)| + |CN(G_2,S_2)| - 1 \leq |S_1| - 2 + |S_2| - 2 - 1 = (|S_1| + |S_2| - 2) - 3 = |S| - 3 < |S| - 2$.

Suppose $G-x-y$ has at least four components. By choosing F to be the union of any two components, (I) holds. Suppose $G-x-y$ has exactly three components. Then there exists a component C such that $|C \cap S| \geq 2$. By choosing F to be C, (I) holds. Suppose $G-x-y$ has exactly two components C and D. If $|C \cap S| \geq 2$ and $|D \cap S| \geq 2$, then by choosing F to be C, (I) holds.

From now on, we can assume that for every edge e in $G-S$, $G-V(e)$ has exactly two components, one of which consists of exactly one vertex denoted by x_e. Note that $x_e \in S$ and $x_e \neq x_f$ for any two distinct edges in $G-S$. Therefore, $|CN(G,S)| \leq |EN(G,S)| \leq |S|$. If $|CN(G,S)| = 1$, then obviously, $|CN(G,S)| < |S|-2$. Suppose $|CN(G,S)| > 1$. Let B be the union of all non-trivial components of $G-S$. Obviously, B is not connected and so is $G[B \cup \bigcup_{e \in E(B)} x_e]$. Since G is 2-connected, we need at least two vertices of S outside $\bigcup_{e \in E(B)} x_e$ to connect B together. Therefore, $|S| \geq |EN(G,S)| + 2$. We have $|CN(G,S)| \leq |EN(G,S)| \leq |S| - 2$. Equality holds if and only if all edges in $G-S$ are independent and $|S \setminus \bigcup_{e \in E(B)} x_e| = 2$. Equivalently, $V(G) := \{x,y\} \cup \bigcup_{i=1}^{k-2} \{x_i, y_i, z_i\} \cup \bigcup_{j=1}^{l} \{a_j\}$, $E(G) := \bigcup_{i=1}^{k-2} \{z_i x_i, z_i y_i, x_i y_i, x_i x, y_i y\} \cup \bigcup_{j=1}^{l} \{a_j x, a_j y\} \cup F$ where $F \subseteq xy \cup \bigcup_{i=1}^{k-2} \{x_i y, y_i x\}$, and $S := \{x,y\} \cup \bigcup_{i=1}^{k-2} \{z_i\}$. □

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