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Oriented threshold graphs

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Abstract

Threshold graphs are a prevalent and widely studied class of simple graphs. We generalize this class of graphs to oriented graphs (directed simple graphs). We give generalizations to four of the most commonly used definitions and show their equivalence in the oriented case. We then enumerate the number of these oriented threshold graphs which relates to the Fibonacci numbers, and finish by finding the number of transitive orientations of threshold graphs.

1 Introduction

1.1 History

Threshold graphs were first seen in several publications in the mid 1970s. Papers in a variety of areas independently developed basic definitions for a class of graphs which gets its name from a 1973 paper titled *Set-packing Problems and Threshold Graphs* by Chvátal and Hammer [1]. These graphs have been found in numerous applications since their introduction; they cover a wide range of subjects including applications in set-packing, parallel processing, resource allocation, scheduling, and psychology. There is a great introduction to threshold graphs and their applications in the book *Threshold Graphs and Related Topics* by Mahadev and Peled, [5].

In recent years, the limit points of threshold graphs (as graphons) have been studied in a paper by Diaconis, Holmes, and Jansen [3]. This gives an interesting result that their limits can be realized as $\{0, 1\}$-valued increasing functions on the unit square.

Another recent result from Cloteaux et al. [2] gives a generalization to directed graphs focusing on degree sequences and unique realizations. This work is extended by Reilly, Scheinerman, and Zhang [6]. These extensions generalize definitions of simple threshold graphs into directed graphs and demonstrate their equivalence with the definitions of Cloteaux et al. These definitions deal predominantly with directed graphs in which 2-cycles (multi-edges in the underlying graph) are permitted in order to obtain unique realizability.

In this paper we will look at oriented simple graphs where we prohibit such 2-cycles and see some surprisingly lovely results.

1.2 Background

Mahadev and Peled in [5] give a thorough treatment of the class of threshold graphs. Here we give the basic definition and some equivalences.

Let G be a graph. We say that G is a *threshold graph* if there exists a threshold $t \in \mathbb{R}$ and an injective vertex weight function $w : V(G) \to \mathbb{R}$ such that $e = (x, y) \in E$ if and only if $w(x) + w(y) > t$.

Although this is a fairly simple definition to work with, there are several equivalences that will be worth considering. To understand them we need few definitions:

A subset of vertices of a graph is a *clique* if all possible edges between vertices in the subset are included in the graph. A graph, $G = (V, E)$, is said to be *split* if the vertex set V can be partitioned into two classes K and I such that K is a clique in G, and I is an independent set in G. The neighborhoods of a graph are said to be *n*ested if any two are comparable as subsets. If H is a graph, a graph G is said to be *H-free* if G contains no induced subgraphs isomorphic to H. A vertex is *dominating* if it is adjacent to all other vertices. A vertex is *isolated* if it has degree 0. A graph is *oriented* if each edge is assigned a direction. The *head* of an oriented edge is the vertex to which the edge points, whereas the *tail* is the vertex from which the edge begins. Notationally, we say $\overrightarrow{xy}$ when the edge xy is oriented with x as the tail and y as the head. In an oriented graph (V, E), the *out-neighborhood* of a vertex v is the set $\{w \in V \mid \overrightarrow{vw} \in E\}$; the *in-neighborhood* of a vertex v is the set $\{u \in V \mid \overrightarrow{uv} \in E\}$; the *in- (out-)degree* of a vertex is the cardinality of its in- (respectively, out-) neighborhood. The notation K_n represents the complete graph on n vertices, P_n denotes the path on n vertices, and C_n denotes the cycle on n vertices. Also $\vec{P}_n$ and $\vec{C}_n$ denote the directed path and directed cycle.

We can now state four characterizations of threshold graphs.

Theorem 1. *[5] The following are equivalent:*

(i) G is a threshold graph.

(ii) G is a split graph and the vertex neighborhoods are nested.

(iii) G is $\{2K_2, C_4, P_4\}$-free.

(iv) The graph G can be constructed by starting with a single vertex and sequentially adding either a dominating vertex or an isolated vertex at each step.

Equivalence (iv) of Theorem 1 allows a very nice constructive bijection between binary sequences of length $n - 1$ and threshold graphs on n vertices. We define a threshold graph by creating a binary sequence $\bar{s} \in \{0,1\}^n$ by setting $s_1 = \star$ (this does not matter since the first vertex is both isolated and dominating), and for $i = 2, 3, \ldots, n$ set $s_i = 1$ if the vertex added is dominating or $s_i = 0$ if it is isolated. Given such a sequence we define $T(\bar{s})$ to be the threshold graph associated with it. In this construction the very first vertex is both isolated and dominating; therefore our classifying it as a 0 or 1 is somewhat misleading. We will always classify the first vertex as $\star$ when giving a threshold graph in its sequential form. We use the convention that the sequence is constructed right to left, thinking of the first vertices added as least significant, as in least significant digits in a number.

2 Oriented Threshold Graphs

There are several definitions for a threshold graph in the undirected case, Theorem 1. We begin by developing an analogous vocabulary for oriented graphs and then state a theorem presenting several equivalent definitions of an oriented threshold graph.

Definition 2. An oriented graph, $G = (V, E)$, is said to be *threshold* if there exists an injective weight function on the vertices $w : V \to \mathbb{R}$ and a threshold value $t \in \mathbb{R}$ such that $\overrightarrow{xy} \in E$ if and only if $|w(x)| + |w(y)| \geq t$ and $w(x) > w(y)$.

Although in the unoriented graph case the weight function need not be injective [5], we choose an injective weight function in the oriented case so that we can think of this as a threshold graph with edges running 'downhill' and not worry about which direction to orient an edge if the weights are equal.

2.1 Background and Oriented Threshold Equivalence

Before we state our main theorem which is directly analogous to Theorem 1, we need to develop vocabulary to state corresponding statements in the oriented case.

The first generalization we will explore is split with nested neighborhoods. These next definitions will help us generalize the concepts of split and nested neighborhoods to oriented graphs where we have not just a total neighborhood, but have in and out neighborhoods as well.

Definition 3. An oriented graph is said to be *oriented split* if the vertex set can be partitioned into three classes, $V = B \cup I \cup T$ (Bottom, Independent, and Top), with the properties:

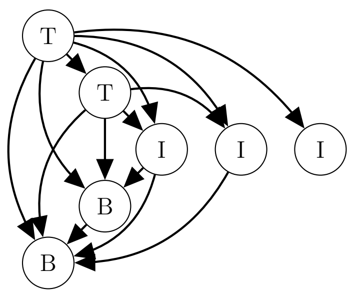
i) I is an independent set;

ii) the graph induced by $B \cup T$ is a tournament;

iii) all edges between T and $B \cup I$ are directed from T;

iv) all edges between B and $T \cup I$ are directed into B.

Figure 2.1: A small oriented split graph



A small example of an oriented split graph may be helpful in understanding this definition; see Figure 2.1

LaMar, prior to working on the unigraphic sequence problem for digraphs, gave a definition for split digraphs in [4]; the definition we just gave fits within his, but is more restrictive. The definition we give only allows the tournament to be split into two parts, a top and a bottom. LaMar's definition allows a third part, in some sense a middle with fewer restrictions on the edges in and out of it. With this stronger definition, we end up with a smaller class of graphs, but we are able to say much more about the structure of our class, both by giving a recursive construction from a ternary sequence, and by defining what it means to have nested neighborhoods in the sense of oriented threshold graphs.

A vertex is a *source* (*sink*) if it has out- (respectively, in-) degree 0. A vertex, v, in an oriented graph (V, E) is called an *out-dominating* (*in-dominating*) vertex if it is a source (respectively, sink) and its out-neighborhood (respectively, in-neighborhood) is $V - v$.

Definition 4. Let $\sigma : V \to 2^V$ be a function from a set to its power set. We say the function σ is *nested* on $S \subseteq V$ if for every $x, y \in S$ we have $\sigma(x) \subseteq \sigma(y) \cup \{y\}$ or $\sigma(y) \subseteq \sigma(x) \cup \{x\}$. We denote $\sigma(x) \subseteq \sigma(y) \cup \{y\}$ by $x \trianglelefteq^\sigma y$.

We say the function σ is *s*trictly nested on $S \subset V$, if for every $x, y \in S$ we have either $\sigma(x) \subseteq \sigma(y)$ or $\sigma(y) \subseteq \sigma(x)$. For $x, y \in V$ we denote $\sigma(x) \subsetneq \sigma(y)$ by $x \triangleleft^\sigma y$.

If σ is nested (respectively, strictly nested) on all of V, we say that σ is nested (respectively, strictly nested).

Recall that in Theorem 1 one of the conditions that makes a graph threshold is that it is split and has nested neighborhoods. That is, the neighborhood function of a threshold graph is a nested function. With this in mind, the proper way to view nested and strictly nested functions in terms of oriented threshold graphs is via the following definition.

Definition 5. Let D be an oriented split graph with clique $K = T \cup B$ and independent set I. We say D has *properly nested neighborhoods* if the following hold for $N, N^+, N^- : V \to 2^V$ the neighborhood, out-neighborhood and in-neighborhood functions:

i) N is nested,

ii) N^+ and N^- are nested on I and for $x, y \in I$ we have if $x \trianglelefteq^N y$ then $x \trianglelefteq^- y$ and $x \trianglelefteq^+ y$ (the N's in the last two inequalities are ommitted to make the notation less cumbersome).

iii) N^+ and N^- are strictly nested on K and for $x, y \in K$ we have $x \triangleleft^+ y$ if and only if $y \triangleleft^- x$.

The properly nested neighborhoods condition states that the total neighborhoods are nested. For vertices in I, the size of in and out neighborhoods are directly correlated, whereas in $B \cup T$ the in and out neighborhoods are inversely correlated. Figure 2.1 gives an example of a graph which has properly nested neighborhoods.

Now, we can generalize Theorem 1 to oriented threshold graphs.

Theorem 6. *The following are equivalent for a graph $G = (V, E)$:*

(a) G is an oriented threshold graph.

(b) G is $\left\{\vec{P}_3, \vec{C}_3\right\}$-free and the underlying undirected graph is $\{2K_2, C_4, P_4\}$-free.

(c) G is a transitive orientation of a threshold graph.

(d) G is an oriented split graph and has properly nested neighborhoods.

(e) G can be constructed from the one vertex empty graph by successively adding an independent vertex, an out-dominating vertex or an in-dominated vertex.

Proof. We will prove the implications in the order they are written. We start with (a) implies (b). Let $w : V(G) \to \mathbb{R}$ be the weight function and $t \in \mathbb{R}$ the threshold value for G. To show the underlying graph is threshold, use the weight function $|w(v)|$ and the same threshold t; this gives $\{2K_2, C_4, P_4\}$-free. Now, $x \to y$ and $y \to z$. By definition then, we know that $|w(x)| + |w(y)| \geq t$ and $|w(y)| + |w(z)| \geq t$ and that $w(x) > w(y) > w(z)$. We need to consider two cases to show that $x \to z$.
Case 1) $w(y) \geq 0$: Then $w(x) > 0$ so $|w(x)| \geq |w(y)|$ therefore $|w(x)| + |w(z)| \geq |w(y)| + |w(z)| \geq t$. So $x \to z$.
Case 2) $w(y) \leq 0$: Then $w(z) < 0$ so $|w(z)| \geq |w(y)|$ therefore $|w(x)| + |w(z)| \geq |w(x)| + |w(y)| \geq t$. So again $x \to z$.

This means that there is an edge between x and z so that G is has no $\vec{P}_3$. Also, the edge is not oriented to form a $\vec{C}_3$.

Next, we show (b) implies (c). Let $\overrightarrow{xy}$ and $\overrightarrow{yz}$ be edges of G. Since G is $\vec{P}_3$ free, either $\overrightarrow{xz}$ or $\overrightarrow{zx}$. Since G is $\vec{C}_3$-free, we must have $\vec{xz}$, making G transitive. By the underlying graph being $\{2K_2, C_4, P_4\}$-free, we have that G is a transitive orientation of a threshold graph.

Continuing on, we show (c) implies (d). Let G be the underlying threshold graph associated with the sequence $\bar{s}$ (Theorem 1). The initial vertex in the sequential construction is given the label $\star$. Now, the collection of $\star$ and the 0s form an independent set; call this set of vertices I. The collection of 1s form a clique; call this set K. Set $T = N^-(\star)$ and $B = N^+(\star)$. Since every 1 was adjacent to $\star$, this partitions K. By the transitivity of the ordering, if $t \in T$ and $b \in B$, then $t \to \star$ and $\star \to b$ so $t \to b$. This gives us the partition $T \cup I \cup B$. We have that all edges between T and B are oriented correctly.

We still need to show that edges between K and I are oriented correctly. That is, we need to show the edges are directed from T to I, and the edges are directed to B from I. To do this we first show that N^+ and N^- are nested on I. Since the underlying graph is threshold, the neighborhood function is nested (this is the nested neighborhoods condition of (ii) in Theorem 1). So, let $i, j \in I$ with $i \trianglelefteq^N j$. Suppose there is $x \in N^+(i) \backslash N^+(j)$. Then $x \in N^-(j)$, since $i \trianglelefteq^N j$. But this means $i \to x \to j$ and transitivity gives $i \to j$; however, that is impossible because $i, j \in I$. So we must have $N^+(i) \subseteq N^+(j)$. A similar argument gives $N^-(i) \subseteq N^-(j)$. This shows property $ii)$ of the properly nested condition, Definition 5.

Using this, and noting that for all $i \in I$ we have $i \trianglelefteq^N \star$, we obtain $N^+(i) \subseteq N^+(\star) = B$ and $N^-(i) \subseteq N^-(\star) = T$, showing that the graph is oriented split.

To show the third condition of properly nested neighborhoods, let $x, y \in K$ with $x \to y$. Then by transitivity $N^+(y) \subsetneq N^+(x)$ (the inclusion is strict because $y \notin N^+(y)$) and $N^-(x) \subsetneq N^-(y)$ (again because $x \notin N^-(x)$) which completes all conditions.

The next implication is (d) implies (e). Let $i \in I$ be minimal in I with respect to total neighborhoods. If $N(i) = \emptyset$ then it is an isolate. If not, we have that either its in-neighborhood is non-empty or its out-neighborhood is non-empty. Say $x \in N^+(i)$. Then, $x \in N^+(j)$ for all $j \in I$ since the neighborhood function is nested on I. Let y be the maximum element in the order given by N^- being strictly nested on K. Then since $x \in N^+(j)$ for all $j \in I$, we have $j \in N^-(x) \subsetneq N^-(y)$ for all $j \in I$. This show y is dominated by I. To show y is dominated by K, suppose for a contradiction, that $y \to z$ for some $z \in K$. Then $y \in N^-(z) \subsetneq N^-(y)$ which is impossible. This means y is an in-dominated vertex. A similar argument shows that if $x \in N^-(i)$ then the maximal vertex with respect to the strictly nested order on N^+ is an out-dominating vertex.

In order to recursively choose an independent, out-dominating, or in-dominated vertex we must show that the removal of such a vertex leaves us with a transitive digraph with properly nested neighborhoods. If the vertex is isolated, its removal has no effect on the neighborhoods or the transitivity of the graph. If the vertex is dominating or dominated, then its removal decreases every neighborhood in exactly the same way, leaving comparability conditions intact. The transitivity also remains because removal of a vertex in any transitive graph leaves the graph transitive.

This gives us a recursive construction of the oriented threshold graph as a sequence of independent, in-dominated, and out dominating vertices as required.

Finally, (e) implies (a). The assumption gives a sequence of zeros, ones, and negative ones, say $(s_i)_{i=1}^n$. If we forget (temporarily) about the sign on the ones, we have a sequence of zeros and ones corresponding to removing the direction on the edges. This underlying graph is constructed by adding isolated or dominating vertices. This means it is an unoriented threshold graph. There is an injective weight function and threshold for this underlying graph, say $(w_i)_{i=1}^n$ and t. It is enough to show then, that the weight function

$$\vec{w}_i = \begin{cases} s_i w_i, & \text{if} s_i \neq 0 \\ w_i, & \text{if} s_i = 0 \end{cases}$$

i gives the correct orientation of the edges.

We make an observation about the weights of the vertices. Notice that the weight of a vertex is directly correlated with the size of its neighborhood since the heigher the weight of a vertex the easier it is to meet the threshold with another vertex. This means the later in the sequence a 1 happens, the higher the weight of the vertex associated to it. Conversely, the later in the sequence a 0 happens the lower the associated weight will be so that it becomes more difficult to meet thresholds and thus will have fewer neighbors.

With this, we see that the above weight function satisfies $|\vec{w}_i| > |\vec{w}_j|$ whenever $i > j$ and $|s_i| = 1$. This means that the orientation of the graph given by the above weight function is the same as the orientation given by the sequential construction.

□

Remark 7. This last equivalence gives a ternary sequence which can be translated into an oriented threshold graph. We call the graph a sequence s produces, $OTG(s)$.

Example 8. Let $\bar{s} = (1, -1, 0, -1, \star)$. The sequential construction yields the graph in Figure 2.2.

Figure 2.2: (Left) $OTG(1, -1, 0, -1, \star)$ and (right) the directed threshold graph corresponding to the weight function $(15, -12, 3, -9, 6)$ and threshold 15.

The vertex weights shown in the figure, $(15, -12, 3, -9, 6)$ with threshold $t = 15$, also gives an isomorphic graph.

3 Sequential Form and Enumeration

A few things before we go further: to draw and think about these oriented threshold graphs, the sequential definition is quite a bit more malleable; we work with it. Recall that in the undirected case, the first vertex drawn is always an independent vertex and we denoted it by $\star$. We will use this convention with oriented threshold graph sequences as well. For another simplification, instead of $+1$ and -1 we simply write $+$ and $-$ (respectively).

Looking more closely at these sequences, things get a little messy. In the undirected case, it is easy to just count the sequences, $\{0,1\}^{n-1}$ ($n-1$ as the first vertex drawn does not matter.) Things are a little more subtle in the case of oriented threshold graphs. Notice (in Figure 3.3) that the sequence $(+\,-\,0\,\star)$ gives an isomorphic graph to the one from the sequence $(-\,+\,0\,\star)$. The isomorphism switches the last two vertices, as shown in the following figure.

Figure 3.3: (Left) $OTG(-1,1,0,\star)$, and (right) $OTG(1,-1,0,\star)$

Define $[n] = \{1,2,3,\cdot,n\}$.

Lemma 9. *Given a sequence $\bar{s} := (s_i)_{i=1}^n$, if there is a $k \in [n]$ such that $|s_k| = |s_{k-1}|$ then the sequence $s' = (s_1, s_2, \cdots, s_{k-2}, s_k, s_{k-1}, s_{k+1}, \cdots, s_n)$ produces a digraph isomorphic to the one produced by s.*

Proof. Clearly if $s_k = s_{k-1}$ we are fine. So without loss of generality assume $s_k = +$ and $s_{k-1} = -$. So there is an edge $\overrightarrow{k(k-1)}$. Now, just note the neighborhoods $N^+(k), N^-(k), N^+(k-1), N^-(k-1)$ do not change when we swap the order of k and $k-1$, as the only edge affected is the one between them, and its order is switched as was needed. □

Remark 10. Since the $\star$ at the beginning of any sequence can be thought of as a $+, -$, or 0, we can always think of $-$'s adjacent to $\star$ as $+$'s.

Using the previous lemma and remark we obtain a 'canonical' representation for any isomorphism class, namely,

$$(+^{p_l}, -^{m_l}, 0^{z_l}, +^{p_{l-1}}, -^{m_{l-1}}, 0^{z_{l-1}}, \cdots, +^{p_1}, -^{m_1}, 0^{z_1}, +^{p_0}, \star)$$

where $z_i \neq 0$ for all i. The notation $+^{p_i}$, $-^{m_i}$, and 0^{z_i} simply mean p_i $+$'s, m_i $-$'s, and z_i 0's (respectively).

Theorem 11. *There is a bijection between isomorphism classes of oriented threshold graphs and sequences of the form*

$$(+^{p_l}, -^{m_l}, 0^{z_l}, +^{p_{l-1}}, -^{m_{l-1}}, 0^{z_{l-1}}, \cdots, +^{p_1}, -^{m_1}, 0^{z_1}, +^{p_0}, \star)$$

where the z_i are positive integers and p_i and m_i are non-negative integers. We call a sequence of this form canonical.

Proof. Let G be a directed threshold graph. Then by Theorem 6 (e) there is a sequence of the characters $+, -$ and 0 corresponding to G. By Lemma 9 any grouping of $+$'s and $-$'s can be rearranged so that all $+$'s are to the left of all $-$'s in the grouping without changing the isomorphism class. Also, by Remark 10, if there is a grouping of $+$'s and $-$'s by $\star$ we can consider them as all $+$'s. Each of these groupings is separated by a grouping of 0's. This gives us a canonical sequence for the graph G.

Now, every ternary sequence gives a unique representation of the form

$$(+^{p_l}, -^{m_l}, 0^{z_l}, +^{p_{l-1}}, -^{m_{l-1}}, 0^{z_{l-1}}, \cdots, +^{p_1}, -^{m_1}, 0^{z_1}, \star).$$

Since each graph gives a ternary sequence, if we can show that two sequences that have different canonical forms give non-isomorphic graphs, we are done.

So let s and t be two different ternary sequences in canonical form. We start with a few trivial cases. If we take the underlying undirected graphs of s and t, and they are non-isomorphic, then s and t themselves cannot be isomorphic. To get to the undirected underlying graphs, we just look at the binary sequences where $+$'s and $-$'s are mapped to 1, and 0's are mapped to 0. If these are not the same, we are done. So assume, s and t have the same length and the same number of 0's; moreover, the indices of the 0's are the same. Let i be the leftmost index in which s and t differ; without loss of generality, say $s_i = +$ and $t_i = -$. Let k be the next index ($k < i$) where $s_k = t_k = 0$ (this exists because s and t are in canonical form, and if it did not, then s and t would be a sequence of $+$'s). Then the grouping of 0's that include s_k and t_k form an independent set, say I. Now, all vertices in I have the same degree, $n - k$.

From here there are two cases: there is another $+$ or $-$ after the grouping of 0's, or $\star$ is the next vertex after the grouping of zeros in which index k lies.

In the first case, there are no other vertices of degree $n - k$ besides those in I. In the sequence s, the number of vertices in the in-neighborhood of the vertices in I is greater than they are in t. This makes the two graphs non-isomorphic.

In the other case, the number of $+$'s and $-$'s are different between the two sequences, meaning the oriented split partition of the vertices is different showing that the sequences represent non-isomorphic graphs. □

Having a ternary canonical representation for oriented threshold graphs gives us an easy way to count the number of isomorphism classes of oriented threshold graphs on n vertices.

Theorem 12. *The number of isomorphism classes of oriented threshold graphs on n vertices is F_{2n} the $2n$ Fibonacci number (where $F_0 = 0, F_1 = 1$.)*

Proof. We find a recursion relation on the classes by looking at the sequences in canonical form. We can always create a new sequence from one in canonical form by augmenting it with a 0 or +, but only sequences that have a 0 or − can be augmented with a − to form a new sequence in canonical form. Let $T(n)$ be the number of sequences in canonical form, and $P(n)$ be the number of sequences in canonical form starting with a +. Because a sequence in canonical form cannot have − before a +, we obtain $T(n) = T(n-1)+T(n-1)+(T(n-1)-P(n-1))$ where the first two terms come from augmenting a 0 or + to an old sequence, and the last term from augmenting a −. Since we could always have augmented a sequence with a +, we have $P(n) = T(n-1)$. This gives us the recurrence $T(n) = 3T(n-1) - T(n-2)$. The initial conditions are that $T(1) = 1$ (being the sequence +) and $T(2) = 2$ (from the sequences $+\star$ and $0\star$).

We look at the Fibonacci sequence; specifically, we look at F_{2n}.

$$\begin{aligned} F_{2n} &= F_{2n-1} + F_{2n-2} \\ &= 2F_{2n-2} + F_{2n-3} \\ &= 3F_{2n-2} - F_{2n-4} \\ &= 3F_{2(n-1)} - F_{2(n-2)} \end{aligned}$$

Also, notice that $F_0 = 1$ and $F_2 = 2$. Therefore we have the same recursion and starting values as the even Fibonacci numbers, and we are done. □

Now, note that (e) from Theorem 6 states that every tertiary sequence is associated to an oriented threshold graph. Since tertiary sequences in cannonical form represent the isomorphism classes, we conclude that the number of isomorphism classes of orientation of threshold graphs which are transitive is F_{2n}.

Further, by using (c) from Theorem 6, we find that every transitive ordering of a threshold graph is oriented threshold. This means that the number of non-isomorphic transitive orientations of a threshold graph can be counted by determining the number of ways to include +'s and −'s in canonical form of the binary sequence representation of the unoriented threshold graph. We do that in the following theorem.

Theorem 13. *Let G be a threshold graph given by the sequence*

$$(+^{p_l}, 0^{z_l}, \ldots, +^{p_1}, 0^{p_1}, +^{p_0}, \star).$$

The number of non-isomorphic transitive orientations of G is

$$\prod_{i=1}^{l} (p_l + 1).$$

Proof. An orientation of G is given by turning some of the +'s into −'s. Canonical form states that we get the same graph if we put −'s at the end of the string of +/−'s. So, for each block of +'s we simply have a choice of where to start putting −'s. There are $p_i + 1$ choices for each block. The last block does not get any −'s. The product of these choices is the total number of orientations which are transitive. □

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