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Completing the design spectra for graphs with six vertices and eight edges

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Abstract

Apart from two possible exceptions, the design spectrum has been determined for every graph with six vertices and at most eight edges. The purpose of this note is to establish the existence of the two missing designs, both of order 32.

1 Introduction

Let G be a simple graph. If the edge set of a simple graph K can be partitioned into edge sets of graphs each isomorphic to G, we say that there exists a *decomposition* of K into G. In the case where K is the complete graph K_n we refer to the decomposition as a *G-design* of order n. The *design spectrum* of G is the set of non-negative integers n for which there exists a G-design of order n.

The design spectrum problem for small graphs has attracted attention. In particular, it has been solved for (i) all graphs with at most five vertices, (ii) all graphs with six vertices and at most seven edges, (iii) all graphs with six vertices and eight edges, with two possible exceptions, and (iv) eleven of the 21 graphs with six vertices and nine edges. See [1] and [2] for details and references. More recently, the spectrum problem was resolved for all of the remaining ten graphs with six vertices and nine edges, [4].

Figure 1: Graphs M_1 and M_2

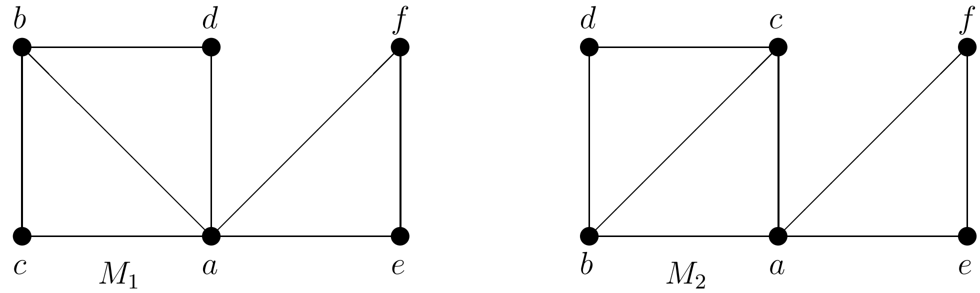
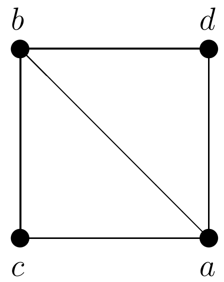


Figure 2: Graph Q



2 Design construction

The two possible exceptions for graphs with six vertices and eight edges are illustrated in Figure 1, and in each case it is order 32 that remains unresolved; see [5], where the graphs are denoted by M_1 and M_2. To complete the task of determining the design spectra for 6-vertex, 8-edge graphs it suffices merely to exhibit designs of order 32 for M_1 and M_2. However, it might be of some interest to explain how they were obtained; so we give brief details.

The number of graphs in each design is 62 and without a vertex of degree 1 there is no convenient automorphism that we can exploit. So we are left in each case with the task of assembling sixty-two 6-tuples of numbers from the set $N = \{0, 1, \ldots, 31\}$.

Let Q denote the graph obtained by removing an edge from a K_4 (Figure 2). Our primary objective is to obtain a decomposition $\mathcal{D}$ of K_{32} labelled with N into 62 triangles and 62 copies of Q, also labelled from N.

A *partial Steiner triple system* PSTS(x) consists of a *point set*, X say, of cardinality x and a collection of *triples*, 3-element subsets of X, such that each unordered pair of elements from X occurs in at most one triple. We create by hill climbing, [6] (or see [3, §2.7.2]), a random PSTS(32), $\mathcal{S}$, with point set N and 62 triples. The triples of $\mathcal{S}$ are the triangles of $\mathcal{D}$.

Next, we create the T-K matrix $\mathbf{M}$ for assembling the graphs Q. Denote by T the set of edges of the labelled K_{32} that are not present as pairs in the triples of $\mathcal{S}$.

To each ordered 4-tuple $k = (a, b, c, d)$, $a, b, c, d \in N$, we associate a graph $Q_k \cong Q$ with its vertices labelled by the elements of k as shown in Figure 2. Let K be the set of ordered 4-tuples k corresponding to distinct labelled Q_k such that all the edges of Q_k belong to T. The rows of $\mathbf{M}$ are indexed by elements of T and the columns by elements of K. The matrix is defined by $\mathbf{M}_{t,k} = 1$ if Q_k contains edge t; otherwise $\mathbf{M}_{t,k} = 0$.

We attempt to solve $\mathbf{Mv} = \mathbf{1}$ for $\mathbf{v}$ a $\{0, 1\}$ vector. If we are unsuccessful, we try another PSTS(32). Otherwise we recover the 4-tuples corresponding to 1s in $\mathbf{v}$ and hence the graphs Q of our decomposition $\mathcal{D}$.

Finally, for $\delta = 2$, 3 we pair off triangles and Qs in $\mathcal{D}$ such that in each pair there is a common label attached to the triangle and a vertex of degree δ in the Q. Thus we have the required decompositions of K_{32} into M_1 when $\delta = 3$ and M_2 when $\delta = 2$. The results are presented in Section 3. Incidentally, by pairing triangles and Qs with no common vertices we obtain a $Q \cup K_3$ design of order 32.

3 The designs

We represent the graphs by ordered 6-tuples $(a, b, c, d, e, f)_{M1}$ and $(a, b, c, d, e, f)_{M2}$, where the letters correspond to vertices as illustrated in Figure 1. With vertex set Z_{32} the decompositions consist of 62 graphs each:

$(8, 9, 3, 16, 23, 0)_{M1}$, $(5, 8, 6, 27, 25, 1)_{M1}$, $(20, 7, 0, 27, 1, 16)_{M1}$,
$(3, 19, 5, 20, 6, 0)_{M1}$, $(14, 20, 5, 22, 1, 4)_{M1}$, $(20, 21, 18, 31, 24, 2)_{M1}$,
$(8, 11, 14, 25, 19, 1)_{M1}$, $(26, 17, 8, 19, 27, 13)_{M1}$, $(30, 18, 8, 19, 0, 27)_{M1}$,
$(15, 11, 2, 27, 29, 0)_{M1}$, $(14, 13, 0, 15, 27, 3)_{M1}$, $(10, 14, 2, 19, 18, 4)_{M1}$,
$(14, 7, 16, 30, 26, 28)_{M1}$, $(9, 18, 14, 26, 22, 0)_{M1}$, $(23, 14, 17, 31, 3, 18)_{M1}$,
$(25, 21, 0, 14, 4, 6)_{M1}$, $(0, 11, 19, 31, 1, 24)_{M1}$, $(12, 10, 0, 16, 20, 26)_{M1}$,
$(26, 2, 0, 21, 1, 15)_{M1}$, $(5, 18, 0, 12, 16, 2)_{M1}$, $(19, 15, 6, 21, 27, 2)_{M1}$,
$(16, 13, 4, 19, 26, 3)_{M1}$, $(22, 16, 15, 31, 27, 1)_{M1}$, $(23, 16, 11, 28, 4, 20)_{M1}$,
$(24, 21, 3, 16, 4, 8)_{M1}$, $(5, 23, 15, 26, 10, 30)_{M1}$, $(28, 15, 3, 18, 0, 4)_{M1}$,
$(12, 24, 15, 27, 14, 6)_{M1}$, $(4, 21, 5, 27, 15, 17)_{M1}$, $(31, 27, 10, 25, 15, 30)_{M1}$,
$(17, 18, 2, 27, 20, 10)_{M1}$, $(12, 4, 9, 31, 21, 7)_{M1}$, $(31, 7, 3, 26, 2, 9)_{M1}$,
$(6, 18, 7, 31, 9, 1)_{M1}$, $(22, 18, 13, 24, 5, 11)_{M1}$, $(13, 31, 5, 28, 20, 8)_{M1}$,
$(6, 2, 13, 28, 16, 30)_{M1}$, $(18, 1, 11, 29, 25, 16)_{M1}$, $(6, 24, 10, 26, 20, 29)_{M1}$,
$(22, 25, 7, 26, 8, 10)_{M1}$, $(26, 4, 11, 30, 29, 10)_{M1}$, $(19, 25, 12, 28, 23, 9)_{M1}$,
$(29, 22, 4, 19, 31, 8)_{M1}$, $(2, 3, 4, 29, 8, 12)_{M1}$, $(23, 25, 2, 29, 27, 6)_{M1}$,
$(21, 22, 6, 23, 28, 8)_{M1}$, $(30, 22, 2, 28, 12, 23)_{M1}$, $(12, 22, 3, 17, 13, 11)_{M1}$,
$(28, 12, 1, 29, 9, 27)_{M1}$, $(7, 1, 2, 13, 8, 15)_{M1}$, $(10, 23, 1, 13, 15, 25)_{M1}$,
$(30, 1, 3, 21, 13, 24)_{M1}$, $(11, 10, 3, 21, 20, 28)_{M1}$, $(9, 17, 5, 21, 15, 20)_{M1}$,
$(29, 7, 5, 17, 13, 21)_{M1}$, $(24, 28, 5, 17, 29, 14)_{M1}$, $(7, 10, 9, 28, 19, 4)_{M1}$,
$(24, 7, 11, 23, 31, 19)_{M1}$, $(17, 11, 6, 30, 0, 16)_{M1}$, $(25, 9, 13, 24, 30, 20)_{M1}$,
$(17, 3, 13, 25, 31, 1)_{M1}$, $(29, 9, 11, 30, 16, 27)_{M1}$,

and

$(16,8,9,3,18,25)_{M2}$, $(27,5,8,6,30,0)_{M2}$, $(0,7,20,27,1,24)_{M2}$,
$(5,3,19,20,25,1)_{M2}$, $(22,14,20,5,0,9)_{M2}$, $(31,20,21,18,19,24)_{M2}$,
$(14,8,11,25,1,4)_{M2}$, $(8,17,26,19,23,0)_{M2}$, $(8,18,30,19,12,2)_{M2}$,
$(27,11,15,2,1,22)_{M2}$, $(0,13,14,15,4,28)_{M2}$, $(19,10,14,2,1,8)_{M2}$,
$(16,7,14,30,17,0)_{M2}$, $(26,9,18,14,1,15)_{M2}$, $(31,14,23,17,2,9)_{M2}$,
$(0,21,25,14,15,29)_{M2}$, $(19,0,11,31,27,2)_{M2}$, $(16,10,12,0,20,1)_{M2}$,
$(21,2,26,0,7,12)_{M2}$, $(12,5,18,0,14,6)_{M2}$, $(6,15,19,21,9,1)_{M2}$,
$(4,13,16,19,6,25)_{M2}$, $(15,16,22,31,17,4)_{M2}$, $(28,16,23,11,14,26)_{M2}$,
$(16,21,24,3,27,29)_{M2}$, $(26,5,23,15,12,20)_{M2}$, $(3,15,28,18,14,27)_{M2}$,
$(27,12,24,15,6,23)_{M2}$, $(5,4,21,27,10,30)_{M2}$, $(10,27,31,25,18,4)_{M2}$,
$(2,17,18,27,20,24)_{M2}$, $(31,4,12,9,8,29)_{M2}$, $(3,7,31,26,18,23)_{M2}$,
$(7,6,18,31,19,4)_{M2}$, $(24,18,22,13,4,8)_{M2}$, $(28,13,31,5,8,21)_{M2}$,
$(13,2,6,28,20,8)_{M2}$, $(29,1,18,11,6,20)_{M2}$, $(10,6,24,26,22,8)_{M2}$,
$(7,22,25,26,8,15)_{M2}$, $(30,4,26,11,6,16)_{M2}$, $(28,19,25,12,9,27)_{M2}$,
$(4,22,29,19,20,23)_{M2}$, $(29,2,3,4,10,26)_{M2}$, $(29,23,25,2,13,21)_{M2}$,
$(23,21,22,6,9,19)_{M2}$, $(28,22,30,2,11,20)_{M2}$, $(17,12,22,3,20,10)_{M2}$,
$(29,12,28,1,14,24)_{M2}$, $(13,1,7,2,11,12)_{M2}$, $(13,10,23,1,24,30)_{M2}$,
$(3,1,30,21,6,0)_{M2}$, $(3,10,11,21,16,26)_{M2}$, $(5,9,17,21,16,2)_{M2}$,
$(17,7,29,5,31,1)_{M2}$, $(5,24,28,17,11,22)_{M2}$, $(9,7,10,28,15,20)_{M2}$,
$(23,7,24,11,30,12)_{M2}$, $(30,11,17,6,31,15)_{M2}$, $(13,9,25,24,26,27)_{M2}$,
$(25,3,17,13,10,15)_{M2}$, $(30,9,29,11,20,25)_{M2}$.

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