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Extreme equitable block-colorings of C_4-decompositions of $K_v - F$

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Abstract

A C_4-decomposition of $K_v - F$, where F is a 1-factor of K_v and C_4 is the cycle of length 4, is a partition P of $E(K_v - F)$ into sets, each element of which induces a C_4 (called a block). A function assigning a color to each block defined by P is said to be an (s, p)-equitable block-coloring if: exactly s colors are used; each vertex v is incident with blocks colored with exactly p colors; and the blocks containing v are shared out as evenly as possible among the p color classes.

Of particular interest is the value of $\chi'_p(v)$, the smallest value of s for which there exists an (s, p)-equitable block-coloring of some C_4-decomposition of $K_v - F$. In this paper the value of $\chi'_p(v)$ is found in the most interesting cases where traditional proof techniques are rendered useless, namely when $\chi'_p(v) > p$. This settles an open problem in a recent paper.

Finally, the study of the structure within such equitable block-colorings is developed. In all cases where $\chi'_p(v) > p$, the problems of finding the value of the smallest color class when it is as large as possible, $\overline{\psi'_1}(C_4, K_v - F)$, and the value of the largest color class when it is as small as possible, $\psi'_s(C_4, K_v - F)$, are both completely settled. This result follows from the solution to another interesting problem, namely that of finding (s, p)-equitable edge-colorings of K_v.

1 Introduction

An H-decomposition of a graph G is an ordered pair (V, B) where V is the vertex set of G and B is a partition of the edges of G into sets, each of which induces a copy of H. For the purposes of this paper, the graphs induced by the elements of B are known as the blocks of the decomposition. (V, B) is said to have an (s, p)-equitable block-coloring $E : B \mapsto C = \{1, 2, \ldots, s\}$ if:

(i) the blocks in B are colored with exactly s colors,

(ii) for each vertex $u \in V(G)$ the blocks containing u are colored using exactly p colors, and

(iii) for each vertex $u \in V(G)$ and for each $\{i, j\} \subset C(E, u)$, $|b(E, u, i) - b(E, u, j)| \leq 1$,

where $C(E, u) = \{i \mid E \text{ colors some block incident with } u \text{ with color } i\}$ and $b(E, u, i)$ is the number of blocks in B containing u that are colored i by E. Such colorings were originally introduced by L. Gionfriddo, M. Gionfriddo and Ragusa in [7], their work being extended by Li and Rodger in [10]. For some values of v, (s, p)-equitable block-colorings of H-decompositions of K_v have also been studied in the case where H is a 4-cycle in [8], 6-cycle in [2] and 8-cycle in [3] (necessarily v is odd in these cases).

In this paper, the focus is on the existence of (s, p)-equitable block-colorings of C_4-decompositions of $K_v - F$, where F is a 1-factor of K_v and C_4 is the cycle of length 4. For any C_4-decomposition $\Sigma = (V, B)$ of $K_v - F$, its spectrum is defined in [10] to be $\Omega_p(\Sigma) = \{s \mid \text{there exists an } (s, p)\text{-equitable block -coloring of } \Sigma\}$. This definition suggests the problem of finding the p-color-spectrum $\Omega_p(v) = \bigcup \Omega_p(\Sigma)$, where the union is taken over the set of all C_4-decompositions, Σ, of $K_v - F$. Two values in $\Omega_p(v)$ are of particular interest, namely the lower p-chromatic index defined to be $\chi'_p(v) = \min \Omega_p(v)$, and the upper p-chromatic index defined to be $\overline{\chi'_p}(v) = \max \Omega_p(v)$.

The primary interest of Li and Rodger in [10] was to find $\chi'_p(v)$ and $\overline{\chi'_p}(v)$ when $p \leq 4$. In so doing, they established $\chi'_4(v)$ when $v \equiv 4t + 2 \pmod{8t}$ in the following result, Theorem 1.1, which includes a non-existence result concerning $(2t, 2t)$-equitable block-colorings. (In another result they also settled the value of $\chi'_4(v)$ for all other values of v; in such cases $\chi'_p(v) = 4$).

Theorem 1.1. *[10] Let $v \equiv 4t + 2 \pmod{8t}$. Then*

1. *there is no C_4-decomposition of $K_v - F$ for which there exists a $(2t, 2t)$-equitable block-coloring, and*

2. *$\chi'_{2t}(v) = 2t + 1$ for $t \in \{1, 2\}$.*

This leaves open the interesting problem of finding $\chi'_{2t}(v)$ when $v \equiv 4t + 2 \pmod{8t}$, noting that Theorem 1.1 just shows that $\chi'_{2t}(v) > 2t$ and settles the case where $t \leq 2$. In this paper we show that there is a $(2t+1, 2t)$-equitable block-coloring of some C_4-decomposition of $K_v - F$ when $v \equiv 4t+2 \pmod{8t}$ (see Theorem 3.1). As a consequence, the value of $\chi'_{2t}(v)$ when $v \equiv 4t+2 \pmod{8t}$ is established in Corollary 3.2, thereby settling the open case left in [10].

Another important focus of this paper is developing the study of the structure within such equitable block-colorings. Here we introduce two concepts in Definitions 1.2 and 1.3 that provide a way to categorize the colorings.

Definition 1.2. The *color vector* of an (s, p)-equitable block-coloring E of an H-decomposition $(V(G), B)$ of a graph G is the vector $V(E) = (c_1(E), c_2(E), \ldots, c_s(E))$ in which for $1 \leq i \leq s$ the number of vertices in G that are incident with a block of color i is $c_i(E)$.

In this paper the colors are named so that $c_1(E) \leq c_2(E) \leq \cdots \leq c_s(E)$. If E is clear then, more simply, c_i is written instead of $c_i(E)$. Regarding the color vector, the values naturally of most interest are c_1 and c_s (but some basic results are presented for c_i for the intermediate components of the vector in Lemmas 2.3 and 2.4). With this in mind we make the following definition.

Definition 1.3. For any graphs G and H and for $1 \leq i \leq s$, define

(i) $\phi(H, G; s, p, i) = \{c_i(E) \mid E$ is an (s, p)-equitable block-coloring of an H-decomposition of $G\}$,

(ii) $\psi'(H, G; s, p, i) = \min \phi(H, G; s, p, i)$, and

(iii) $\overline{\psi'}(H, G; s, p, i) = \max \phi(H, G; s, p, i)$.

In Corollary 3.3 we focus on the smallest and largest color classes finding the values of $\overline{\psi'}(C_4, K_v - F; 2t+1, 2t, 1)$ and $\psi'(C_4, K_v - F; 2t+1, 2t, s)$. Since the cases where $(s, p) = (2t+1, 2t)$ are the sole focus of this paper, for convenience define $\psi'(H, G; 2t+1, 2t, i) = \psi'_i(H, G)$, and $\overline{\psi'}(H, G; 2t+1, 2t, i) = \overline{\psi'_i}(H, G)$.

It is worth noting here that if the blocks are defined to be K_2, then we would be seeking a type of edge-coloring that generalizes the well-studied equitable edge-colorings, each of which is easily seen to be equivalent to an (s, s)-equitable edge-coloring (so each block is a copy of K_2). Edge-colorings which are proper are certainly equitable, but equitable edge-colorings become particularly interesting when the number of colors being used to color $E(G)$ is less than $\chi'(G)$ (for example, see [1, 6, 9, 11] for some results and applications). Interchanging colors along paths with alternately colored edges is a traditionally powerful technique for finding such edge-colorings, but they are rendered useless in this more general setting whenever it is required that $s > p$, as is the situation for results in this paper. Not only are these edge-colorings challenging to produce in themselves, but also (s, p)-equitable edge-colorings are relevant here because of the correspondence described in Lemma 2.1; so we focus on finding the values of $\overline{\psi'_1}(K_2, K_v)$ and $\psi'_s(K_2, K_v)$ (see Corollary 3.3).

It is also interesting that fair holey decompositions are examples of these colorings. For example, in [5] (s,p)-equitable block colorings of the complete multipartite graph $K(n,r)$ (n vertices in each of r parts) are found where the blocks are holey 1-factors (i.e. matchings of size $n(r-1)/2$ in which each matching saturates all vertices except for those in one part called the hole of the matching); so a consequence is that $s = nr$ and $p = n(r-1)$. A similar result is found in [4] in which the blocks of the decomposition of $K(n,r)$ are cycles of length $n(r-1)$.

The following notation will be useful in our results. Define $\lceil x\rceil^o$ to be the smallest odd integer greater than or equal to x, $\lceil x\rceil^e$ to be the smallest even integer greater than or equal to x, $\lceil x\rceil^{d4}$ to be the smallest integer divisible by 4 and greater than or equal to x, $\lfloor x\rfloor_o$ to be the largest odd integer less than or equal to x , $\lfloor x\rfloor_e$ to be the largest even integer less than or equal to x, and $\lfloor x\rfloor_{d4}$ to be the largest integer divisible by 4 and less than or equal to x. For a vertex set R, let $K[R]$ denote the complete graph defined on R. In what follows, a color i is said to appear at a vertex u if at least one block incident with u is colored i.

2 General Results

Define $G \times 2$ to be the graph with vertex set $\{(u,1),(u,2) \mid u \in V(G)\}$ and edge set $\{\{(u,i),(w,j)\} \mid 1 \le i,j \le 2 \text{ and } \{u,w\} \in E(G)\}$. As Lemma 2.1 suggests, when studying C_4-decompositions of $K_v - F$, edge-colorings of the graph $K_{v/2}$ are pertinent and useful.

Lemma 2.1. *[10] If there exists an (s,p)-equitable edge-coloring E of G then there exists an (s,p)-equitable C_4-coloring E' of $G \times 2 - F$ for some 1-factor F of $G \times 2$.*

Furthermore, note by Lemma 2.1 that $2c_i(E) = c_i(E')$ for $1 \le i \le s$. In Theorem 3.1, as a result of Lemma 2.1, we show that there exists a $(2t+1, 2t)$-equitable block-coloring of some C_4-decomposition of $K_{v'} - F$ when $v' \equiv 4t+2 \pmod{8t}$, by showing that there exists a $(2t+1, 2t)$-equitable edge-coloring of K_v where $v = v'/2$.

In any (s,p)-equitable block-coloring E of an H-decomposition of G, for each $u \in V(G)$, let $b(H, G; E, u, i)$ denote the number of blocks incident with u that are colored i.

Lemma 2.2. *Let $v \equiv 1 \pmod p$ and E be an (s,p)-equitable edge-coloring of K_v. Then $b(K_2, K_v; E, u, i) = \frac{v-1}{p}$ for all $u \in V(K_v)$ and $i \in C(E,u)$.*

Proof. Let $u \in V(K_v)$, E be an (s,p)-equitable edge-coloring of K_v, and let $i \in C(E,u)$. The edges incident with u are colored with p colors, each color appearing on $\lfloor\frac{d(u)}{p}\rfloor$ or $\lceil\frac{d(u)}{p}\rceil$ edges. So since $d(u) = v-1$ and $v \equiv 1 \pmod p$, $\frac{v-1}{p} = \lfloor\frac{d(u)}{p}\rfloor = \lceil\frac{d(u)}{p}\rceil = b(K_2, K_v; E, u, i)$. □

The particular case of interest here is a $(2t+1, 2t)$-equitable edge-coloring, E of K_v where $v \equiv 2t+1 \pmod{4t}$. So in this situation (and, indeed, whenever

$v \equiv p+1 \pmod{2p}$) Lemma 2.2 implies that $b(K_2, K_v; E, u, i)$ is constant for all $u \in V(K_v)$ and all $i \in C(E, u)$, regardless of the choice of E; so in such cases it makes sense to define $b(v) = b(K_2, K_v; E, u, i) = (v-1)/p$. We now get a series of lemmas that restrict parameters of interest in regards to equitable edge-colorings of K_v and equitable C_4-colorings of $K_{v'} - F$ where $v' \equiv 4t+2 \pmod{8t}$ and $v = v'/2$ with some more general results as well.

Lemma 2.3. *Let $v \equiv p+1 \pmod{2p}$. In any (s,p)-equitable edge-coloring E of K_v, for $1 \le i \le s$*

(i) $c_i(E)$ must be even,

(ii) $c_i(E) \ge b(v) + 1 = \frac{v-1}{p} + 1$, and

(iii) if v is odd, then $c_i(E) \le v - 1$.

Proof. Let $1 \le i \le s$ and E be an (s,p)-equitable edge-coloring of K_v. Since $v \equiv p+1 \pmod{2p}$, let $v = p + 1 + 2px$, where x is an integer. Then by Lemma 2.2, $b(v) = \frac{v-1}{p} = \frac{p+2px}{p} = 1 + 2x$, so $b(v)$ is odd. If $c_i(E)$ is odd, the subgraph induced by $W = \{u \in V(K_v) | i \in C(E,u)\}$ would be an odd-regular graph with an odd number of vertices, which cannot exist, so (i) follows. For $u \in V(K_v)$ and $i \in C(E,u)$, u is joined to $b(v)$ neighbors with an edge colored i, so at least $b(v)+1$ vertices are incident with an edge colored i; so (ii) follows. Clearly $c_i \le v$, so (iii) follows since c_i was just shown to be even and v is clearly odd. $\square$

Lemma 2.4. *Let $v' \equiv 4t+2 \pmod{8t}$. In any $(2t+1, 2t)$-equitable C_4-coloring E' of $K_{v'} - F$, for $1 \le i \le 2t+1$, 4 divides $c_i(E')$.*

Proof. Let $v' = 4t + 2 + 8tx$ for some integer x. For $u \in V(K_{v'} - F)$, $d_{K_{v'}-F}(u) = 8tx + 4t$, so the number of blocks in any C_4-decomposition of $K_{v'} - F$ incident with u is $d_{K_{v'}-F}(u)/2 = 4tx + 2t$. Let E' be a $(2t+1, 2t)$-equitable C_4-coloring of $K_{v'} - F$ and $i \in C(E', u)$. Then

$$b(C_4, K_{v'} - F; E', u, i) = \frac{4tx + 2t}{2t} = 2x + 1.$$

Note then the number of edges in the blocks in each color class is

$$2c_i(E')b(C_4, K_{v'} - F; E', u, i)/2 = c_i(E')(2x+1).$$

Therefore 4 divides $c_i(E')$ since E' is a C_4-coloring.. $\square$

Lemma 2.5. *Let E be a $(2t+1, 2t)$-equitable block-coloring of a graph G with $v = |V(G)|$ vertices. Then,*

$$\sum_{i=1}^{2t+1} c_i(E) = 2tv.$$

Proof. Since the number of colors appearing at each vertex is $2t$ and c_i is the number of vertices where a block of color i appears, the above holds. $\square$

Lemma 2.6. *Let $v \equiv 2t+1 \pmod{4t}$. In any $(2t+1, 2t)$-equitable edge-coloring E of K_v,*

(i) $c_1(E) \le \lfloor \frac{2tv}{2t+1} \rfloor_e$ *and*

(ii) $c_{2t+1}(E) \ge \lceil \frac{2tv}{2t+1} \rceil^e$.

Proof. Note by Lemma 2.5 the average of the integers $c_1, \dots, c_{2t+1}$ is

$$\frac{\sum_{i=1}^{2t+1} c_i}{2t+1} = \frac{2tv}{2t+1}.$$

By Lemma 2.3, c_i is even and by definition $c_1 \le c_i$ for $1 \le i \le 2t+1$, so $c_1 \le \lfloor \frac{2tv}{2t+1} \rfloor_e$. Similarly by Lemma 2.3, c_i is even and by definition $c_{2t+1} \ge c_i$ for $1 \le i \le 2t+1$, so it follows that $c_{2t+1} \ge \lceil \frac{2tv}{2t+1} \rceil^e$. □

Lemma 2.7. *Let $v' \equiv 4t+2 \pmod{8t}$. In any $(2t+1, 2t)$-equitable C_4-coloring E' of $K_{v'} - F$,*

(i) $c_1(E') \le \lfloor \frac{2tv'}{2t+1} \rfloor_{d4}$ *and*

(ii) $c_{2t+1}(E') \ge \lceil \frac{2tv'}{2t+1} \rceil^{d4}$.

Proof. Note by Lemma 2.5 the average of the integers $c_1, \dots, c_{2t+1}$ is

$$\frac{\sum_{i=1}^{2t+1} c_i}{2t+1} = \frac{2tv'}{2t+1}.$$

By Lemma 2.4 for $1 \le i \le 2t+1$, c_i is divisible by 4. By definition $c_1 \le c_i$ for $1 \le i \le 2t+1$, so $c_1 \le \lfloor \frac{2tv'}{2t+1} \rfloor_{d4}$. Similarly by definition $c_{2t+1} \ge c_i$ for $1 \le i \le 2t+1$, so it follows that $c_{2t+1} \ge \lceil \frac{2tv'}{2t+1} \rceil^{d4}$. □

Lemma 2.8. *Let $v \equiv 2t+1 \pmod{4t}$ and let E be a $(2t+1, 2t)$-equitable edge-coloring of K_v. If $|c_1(E) - c_{2t+1}(E)| \in \{0, 2\}$ then*

(i) $c_1(E) = \overline{\psi'_1}(K_2, K_v) = \lfloor \frac{2tv}{2t+1} \rfloor_e$, *and*

(ii) $c_{2t+1}(E) = \psi'_{2t+1}(K_2, K_v) = \lceil \frac{2tv}{2t+1} \rceil^e$.

Proof. Since E is clear we let $c_i = c_i(E)$ for $1 \le i \le 2t+1$. First suppose that $|c_1 - c_{2t+1}| = 0$. Since we have named the colors so that $c_i \le c_j$ for $1 \le i < j \le 2t+1$, it follows that $c_i = c_j$ for $1 \le i < j \le 2t+1$. Then by Lemma 2.5,

$$c_i = \frac{\sum_{i=1}^{2t+1} c_i}{2t+1} = \frac{2tv}{2t+1}.$$

Therefore by Lemma 2.3, $\frac{2tv}{2t+1}$ is an even integer and $c_i = \frac{2tv}{2t+1} = \lfloor \frac{2tv}{2t+1} \rfloor_e = \lceil \frac{2tv}{2t+1} \rceil^e$. Hence by Lemma 2.6, $c_1 = \overline{\psi'_1}(K_2, K_v) = \frac{2tv}{2t+1}$ and $c_{2t+1} = \psi'_{2t+1}(K_2, K_v) = \frac{2tv}{2t+1}$.

Now suppose that $|c_1 - c_{2t+1}| = 2$. By the naming of the colors, we have that $c_1 = c_{2t+1} - 2$. Note by Lemma 2.5, $\frac{2tv}{2t+1}$ is the average of the integers $c_1, \ldots, c_{2t+1}$. So if $c_1 < c_{2t+1}$ then,

$$c_1 < \frac{2tv}{2t+1} < c_{2t+1}. \tag{2.1}$$

If $\frac{2tv}{2t+1}$ is an integer then it is clearly even; but then by (2.1) this is a contradiction, because by Lemma 2.3 c_{2t+1} is an even integer.

Therefore $\frac{2tv}{2t+1}$ is not an integer and $\left\lfloor \frac{2tv}{2t+1} \right\rfloor_e = \left\lceil \frac{2tv}{2t+1} \right\rceil^e - 2$. So by Lemma 2.6 we have that

$$\begin{aligned} c_{2t+1} - 2 = c_1 \leq \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e &= \left\lceil \frac{2tv}{2t+1} \right\rceil^e - 2, \text{ so} \\ c_{2t+1} \leq \left\lceil \frac{2tv}{2t+1} \right\rceil^e &\leq c_{2t+1}, \end{aligned}$$

so $c_{2t+1} = \psi'_{2t+1}(K_2, K_v) = \left\lceil \frac{2tv}{2t+1} \right\rceil^e$. It also follows now that

$$c_1 = c_{2t+1} - 2 = \left\lceil \frac{2tv}{2t+1} \right\rceil^e - 2 = \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e,$$

so by Lemma 2.6, $c_1 = \overline{\psi'_1}(K_2, K_v) = \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e$. □

Lemma 2.9. *Let $v' \equiv 4t+2 \pmod{8t}$ and let E' be a $(2t+1, 2t)$-equitable C_4-coloring of $K_{v'} - F$. If $|c_1(E') - c_{2t+1}(E')| \in \{0, 4\}$ then*

(i) $c_1(E') = \overline{\psi'_1}(C_4, K_{v'} - F) = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$, *and*

(ii) $c_{2t+1}(E') = \psi'_{2t+1}(C_4, K_{v'} - F) = \left\lceil \frac{2tv}{2t+1} \right\rceil^{d4}$.

Proof. First suppose that $|c_1(E') - c_{2t+1}(E')| = 0$. Since we have named the colors so that $c_i \leq c_j$ for $1 \leq i < j \leq 2t+1$, it follows that $c_i = c_j$ for $1 \leq i < j \leq 2t+1$. Then by Lemma 2.5

$$c_i = \frac{\sum_{i=1}^{2t+1} c_i(E')}{2t+1} = \frac{2tv'}{2t+1}.$$

Therefore by Lemma 2.4, $\frac{2tv'}{2t+1}$ is divisible by 4 and $c_i = \frac{2tv'}{2t+1} = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4} = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$. Hence by Lemma 2.7 $c_1(E') = \overline{\psi'_1}(C_4, K_{v'} - F) = \frac{2tv'}{2t+1}$ and $c_{2t+1}(E') = c_{2t+1}(E') = \psi'_s(C_4, K_{v'} - F) = \psi'_{2t+1}(C_4, K_{v'} - F) = \frac{2tv'}{2t+1}$.

Now suppose that $|c_1(E') - c_{2t+1}(E')| = 4$. By the naming of the colors, we have that $c_1(E') = c_{2t+1}(E') - 4$. Note by Lemma 2.5, $\frac{2tv'}{2t+1}$ is the average of the integers $c_1, \ldots, c_{2t+1}$. So if $c_1(E') < c_{2t+1}(E')$ then

$$c_1(E') < \frac{2tv'}{2t+1} < c_{2t+1}(E'). \tag{2.2}$$

If $\frac{2tv'}{2t+1}$ is an integer then it is clearly divisible by 4 since $v' \equiv 4t+2 \pmod{8t}$; but then by (2.2) this is a contradiction, because by Lemma 2.4 $c_{2t+1}(E')$ is divisible by 4.

Therefore $\frac{2tv'}{2t+1}$ is not an integer and $\left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4} = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} - 4$. So by Lemma 2.7 we have that

$$c_{2t+1}(E') - 4 = c_1(E') \leq \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4} = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} - 4, \text{ so}$$
$$c_{2t+1}(E') \leq \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} \leq c_{2t+1}(E'),$$

so $c_{2t+1}(E') = \psi'_{2t+1}(C_4, K_{v'} - F) = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$. It also follows now that

$$c_1(E') = c_{2t+1}(E') - 4 = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4} - 4 = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4},$$

so by Lemma 2.7, $c_1(E') = \overline{\psi'_1}(C_4, K_{v'} - F) = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$. □

3 Main Theorem

The following theorem establishes the value of $\chi'_{2t}(v')$ for $v' \equiv 4t+2 \pmod{8t}$, settling the open case left in [10] (see Corollary 3.2). In so doing, with $v = v'/2$, an extreme equitable edge-coloring is produced which as stated in Corollary 3.3 establishes the value of $\overline{\psi'_1}(K_2, K_v)$ (the largest value that the smallest element of the color vector can attain), and $\psi'_{2t+1}(K_2, K_v)$ (the smallest value that the largest element of the color vector can attain). Using Lemma 2.1, this also establishes the values of $\overline{\psi'_1}(C_4, K_{v'} - F)$ and $\psi'_{2t+1}(C_4, K_{v'} - F)$ as stated in Corollary 3.4.

Theorem 3.1. *Let $v' \equiv 4t+2 \pmod{8t}$ Then there exists a $(2t+1, 2t)$-equitable block-coloring of some C_4-decomposition of $K_{v'} - F$.*

Proof. By Lemma 2.1 it need only be shown that there exists a $(2t+1, 2t)$-equitable edge-coloring of K_v where $v = \frac{v'}{2}$, thus we let $v = 2t+1+4tx$. We first describe the coloring E, show E is well defined, and then show E is a $(2t+1, 2t)$-equitable edge-coloring.

We begin by partitioning the vertices into $2t+1$ groups such that

1. each group has an odd number of vertices and
2. the number of vertices in any two groups differs by at most two.

Therefore the size of each group is $l(v) = \left\lceil \frac{v}{2t+1} \right\rceil^{o}$ or $s(v) = \left\lfloor \frac{v}{2t+1} \right\rfloor_{o}$. Note by construction, the number of groups with $s(v)$ vertices, which we refer to as small groups, is

$$S_v = \frac{l(v)(2t+1) - v}{2} = \frac{2l(v)t + l(v) - (2t+1+4tx)}{2} = l(v)t + \frac{l(v)}{2} - 2tx - t - \frac{1}{2}.$$

Note, S_v is easily seen to be an integer. The number of groups with $l(v)$ vertices, which we refer to as large groups, is $L_v = 2t + 1 - S_v$. Note as well for calculation purposes,

$$b(v) = \frac{v-1}{p} = \frac{2t+1+4tx-1}{2t} = 2x+1.$$

Let the groups with $s(v)$ vertices be named $P_1, \ldots, P_{S_v}$ and the groups with $l(v)$ vertices be named $P_{S_v+1}, \ldots, P_{2t+1}$. For $1 \leq j \leq t$ and $1 \leq i \leq 2t+1$ we color all the edges between the vertices in group P_m and the vertices in group P_n with color i for $m \equiv i + j \pmod{2t+1}$ and $n \equiv i - j \pmod{2t+1}$. Clearly the coloring of all edges between the groups is well defined.

To color edges joining vertices within the groups, first note that $b(v)$, $s(v)$ and $l(v)$ are all odd (so $b(v) - s(v)$ and $b(v) - l(v)$ are even) and it is well known that there exists a 2-factorization of $K[P_i]$ for $1 \leq i \leq 2t+1$. The 2-factors in such 2-factorizations are combined as follows. For $1 \leq l \leq 2t+1$ and $1 \leq j \leq S_v$ with $l \neq j$, if the edges joining P_l to P_j are colored i then color the edges in one $\big(b(v) - s(v)\big)$-factor in $K[P_l]$ with color i; so now there are exactly $b(v)$ edges of color i incident with each vertex in P_l. Similarly, for $1 \leq l \leq 2t+1$ and $S_v + 1 \leq k \leq 2t+1$ with $l \neq k$, if the edges joining P_l to P_k are colored i, then we color the edges in one $\big(b(v) - l(v)\big)$-factor in $K[P_l]$ with color i; so it is also the case that now there are exactly $b(v)$ edges of color i incident with each vertex in P_l. So for every vertex v and for every color i on an edge incident with v:

$$v \text{ is incident with exactly } b(v) \text{ edges colored } i. \tag{3.1}$$

Notice that 3.1 implies that for each v, blocks incident with v have been colored with $\deg(v)/b(v) = p$ colors as required. By considering two cases in turn, we now show that this construction is well-defined, and that all the edges in $K[P_i]$ for $1 \leq i \leq 2t+1$ have been colored.

Case 1: Suppose $l(v) = s(v)$, so the number of vertices in each group is $l(v) = \frac{v}{2t+1}$, and $K[P_i] \cong K_{l(v)}$ for $1 \leq i \leq 2t+1$. It suffices to show that each vertex in each $K_{l(v)}$ has degree equal to the sum of the degrees of factors defined in the coloring. Therefore since each P_i is joined to P_j for all $j \neq i$, the sum of the degrees of the factors is

$$\begin{aligned}
\sum_{1}^{2t} \big(b(v) - l(v)\big) &= 2t\big(b(v) - l(v)\big) \\
&= 4tx + 2t - \big(4t^2 + 2t + 8t^2x\big) / (2t+1) \\
&= (4tx + 2t + 1 - 2t - 1) / (2t+1) \\
&= \frac{v}{2t+1} - 1
\end{aligned}$$

which is exactly the degree of each vertex in $K_{l(v)}$. Therefore the coloring of the complete graphs induced by each group is well defined.

Case 2: Suppose $l(v) \neq s(v)$. Then $\frac{v}{2t+1}$ is not an odd integer, so $s(v) = l(v) - 2$.

First we show that each vertex in $K[P_i]$ for $1 \leq i \leq S_v$ has degree equal to the sum of the degrees of its factors defined in the coloring. Since each P_i for $1 \leq i \leq S_v$ is joined to all small groups except itself and to all large groups, the sum of the degrees of the factors is

$$\begin{aligned}
\big(S_v - 1\big)(b(v) - s(v)) + L_v\big(b(v) - l(v)\big)a \\
&= \big(S_v - 1\big)\big(b(v) - (l(v) - 2)\big) + L_v\big(b(v) - l(v)\big) \\
&= \big(b(v) - l(v)\big)\big(S_v - 1 + L_v\big) + 2\big(S_v - 1\big) \\
&= \big(b(v) - l(v)\big)\big(S_v - 1 + 2t + 1 - S_v\big) + 2\big(S_v - 1\big) \\
&= \big(b(v) - l(v)\big)2t + 2\big(S_v - 1\big) \\
&= \big(2x + 1 - l(v)\big)2t + 2\left(l(v)t + \frac{l(v)}{2} - 2tx - t - \frac{3}{2}\right) \\
&= 4tx + 2t - 2tl(v) + 2tl(v) + l(v) - 4tx - 2t - 3 \\
&= l(v) - 3 = s(v) - 1
\end{aligned}$$

which is exactly the degree of each vertex in $K[P_i]$ for $1 \leq i \leq S_v$. Therefore the coloring of each complete graph induced on the vertices of each small group is well defined.

We now show that each vertex in $K[P_i]$ for $S_v + 1 \leq i \leq 2t + 1$ has degree equal to the sum of the degrees of its factors defined in the coloring. Since each P_i for $S_v \leq i \leq 2t + 1$ is joined to all small groups and all large groups except itself, the sum of the degrees of the factors is

$$\begin{aligned}
S_v(b(v) - s(v)) + \big(L_v - 1\big)\big(b(v) - l(v)\big) \\
&= S_v\big(b(v) - (l(v) - 2)\big) + \big(L_v - 1\big)\big(b(v) - l(v)\big) \\
&= \big(b(v) - l(v)\big)\big(S_v + L_v - 1\big) + 2S_v \\
&= \big(b(v) - l(v)\big)\big(S_v + 2t + 1 - S_v - 1\big) + 2S_v \\
&= \big(b(v) - l(v)\big)2t + 2S_v \\
&= \big(2x + 1 - l(v)\big)2t + 2\left(l(v)t + \frac{l(v)}{2} - 2tx - t - \frac{1}{2}\right) \\
&= 4tx + 2t - 2tl(v) + 2tl(v) + l(v) - 4tx - 2t - 1 \\
&= l(v) - 1
\end{aligned}$$

which is exactly the degree of each vertex in $K[P_i]$ for $S_v + 1 \leq i \leq 2t + 1$. Therefore the coloring of each complete graph induced on the vertices of each large group is well defined.

In both cases the number of colors used is exactly $p = 2t + 1$, so it remains to show there are exactly $s = 2t$ colors appearing at each vertex. We will do this by showing that for $1 \leq i \leq 2t + 1$:

(a) color i is does not appear at any vertex of P_i and

(b) for each color $h \neq i$, color h appears at all vertices of P_i.

Note that the edges colored i between parts join P_{i+j} and P_{i-j} for $1 \leq j \leq t$. In particular, there are no edges colored i between parts which are incident with vertices in P_i. Therefore the construction also ensures that there is no factor colored i in $K[P_i]$, so it follows that (a) holds. Note that for $1 \leq h \leq 2t+1$ with $h \neq i$, there exist j', $1 \leq j' \leq t$, for which $i \equiv h \pm j' \pmod{2t+1}$. So each vertex in P_i has an edge of color h incident with it by construction, so (b) holds. Finally note by 3.1, for each $u \in V(K_v)$ and $i \in C(E,u)$ there are $b(v)$ edges of color i incident with u, so the coloring is equitable. Therefore we have formed a $(2t+1, 2t)$-equitable edge-coloring of K_v. □

Corollary 3.2. *Let $v' \equiv 4t+2 \pmod{8t}$. Then $\chi'_{2t}(v') = 2t+1$.*

Proof. By Theorem 1.1 we know that $\chi'_{2t}(v') > 2t$. So, using the edge-coloring produced in the proof of Theorem 3.1, it follows by Lemma 2.1 that $\chi'_{2t}(v') =$ 2t+1. □

Corollary 3.3. *Let $v \equiv 2t+1 \pmod{4t}$. Then*

(i) $\overline{\psi'_1}(K_2, K_v) = \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e$, *and*

(ii) $\psi'_{2t+1}(K_2, K_v) = \left\lceil \frac{2tv}{2t+1} \right\rceil^e$.

Proof. By the proof of Theorem 3.1, there exists a $(2t+1, 2t)$-equitable edge-coloring E of K_v such that $|c_1(E) - c_{2t+1}(E)| \in \{0, 2\}$. So by Lemma 2.8, $\overline{\psi'_1}(K_2, K_v) = \left\lfloor \frac{2tv}{2t+1} \right\rfloor_e$ and $\psi'_{2t+1}(K_2, K_v) = \left\lceil \frac{2tv}{2t+1} \right\rceil^e$. □

Corollary 3.4. *Let $v' \equiv 4t+2 \pmod{8t}$. Then*

(i) $\overline{\psi'_1}(C_4, K_{v'} - F) = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$, *and*

(ii) $\psi'_{2t+1}(C_4, K_{v'} - F) = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$.

Proof. Let $v = v'/2$. By the proof of Theorem 3.1, there exists a $(2t+1, 2t)$-equitable edge-coloring E of K_v such that $|c_1(E) - c_{2t+1}(E)| \in \{0, 2\}$. Thus by Lemma 2.1 there exists a C_4-coloring E' of $K_{v'} - F$ and $2c_i(E) = c_i(E')$ for $1 \leq i \leq 2t+1$, so $|c_1(E') - c_2(E')| = |2c_1(E) - 2c_{2t+1}(E)| = 2|c_1(E) - c_{2t+1}(E)| \in \{0, 4\}$. So by Lemma 2.9, $\overline{\psi'_1}(C_4, K_{v'} - F) = \left\lfloor \frac{2tv'}{2t+1} \right\rfloor_{d4}$ and $\psi'_{2t+1}(C_4, K_{v'} - F) = \left\lceil \frac{2tv'}{2t+1} \right\rceil^{d4}$. □

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