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On the q-analog of the revolving door algorithm

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Abstract

We obtain a recurrence to generate a sequence containing all $n \times k$ column reduced Echelon forms over the finite field $\mathbb{F}_q$ with q elements such that two consecutive Echelon forms differ in exactly one position. The corresponding sequence of subspaces generated by the Echelon forms defines a cyclic minimal change sequence on the Grassmannian containing all k-dimensional subspaces of $\mathbb{F}_q^n$ with respect to the injection metric of subspaces. Furthermore, plugging $q = 1$ into the recurrence yields a revolving door algorithm of the set of k-element subsets on a set with n elements.

1 Introduction

A *Gray code* of $\{0, 1\}^k$ is a sequence of all binary vectors of length k such that two consecutive vectors differs in exactly one bit. In addition, if the first and the last vector within the sequence also differ in position, the Gray code is called *cyclic* [5, 7].

The concept of a Gray code can be generalized in the following sense. If X denotes a finite set on which a metric d is defined, a *minimal change sequence* of X is a sequence of all objects of X such that the distance of two consecutive elements is minimal. Furthermore, if the distance of the first and the last element is also minimal, then the minimal change sequence is called *cyclic* [8].

A well-known cyclic minimal change sequence on the set of k-element subsets of a set with n elements, abbreviated by $S(n, k)$, with respect to the metric $d_A(S, T) := k - |S \cap T|$ for $S, T \in S(n, k)$, is called a *revolving door algorithm* [7, 9]. Further examples of Gray codes can be found in [1, 2, 3].

Concepts, theories, and discrete structures based on finite sets and their subsets turn into a combinatorial *q-analog* if they are considered over vector spaces over a finite field $\mathbb{F}_q$ with q elements. In this case "subsets" of a finite set become "subspaces" of a vector space and "cardinalities" of subsets become "dimensions" of subspaces.

If $S_q(n,k)$ is the set of k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$, called the *Grassmannian*, the q-analog of a revolving door algorithm on $S_q(n,k)$ which yields a cyclic minimal change sequence with respect to the injection metric $d_I(S,T) := k - \dim(S \cap T)$ for $S,T \in S_q(n,k)$ was constructed in [10].

If $E_q(n,k)$ denotes the set of $n \times k$ column reduced Echelon forms over $\mathbb{F}_q$, the goal of this paper is to recursively construct a minimal change sequence on $E_q(n,k)$ with respect to the Hamming metric $d_H(A,B) := |\{(i,j) \mid a_{i,j} \neq b_{i,j}\}|$ of matrices $A = [a_{i,j}], B = [b_{i,j}] \in \mathbb{F}_q^{n\times k}$. This sequence generalizes the recurrence given in [4] for arbitrary field sizes.

We show that the resulting recurrence formula yields a cyclic minimal change sequence on $S(n,k)$ with respect to d_A by plugging $q = 1$ into the recurrence.

Furthermore, the corresponding sequence of subspaces which are generated by the column vectors of the column reduced Echelon forms also yields a cyclic minimal change sequence on $S_q(n,k)$ with respect to d_I.

2 On Column Reduced Echelon Forms

In this section we briefly recall some basic facts on Echelon forms in order to deduce a first recursive formula for the set $E_q(n,k)$ of all $n \times k$ column reduced Echelon forms.

A k-dimensional subspace of the canonical n-dimensional vector space $\mathbb{F}_q^n$ can be represented by an $n \times k$ matrix over $\mathbb{F}_q$ where its columns denote a basis of the subspace. By Gaussian elimination we can turn any basis of the k-dimensional subspace into a unique basis having the following properties, called the *column reduced Echelon form* [6]:

- the first nonzero number upwards in each column, called the *pivot coefficient*, is 1 and it is strictly above the pivot coefficient of the columns right next to it;
- each pivot coefficient is the only non-zero entry in its row.

Figure 1 shows the general shape of an Echelon form where the stars indicate finite field elements.

In order to generate all matrices of $E_q(n,k)$, we use a recurrence arising by partitioning all elements of $E_q(n,k)$ into two classes:

- column reduced Echelon forms containing a pivot element in the top row, say

$$\left[\begin{array}{c|c} 1 & \\ \hline & A \end{array}\right] := \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & a_{0,0} & \cdots & a_{0,k-2} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n-2,0} & \cdots & a_{n-2,k-2} \end{array}\right]$$

$$
\begin{bmatrix}
* & * & & * \\
\vdots & \vdots & & \vdots \\
* & * & & * \\
\boxed{1} & 0 & & 0 \\
 & * & & * \\
 & \vdots & & \vdots \\
 & * & & * \\
 & \boxed{1} & & 0 \\
 & & & * \\
 & & \ddots & \vdots \\
 & & & * \\
 & & & \boxed{1} \\
 & & & 0 \\
 & & & \vdots \\
 & & & 0
\end{bmatrix}
$$

Figure 1: The general shape of a column reduced Echelon form

where $A = [a_{i,j}] \in E_q(n-1, k-1)$, and

- column reduced Echelon forms *not* containing a pivot element in the top row, say

$$\left[\begin{array}{c} C \\ \hline A \end{array}\right] := \left[\begin{array}{ccc} c_0 & \cdots & c_{k-1} \\ \hline a_{0,0} & \cdots & a_{0,k-1} \\ \vdots & & \vdots \\ a_{n-2,0} & \cdots & a_{n-2,k-1} \end{array}\right]$$

where $A = [a_{i,j}] \in E_q(n-1, k)$ and $C = [c_0, \ldots, c_{k-1}] \in \mathbb{F}_q^{1\times k}$.

The decomposition of $E_q(n, k)$ into these two classes immediately yields the following recurrence formula for $0 < k \leq n$:

$$
\begin{aligned}
E_q(n,k) = & \left\{ \left[\begin{array}{c|c} 1 & \\ \hline & A \end{array}\right] \mid A \in E_q(n-1,k-1) \right\} \\
& \cup \left\{ \left[\begin{array}{c} C \\ \hline A \end{array}\right] \mid A \in E_q(n-1,k), C \in \mathbb{F}_q^{1\times k} \right\}.
\end{aligned}
$$

Iterating this formula in the right part the number n decreases by one in each step until the set $E_q(k, k)$ is reached, which contains the only one $k \times k$ column reduced Echelon form over $\mathbb{F}_q$ given by the identity matrix entries having 1 on the diagonal and 0 otherwise:

$$E_q(k,k) = \left\{ U_k := \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \right\}.$$

On the other hand, iterating the recurrence for $E_q(n, k)$ in the left part the values n and k decrease simultaneously until $E_q(n - k, 0)$ is reached. Since the only trivial

0-dimensional subspace of $\mathbb{F}_q^n$ contains the zero vector we use the convention that $E_q(n,0)$ contains the "empty" matrix $\emptyset$ with n rows and zero columns. By definition extending the empty matrix $\emptyset \in E_q(n,0)$ by an additional column containing a pivot element yields the following column reduced $(n+1) \times 1$ Echelon form

$$\left[\begin{array}{c|c} 1 & \\ \hline & \emptyset \end{array}\right] = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in E_q(n+1,1).$$

3 Minimal Change Sequences

In this section we consider operations on sequences of matrices and describe general results how to form new minimal change sequences from given ones. These general results will be combined to obtain a recurrence for a minimal change sequence of column reduced Echelon forms in the subsequent section.

A series $A = A_0, \ldots, A_{r-1}$ of $\alpha \times \beta$ matrices over $\mathbb{F}_q$ is called a *sequence* and it is abbreviated by $A \sqsubset \mathbb{F}_q^{\alpha\times\beta}$. The i-th element of the sequence A is denoted by A_i. The number r of elements in the sequence A is called the *length* of A and it is denoted by $|A|$.

A sequence A is called a *minimal change sequence* if and only if two consecutive matrices A_i and A_{i+1} differ in exactly one position for all $0 \leq i < |A| - 1$.

For $A \sqsubset \mathbb{F}_q^{\alpha\times\beta}$, $B \sqsubset \mathbb{F}_q^{\alpha\times\beta}$, $C \in \mathbb{F}_q^{\gamma\times\beta}$, and $i \geq 0$ we define the following operations:

- reflection:
$$A^R := A_{|A|-1}, \ldots, A_0 \sqsubset \mathbb{F}_q^{\alpha\times\beta},$$
- conditioned reflection:
$$A^{\langle i\rangle} := \begin{cases} A & \text{if } i \text{ is zero or even} \\ A^R & \text{otherwise} \end{cases} \sqsubset \mathbb{F}_q^{\alpha\times\beta},$$
- shortening:
$$A^S := A_0, \ldots, A_{|A|-2} \sqsubset \mathbb{F}_q^{\alpha\times\beta},$$
- extension:
$$A^E := \left[\begin{array}{c|c} 1 & \\ \hline & A_0 \end{array}\right], \ldots, \left[\begin{array}{c|c} 1 & \\ \hline & A_{|A|-1} \end{array}\right] \sqsubset \mathbb{F}_q^{(\alpha+1)\times(\beta+1)},$$
- concatenation:
$$A|B := A_0, \ldots, A_{|A|-1}, B_0, \ldots, B_{|B|-1} \sqsubset \mathbb{F}_q^{\alpha\times\beta},$$
- left augmentation:
$$CA := \left[\begin{array}{c} C \\ \hline A_0 \end{array}\right], \ldots, \left[\begin{array}{c} C \\ \hline A_{|A|-1} \end{array}\right] \sqsubset \mathbb{F}_q^{(\alpha+\gamma)\times\beta},$$

- right augmentation:

$$AC := \left[\frac{A_0}{C}\right], \ldots, \left[\frac{A_{|A|-1}}{C}\right] \sqsubset \mathbb{F}_q^{(\alpha+\gamma)\times\beta}.$$

Furthermore, for a series of sequences $A^{(0)}, A^{(1)}, \ldots, A^{(m-1)} \sqsubset \mathbb{F}_q^{\alpha\times\beta}$ we introduce the notation

$$\prod_{i=0}^{m-1} A^{(i)} := A^{(0)}|A^{(1)}|\ldots|A^{(m-1)}.$$

Lemma 3.1. *Let $A \sqsubset \mathbb{F}_q^{\alpha\times\beta}$ and $B \sqsubset \mathbb{F}_q^{\gamma\times\beta}$ be minimal change sequences, let $C, D \in \mathbb{F}_q^{\gamma\times\beta}$ matrices differing in exactly one position, and let $i \geq 0$ be an integer. Then the following sequences have the minimal change property:*

1. *A^R, A^S, A^E, $A^{\langle i\rangle}$, AC, and CA,*
2. *$AC|A^RD$,*
3. *$\prod_{i=0}^{|B|-1} A^{\langle i\rangle} B_i$.*

Proof. The first item is obvious from the definition of the operations. The second item is also easy to see. Each part is a minimal change sequence and at the transition from AC to A^RD the elements differ in one position in the lower matrix segment C and D, respectively. The third item is an iteration of the second one. □

4 The Recurrence Formula

Theorem 4.1. *Let q be a prime power and let $G_q(k) \sqsubset \mathbb{F}_q^{1\times k}$ denote a minimal change sequence for all row vectors of length $k > 0$ over $\mathbb{F}_q$ whose first element is the unit vector $10\ldots0$ and whose last element is the all-zero vector $0\ldots0$. Then for $0 < k < n$ the recurrence*

$$\Gamma_q(n,k) := A|B|C$$

with

$$\begin{aligned} A &:= \Gamma_q(n-1,k-1)^E \\ B &:= \prod_{i=0}^{|\Gamma_q(n-1,k)|-1} (G_q(k)^S)^{\langle i\rangle}\Gamma_q(n-1,k)_i \\ C &:= G_q(k)_{|G_q(k)|-1}\Gamma_q(n-1,k)^R \end{aligned}$$

and initial values

$$\Gamma_q(n,0)^E := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \sqsubset \mathbb{F}_q^{(n+1)\times 1}$$

and

$$\Gamma_q(k,k) := U_k \sqsubset \mathbb{F}_q^{k\times k}$$

defines a minimal change sequence on the set $E_q(n,k)$ of column reduced $n\times k$ Echelon forms over $\mathbb{F}_q$ with respect to the Hamming metric of matrices.

Proof. (i) We note that a Gray code $G_q(k)$ with the required property does exist and can be obtained from a recursive construction [7].

(ii) The elements of $\Gamma_q(n,k)$ are the same as $E_q(n,k)$: the partial sequence A contains the matrices of the left part and $B|C$ corresponds to the right part of the recurrence for $E_q(n,k)$ introduced in Section 2. Hence, we get

$$|\Gamma_q(n,k)| = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the number of k-subspaces of an n-dimensional vector space over the finite field with q elements.

(iii) We determine the first and the last element of the sequence. The first element of $\Gamma_q(n,k)$ arises by A by extending the first element of $\Gamma_q(n-1,k-1)$. Starting with $\Gamma_q(n-k,0)$ containing only the "empty" matrix we recursively obtain

$$\Gamma_q(n,k)_0 = \left[\begin{array}{c} U_k \\ \hline \mathbf{0}_{n-k,k} \end{array}\right]$$

where $\mathbf{0}_{n-k,k}$ denotes the $(n-k)\times k$ zero matrix. The last element of $\Gamma_q(n,k)$ arises by C from the last element of $\Gamma_q(n-1,k)^R$ which is the first element of $\Gamma_q(n-1,k)$ augmented by an additional zero row $G_q(k)_{q^k-1} = \mathbf{0}_{1,k}$:

$$\Gamma_q(n,k)_{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q -1} = \left[\begin{array}{c} \mathbf{0}_{1,k} \\ \hline U_k \\ \hline \mathbf{0}_{n-1-k,k} \end{array}\right].$$

(iv) Finally, we show the minimal change property of the recurrence formula for $\Gamma_q(n,k)$ by induction. Assuming that $\Gamma_q(n-1,k-1)$ and $\Gamma_q(n-1,k)$ are minimal change sequences it is immediate to see from Lemma 3.1 that the partial sequences A, B, and C are all minimal change sequences. To obtain a minimal change sequence by $A|B|C$ we must verify the minimal change property at the transition of the partial sequences.

$A|B$: The last element of A is given by

$$\left[\begin{array}{c|c} 1 & \\ \hline & \begin{array}{c} \mathbf{0}_{1,k-1} \\ \hline U_{k-1} \\ \hline \mathbf{0}_{n-k,k-1} \end{array} \end{array}\right] = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \boxed{0} & 0 & \cdots & 0 \\ \hline 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right]$$

and the first element of B is defined to be

$$\left[\begin{array}{c} G_q(k)_0 \\ \hline U_{k-1} \\ \hline \mathbf{0}_{n-k,k} \end{array}\right] = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \boxed{1} & 0 & \cdots & 0 \\ \hline 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right].$$

Both matrices differ in exactly one position in the first column and second row.

$B|C$: The first element of C is given by

$$\left[\begin{array}{c} G_q(k)_{q^k-1} \\ \hline \mathbf{0}_{1,k} \\ \hline U_k \\ \hline \mathbf{0}_{n-2-k,k} \end{array}\right].$$

The last element of B depends on the parity of the number of elements in $\Gamma_q(n-1,k)$. If the number of elements is even the last element of B is

$$\left[\begin{array}{c} G_q(k)_0 \\ \hline \mathbf{0}_{1,k} \\ \hline U_k \\ \hline \mathbf{0}_{n-2-k,k} \end{array}\right]$$

and if the number of elements is odd the last element of B is

$$\left[\begin{array}{c} G_q(k)_{q^k-2} \\ \hline \mathbf{0}_{1,k} \\ \hline U_k \\ \hline \mathbf{0}_{n-2-k,k} \end{array}\right].$$

Hence, the last element of B and the first element of C differ in the first column and first row if the number of elements in $\Gamma_q(n-1,k)$ is even and in the first column and the last row if the number of elements is odd. □

Example 4.1. We list $\Gamma_3(n,k)$ for $1 \le k \le 2$ and $k \le n \le 3$:

$$\Gamma_3(1,1) = \left[\begin{array}{c} 1 \end{array}\right]$$

$$\Gamma_3(2,1) = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\Gamma_3(1,0)^E}, \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{(G_3(1)^S)\Gamma_3(1,1)_0}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{0\Gamma_3(1,1)^R}$$

$$\Gamma_3(2,2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Gamma_3(3,1) = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\Gamma_3(2,0)^E},$$

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{(G_3(1)^S)\Gamma_3(1,1)_0}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{(G_3(1)^S)^R\Gamma_3(1,1)_1},$$

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}_{(G_3(1)^S)\Gamma_3(1,1)_2}, \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{(G_3(1)^S)^R\Gamma_3(1,1)_3},$$

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{0\Gamma_3(2,1)^R}$$

$$\Gamma_3(3,2) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\Gamma_3(2,1)^E},$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{(G_3(2)^S)\Gamma_3(2,2)_0},$$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{00\Gamma_3(2,2)^R}$$

5 A Gray Code on the Grassmannian

If $\langle K\rangle$ denotes the subspace generated by the column vectors of the matrix K, the sequence $\Gamma_q(n,k)$ obviously yields a sequence on all k-dimensional subspaces of $\mathbb{F}_q^n$ by

$$\langle\Gamma_q(n,k)\rangle := \langle\Gamma_q(n,k)_0\rangle, \langle\Gamma_q(n,k)_1\rangle, \ldots.$$

Lemma 5.1. *Let S, T be two column reduced $n \times k$ Echelon forms over $\mathbb{F}_q$ with Hamming distance $d_H(S,T) = 1$. Then the injection distance satisfies $d_I(\langle S\rangle, \langle T\rangle) = 1$.*

Proof. If S and T differ by one entry they also differ in exactly one column. The intersection of both subspaces $\langle S\rangle$ and $\langle T\rangle$ is therefore $k-1$-dimensional and $d_I(\langle S\rangle, \langle T\rangle) = k - (k-1) = 1$. □

Lemma 5.2. *The first and the last subspace of the sequence $\langle\Gamma_q(n,k)\rangle$ satisfy*

$$d_I(\langle\Gamma_q(n,k)_0\rangle, \langle\Gamma_q(n,k)_{\left[{n \atop k}\right]_q - 1}\rangle = 1.$$

Proof. If u_i is the i-th unit vector in $\mathbb{F}_q^n$ (starting with index 0) we get the structure of both generator matrices from the proof of Theorem 4.1:

$$\Gamma_q(n,k)_0 = \left[\begin{array}{c} U_k \\ \hline \mathbf{0}_{n-k,k} \end{array}\right] = [u_0, u_1, \ldots, u_{k-1}],$$

$$\Gamma_q(n,k)_{\left[{n \atop k}\right]_q - 1} = \left[\begin{array}{c} \mathbf{0}_{1,k} \\ \hline U_k \\ \hline \mathbf{0}_{n-1-k,k} \end{array}\right] = [u_1, \ldots, u_{k-1}, u_k].$$

The intersection of the corresponding subspaces is $\langle u_1, \ldots, u_{k-1}\rangle$ and hence we get $d_I(\langle\Gamma_q(n,k)_0\rangle, \langle\Gamma_q(n,k)_{\left[{n \atop k}\right]_q - 1}\rangle = k - (k-1) = 1$. □

Corollary 5.1. *For all prime powers q and integers $1 \leq k \leq n$, the sequence $\langle\Gamma_q(n,k)\rangle$ defines a cyclic minimal change sequence on the set of all k-dimensional subspaces of $\mathbb{F}_q^n$ with respect to the injection distance of subspaces.*

In [10] Schwartz has already constructed a cyclic minimal change sequence on the Grassmannian. The proposed sequence $\langle\Gamma_q(n,k)\rangle$ has the further property that the corresponding sequence of generating matrices is also a minimal change sequence with respect to the Hamming metric of matrices.

6 The Case $q = 1$

Plugging $q = 1$ into the column reduced Echelon forms only yields matrices consisting of unit vectors $[u_{i_0}, \ldots, u_{i_{k-1}}]$ with corresponding index set $\{i_0, \ldots, i_{k-1}\} \subseteq \{0, \ldots, n-1\}$. Furthermore, the recurrence of Theorem 4.1 reduces to

$$\Gamma_1(n,k) := \Gamma_1(n-1,k-1)^E | \mathbf{0}_{1,k}\Gamma_1(n-1,k)^R$$

since no nonzero vectors exist in $G_1(k)$. Hence the middle part B vanishes. Translating this matrix representation to characteristic vectors of the corresponding index sets, we obtain the following recursion which is a minimal change sequence on subsets:

Theorem 6.1. *For integers $1 \leq k \leq n$ the sequence*

$$\gamma(n,k) := 1\gamma(n-1,k-1)|0\gamma(n-1,k)^R$$

with initial values $\gamma(k,k) = 1^k = 1\ldots1$ and $\gamma(n,0) = 0^n = 0\ldots0$ defines a minimal change sequence on the set of k-element subsets of the canonic n-set with respect to the distance d_A.

Proof. It is clear that $\gamma(n,k)$ really generates all k-subsets of the canonic n-element set (i.e. all binary vectors of length n and weight k).

By induction it follows that the first element of the sequence is

$$1^k0^{n-k} = \underline{1}1^{k-1}\underline{0}0^{n-1-k}$$

and the last element is

$$01^k0^{(n-1)-k} = \underline{0}1^{k-1}\underline{1}0^{(n-1)-k}.$$

The Hamming distance of both vectors is exactly 2 which means a distance of 1 if we consider the vectors as subsets.

Assuming that $\gamma(n-1,k-1)$ and $\gamma(n-1,k)$ satisfy the minimal change property, it is sufficient to verify this property at the transition from the first half $1\gamma(n-1,k-1)$ to the second half $0\gamma(n-1,k)^R$ in order to show that $\gamma(n,k)$ is a minimal change sequence.

We know that the last element of the partial sequence $1\gamma(n-1,k-1)$ is given by

$$101^{k-1}0^{(n-2)-(k-1)} = \underline{1}01^{k-1}\underline{0}0^{n-2-k}$$

and the first element of $0\gamma(n-1,k)^R$ is

$$001^k0^{(n-2)-k} = \underline{0}01^{k-1}\underline{1}0^{n-2-k}$$

Again the underlined positions indicate a distance of 1, proving that the sequence is a cyclic minimal change sequence. $\square$

Example 6.1. For $n=4$ and $k=2$ we get the sequence

$$\Gamma_1(4,2) = \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\\0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\1&0\\0&1\\0&0\end{bmatrix}$$

defining the following cyclic minimal change sequence on subsets:

$$\gamma(4,2) = 1100, 1001, 1010, 0011, 0101, 0110.$$

The corresponding representation as index sets is

$$\{0,1\}, \{0,3\}, \{0,2\}, \{2,3\}, \{1,3\}, \{1,2\}$$

which is a revolving door algorithm on the set of 2-element subsets of the canonic 4-set.

7 Final Remarks

For enumerative coding and decoding of the introduced minimal change sequence on column reduced Echelon forms algorithms for rank, unrank, and successor (see [8]) can be implemented. Facilitating the recurrence equation for $\Gamma_q(n,k)$ the algorithms for rank, unrank, and successor can be derived quite analogously as it is described in [4].

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