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On the vertex irregular total labeling for subdivision of trees

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Abstract

Let $G = (V, E)$ be a simple, connected and undirected graph with nonempty vertex set $V(G)$ and edge set $E(G)$. We define a labeling $\phi : V \cup E \to \{1, 2, 3, \ldots, k\}$ to be a *vertex irregular total k-labeling* of G if for every two different vertices x and y of G, their weights $w(x)$ and $w(y)$ are distinct, where the weight $w(x)$ of a vertex $x \in V$ is $w(x) = \phi(x) + \sum_{xy \in E(G)} \phi(xy)$. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, denoted by $\text{tvs}(G)$. The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each edge $e = uv$ with the path (u, r_e, v) of length 2, where r_e is a new vertex (called a subdivision vertex) corresponding to the edge e. Let T be a tree. Let $E(T) = E_1 \cup E_2$ be the set of edges in T where $E_1(T) = \{e_1, e_2, \ldots, e_{n_1}\}$ and $E_2(T) = \{e^1, e^2, \ldots, e^{n_2}\}$ are the sets of pendant edges and interior edges, respectively. Let $S(T; r_i; s_j)$ be the subdivision tree obtained from T by replacing each edge $e_i \in E_1$ with a path of length $r_i + 1$ and each edge $e^j \in E_2$ with a path of length $s_j + 1$, for $i \in [1, n_1]$ and $j \in [1, n_2]$. In 2010, Nurdin et al. conjectured that $\text{tvs}(T) = \max\{t_1, t_2, t_3\}$, where $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i + 1) \rceil$ and n_i is the number of vertices of degree $i \in [1, 3]$. In this paper, we show that the total vertex irregularity strength of $S(T; r_i; s_j)$ is equal to t_2, where the value of t_2 is calculated for $S(T; r_i; s_j)$.

1 Introduction

Graph theory has experienced a fast development during the last 60 years. Among all the different kinds of problems that appear while studying graph theory, one that has been growing strongly during the last three decades is the area that studies labelings of graphs. Let us consider a connected and undirected graph $G = (V, E)$ without loops and parallel edges. The set of vertices and edges of this graph are denoted by $V(G)$ and $E(G)$, respectively. Wallis [17] defined a *labeling* of G as a mapping that carries a set of graph elements into a set of integers, called labels. If the labeling on a graph G is applied to the union of its vertex and edge sets, then such a labeling is called a *total labeling.* Baca et al. [5] defined *a vertex irregular total k-labeling* on graph G as a mapping $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that the total vertex-weights $w(x) = \phi(x) + \sum_{xy \in E(G)} \phi(xy)$ are different for all vertices, that is, $w(x) \neq w(y)$ for all different vertices $x, y \in V$. Furthermore, they defined the *total vertex irregularity strength* $\mathrm{tvs}(G)$ of a graph G as the minimum k for which G has a vertex irregular total k-labeling.

Finding the irregularity strength of a graph seems to be rather hard even for simple graphs; see [1, 2, 3, 6, 7, 8, 12]. In [5], Baca et al. proved that for any tree with m pendant vertices and no vertices of degree 2, $\lceil (m + 1)/2 \rceil \leq \mathrm{tvs}(T) \leq m$. Anholcer et al. [4] in 2011, then, improved the lower bound for such a tree T with no isolated vertices by showing that $\mathrm{tvs}(T) = \lceil (m+1)/2 \rceil$. Stronger results for trees were proved by Nurdin et al. [10]. They gave the following theorem.

Theorem 1.1 *[10] Let T be any tree having n_i vertices of degree i, $(i = 1, 2, \ldots, \Delta)$, where $\Delta = \Delta(T)$ is the maximum degree in T. Then*

$$\mathrm{tvs}(T) \geq \max\left\{ \left\lceil \frac{1+n_1}{2} \right\rceil, \left\lceil \frac{1+n_1+n_2}{3} \right\rceil, \ldots, \left\lceil \frac{1+n_1+n_2+\cdots+n_\Delta}{\Delta+1} \right\rceil \right\}.$$

Studying the total vertex irregularity strength of a general tree is an NP-complete problem. So there is no efficient algorithm to determine the total vertex irregularity strength of a general tree. Nurdin et al. [10] conjectured that the total vertex irregularity strength of any tree T is only determined by the number of its vertices of degrees $1, 2$, and 3. More precisely, they gave the following conjecture.

Conjecture 1 *[10] Let T be a tree with maximum degree Δ. Let n_i be the number of vertices of degree $i \leq \Delta$ in T. Then*

$$\mathrm{tvs}(T) = \max\{t_1, t_2, t_3\}, \text{ where } t_i = \left\lceil \frac{1 + \sum_{k=1}^{i} n_k}{(i+1)} \right\rceil \text{ for } i \in \{1, 2, 3\}. \tag{1}$$

For an integer $r \geq 1$, the *subdivision* graph $S(G, r)$ of a graph G is the graph obtained from G by replacing every edge $e = uv$ with a path $(u, x_1, x_2, \ldots, x_r, v)$ of length $r + 1$. The new vertices $x_1, x_2, \ldots, x_r$ are called *subdivision vertices* corresponding to the edge uv. If $r = 1$, then we write $S(G)$ for short. Let T be a tree and let $E(T) = E_1 \cup E_2$ be the set of edges in T where $E_1(T) = \{e_1, e_2, \ldots, e_{n_1}\}$ and $E_2(T) = \{e^1, e^2, \ldots, e^{n_2}\}$ are the sets of pendant edges and interior edges, respectively. Now define $S(T; r_i; s_j)$ to be the subdivision tree obtained from T by replacing each edge $e_i \in E_1$ with a path of length $r_i + 1$ and each edge $e^j \in E_2$ with a path of length $s_j + 1$, for $i \in [1, n_1]$ and $j \in [1, n_2]$. If $r_i = s_j = r$ for all i and j, then we write $S(T; r_i; s_j)$ as $S(T, r)$. Furthermore, if $r_i = s_j = 1$ for all i and j, then we write $S(T; r_i; s_j)$ as $S(T)$.

Consider $S(T; r_i; s_j)$. In order to define the total labeling of $S(T; r_i; s_j)$, we need some additional definitions. A vertex of degree at least 3 in a tree T will be called a *major vertex* of T. Any pendant vertex u of T is said to be a *terminal vertex* of a major vertex v of T if $d(u, v) < d(u, w)$ for every major vertex w of T. The terminal degree of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree.

A *branch path* is a path between a terminal vertex and its exterior major vertex. Furthermore, we define an *interior path* as a path between two major vertices that is passing vertices of degree 2 only. Let $\mathcal{P} = \{P^1, P^2, \ldots, P^{n_1}\}$ be the ordered set of n_1 branch paths where $|P^i| \geq |P^{i+1}|$ for each i, and let $\mathcal{Q} = \{Q^1, Q^2, \ldots, Q^{n_2}\}$ be the ordered set of interior paths with $|Q^j| \geq |Q^{j+1}|$ for each j. See Figure 1 for an illustration.

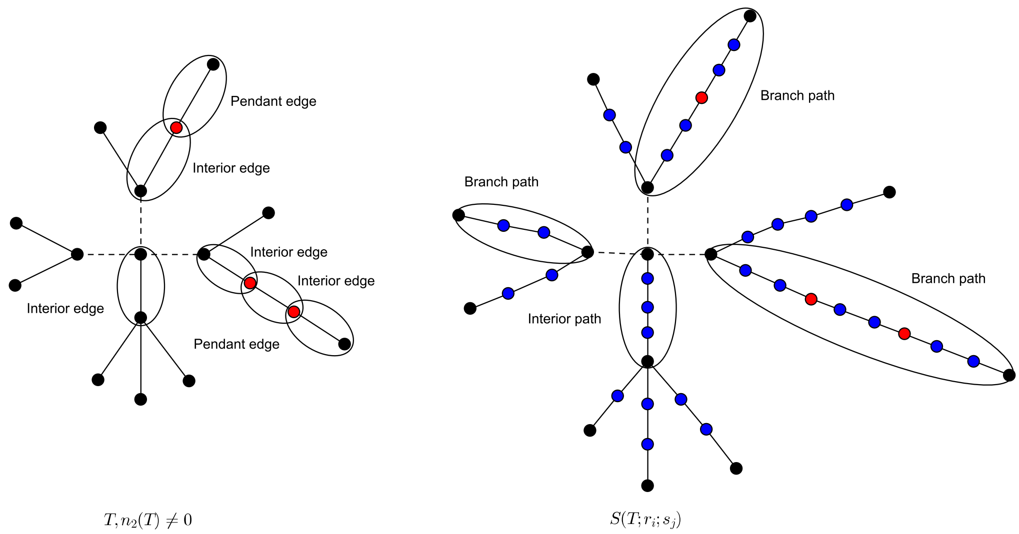

Figure 1: T and $S(T; r_i; s_j)$.

Many researchers restricted their studies in finding the total vertex irregularity strength of trees for certain classes. Susilawati, Baskoro and Simanjuntak in [14, 15] determined the total vertex irregularity strength of trees with maximum degree four or five. They also determined the total vertex irregularity strength for subdivision of several classes of trees such as subdivision of a caterpillar, subdivision of a fire cracker and subdivision of an amalgamation of stars [13]. Recently, they studied the vertex irregularity strength for trees having many vertices of degree 2 [16]. Furthermore, for some particular classes of trees such as paths and caterpillars [10], linear forests [9], firecrackers and banana trees [11], their values are known. These results supported Conjecture 1.

In this paper, we investigate the total vertex irregularity strength of an irregular subdivision of any tree $S(T; r_i; s_j)$. This paper adds further support to Conjecture 1 by showing that such a subdivision of any tree has total vertex irregularity strength equal to t_2. Note that any subdivision of a tree is also a tree.

2 Main Results

The following lemma will be used to prove the next theorem.

Lemma 2.1 *Let T be a tree with p vertices and q edges. If $S(T; r_i; s_j)$ is an irregular subdivision of T where $r_i, s_j \geq 1$ for each i and j, then $t_2 = \max\{t_1, t_2, \ldots, t_\Delta\}$, where all the t_i are calculated for $S(T; r_i; s_j)$.*

Proof. Consider a tree T with p vertices and q edges. Let n_i be the number of vertices of degree i in T. By using the facts that $p = \sum_{i=1}^{\Delta} n_i$ and $q = p - 1$ such that $q = \sum_{i=1}^{\Delta} n_i - 1$, we have:

$$n_1 = 2 + \sum_{i=2}^{\Delta}(i-2)n_i. \tag{2}$$

By substituting the value of n_1 from Equation (2) into Equation (1), we obtain

$$\begin{aligned} t_i &= \left\lceil \frac{1 + \sum_{k=1}^{i} n_k}{(i+1)} \right\rceil = \left\lceil \frac{1 + (2 + \sum_{k=2}^{\Delta}(k-2)n_k) + \sum_{k=2}^{i} n_k}{(i+1)} \right\rceil \\ &= \left\lceil \frac{3 + \left(\sum_{k=2}^{i}(k-2)n_k + \sum_{j=i+1}^{\Delta}(j-2)n_j \right) + \sum_{k=2}^{i} n_k}{(i+1)} \right\rceil \\ &= \left\lceil \frac{3 + \sum_{k=2}^{i}(k-1)n_k + \sum_{j=i+1}^{\Delta}(j-2)n_j}{(i+1)} \right\rceil. \end{aligned} \tag{3}$$

Now we consider $S(T; r_i; s_j)$. Since $r_i, s_j \geq 1$ and each edge of T is subdivided, we have:

$$n_2(S(T; r_i; s_j)) \geq \sum_{i=1}^{\Delta} n_i(T) - 1. \tag{4}$$

After the subdivision of tree T, only the number of vertices of degree 2 (n_2) in $S(T; r_i; s_j)$ is changed. The other values of the n_i ($i \neq 2$) remain the same. Recall equation (1) and calculate all the t_i for $S(T; r_i; s_j)$. By Inequality (4), it is clear that $t_2 \geq t_1$. Now, for $i \geq 3$ and t_i in (3), we will show that $t_2 - t_i$ is nonnegative.

$$
\begin{aligned}
& t_2(S(T;r_i;s_j)) - t_i(S(T;r_i;s_j)) \\
&= \left\lceil \frac{1+n_1+n_2}{3} \right\rceil - \left\lceil \frac{3+\sum_{k=2}^{i}(k-1)n_k + \sum_{j=i+1}^{\Delta}(j-2)n_j}{(i+1)} \right\rceil \\
&= \left\lceil \frac{(i+1)(1+n_1+n_2)}{3(i+1)} \right\rceil - \left\lceil \frac{3(3+\sum_{k=2}^{i}(k-1)n_k + \sum_{j=i+1}^{\Delta}(j-2)n_j)}{3(i+1)} \right\rceil \\
&\geq \left\lceil \frac{(i+1)+(i+1)n_1+(i+1)(\sum_{j=1}^{\Delta} n_j - 1)}{3(i+1)} \right\rceil \\
&\quad - \left\lceil \frac{9+3\sum_{k=2}^{3}(k-1)n_k + 3\sum_{j=4}^{\Delta}(j-2)n_j)}{3(i+1)} \right\rceil \\
&= \left\lceil \frac{2(i+1)n_1+(i+1)\sum_{k=2}^{\Delta} n_k}{3(i+1)} \right\rceil \\
&\quad - \left\lceil \frac{9+3\sum_{k=2}^{3}(k-1)n_k + 3\sum_{j=4}^{\Delta}(j-2)n_j}{3(i+1)} \right\rceil.
\end{aligned}
$$

Since $2(i+1)n_1 + (i+1)\sum_{k=2}^{\Delta} n_k > 9 + 3\sum_{k=2}^{3}(k-1)n_k + 3\sum_{j=4}^{\Delta}(j-2)n_j$, for $i \geq 3$ we have $t_2 - t_i \geq 0$. Hence $t_2 = \max\{t_1, t_2, \ldots, t_\Delta\}$. □

Theorem 2.1 *Let T be a tree with p vertices and q edges. If $S(T;r_i;s_j)$ is an irregular subdivision of T where $r_i, s_j \geq 1$ for each i and j, then* $\mathrm{tvs}(S(T;r_i;s_j)) = t_2$*, where all t_is are calculated for $S(T;r_i;s_j)$.*

Proof. By Lemma 2.1 and Theorem 1.1, we have $\mathrm{tvs}(S(T;r_i;s_j)) \geq t_2$, where t_2 is calculated in $S(T;r_i;s_j)$. To prove the upper bound, we define a total t_2-mapping ϕ of $S(T;r_i;s_j)$ as follows.

Let V_B and E_B be the set of vertices and edges of $S(T;r_i;s_j)$ in branch paths, respectively. Their members are as follows.

$$
\begin{aligned}
V_B &= \{v_{i,j} \mid 1 \leq i \leq n_1, 1 \leq j \leq k_i + 1\} \cup \{w_{t,l} | 3 \leq t \leq \Delta, 1 \leq l \leq n_k\}, \\
E_B &= \{v_{i,j}v_{i,j+1} \mid 1 \leq i \leq n_1, 1 \leq j \leq k_i\} \cup \{v_{i,k_i+1}w_{t,l} | 1 \leq i \leq n_1\}.
\end{aligned}
$$

First, we label the edges in all branch paths by the following steps. Note that k_i is the number of vertices of degree 2 in the branch path P^i starting from the pendant vertex $v_{i,1}$ for $i \in [1, n_1]$. Vertices $w_{t,l}$ are the external major vertices. Order all branch paths in $S(T;r_i;s_j)$ such that $k_1 \geq k_2 \geq \cdots \geq k_{n_1} \geq 1$ and let $k = k_1 + k_2 + \cdots + k_{n_1} + n_1 + 1$.

For $1 \leq j \leq k_1$, we define

$$
\begin{aligned}
a_j &= |\{k_i \mid k_i = j, 1 \leq i \leq n_1\}|, \\
z_0 &= n_1, \\
z_j &= n_1 - \sum_{s=1}^{j} a_s.
\end{aligned}
$$

Label the edges in all branch paths by the following steps.

1. For $1 \le i \le z_0$, $\phi(v_{i,1}v_{i,2}) = i$.
2. For $2 \le j \le k_1$ and $1 \le i \le z_{j-1}$, $\phi(v_{i,j}v_{i,j+1}) = \left\lceil \frac{(1+i+\sum_{r=0}^{j-2} z_r) - \phi(v_{i,j-1}v_{i,j})}{2} \right\rceil$.
3. For $1 \le i \le n_1$, $\phi(v_{i,k_i+1}w_{t,l}) = t_2$ where $v_{i,k_i+1}w_{t,l}$ is an edge for some t, l.

Label the vertices in all branch paths by the following steps.

1. For $1 \le i \le z_0$, $\phi(v_{i,1}) = 1$.
2. For $2 \le j \le k_1 + 1$ and $1 \le i \le z_{j-1}$, $\phi(v_{i,j}) = \left\lfloor \frac{(1+i+\sum_{r=0}^{j-2} z_r) - \phi(v_{i,j-1}v_{i,j})}{2} \right\rfloor$.

Now, consider all the interior paths of $S(T; r_i; s_j)$. Note that l_j is the number of vertices of degree two in the j^{th} interior path Q^j. Order all the interior paths in $S(T; r_i; s_j)$ such that $l_1 \ge l_2 \ge \cdots \ge l_{n_2} \ge 1$.

Let V_I and E_I be the vertex and edge sets of all interior paths, respectively. Their members are as follows.

$$\begin{aligned} V_I &= \{y_{m,n} | \ 1 \le m \le n_2 \text{ and } 1 \le n \le l_m\} \text{ and} \\ E_I &= \{w_{t,l}y_{m,1} | 1 \le m \le n_2\} \cup \{y_{m,n}y_{m,n+1} | \ 1 \le m \le n_2, 1 \le n \le l_m - 1\} \cup \\ &\qquad \{y_{m,l_m}w_{t,l} | 1 \le m \le n_2\}. \end{aligned}$$

See Figure 2 for an illustration.

Label the edges in all interior paths of $S(T; r_i; s_j)$ by the following steps.

For $1 \le m \le n_2 - 1$, let $z_0 = k$ and $z_m = z_{m-1} + l_m$.

1. For $1 \le m \le n_2$, define $\phi(w_{t,l}y_{m,1}) = t_2$ and $\phi(y_{m,l_m}w_{t',l'}) = t_2$ for some t, t', l and l'.
2. For $1 \le m \le n_2$ and $1 \le n \le l_m - 1$, define
$$\phi(y_{m,n}y_{m,n+1}) = \left\lceil \frac{n + z_{m-1} - \phi(y_{m,n-1}y_{m,n})}{2} \right\rceil.$$
Note that $y_{m,0} = w_{t,l}$ for some t, l.

Label the vertices in all interior paths as follows.

For $1 \le m \le n_2$ and $1 \le n \le l_m + 1$, we define
$$\phi(y_{m,n}) = \left\lfloor \frac{n + z_{m-1} - \phi(y_{m,n-1}y_{m,n})}{2} \right\rfloor.$$

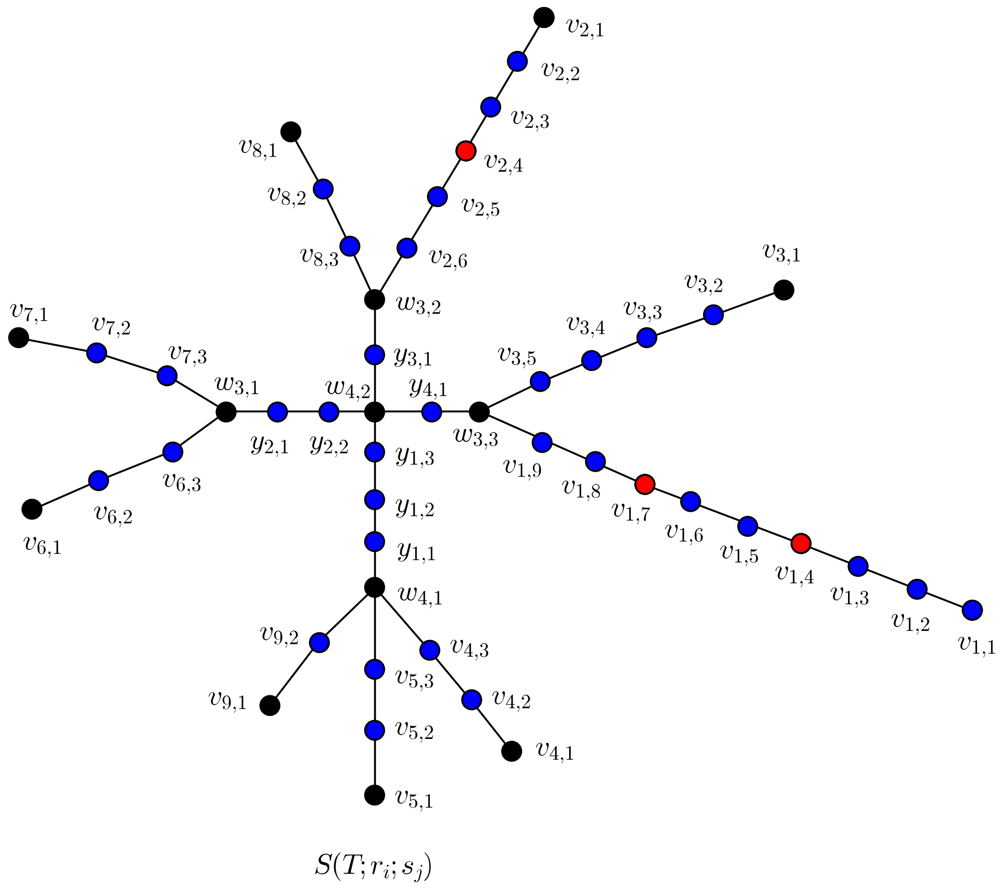

Figure 2: $S(T;r_i;s_j)$.

Label all vertices of degree $k \in \{3,4,\ldots,\Delta\}$ as follows.

For $s \in \{3,4,\ldots,\Delta\}$ and $l = 1,2,\ldots,n_s$, we define $\phi(w_{s,l}) = (n_1+1)+l+n_{s-1}-st_2$.

Under the labeling ϕ, the total weights of all vertices in $S(T;r_i;s_j)$ are described as follows.

- All vertices in V_B admit consecutive weights from 2 to $k+1$.
- All vertices in V_I admit consecutive weights from $k+2$ to $k+\sum_{i=1}^{w} l_i + 2$
- All vertices of the form $w_{s,l}$ have different weights $st_2 + \sum_{i=1}^{n_1} k_i + 1 + n_1$, where $l = 1,2,\ldots,n_s$.

Therefore the proof is complete. $\square$

Now we give an alternative function λ to label $S(T)$ with $n_2(T) = 0$. We define a function $\lambda : V \cup E \to \{1,2,\ldots,t_2\}$. Let $V = V_1 \cup V_2 \cup V_\Delta$ be the set of all vertices of $S(T)$ and let $|V| = n$, where

$$
\begin{aligned}
V_1 &= \{v_{i,j} \mid i = 1, 2, \ldots, n_1, j = 1, 2\} \text{ is the set of all vertices of degree} \\
&\qquad \text{one or two in pendant edges of } S(T); \\
V_2 &= \{y_{m,1} \mid m = 1, 2, \ldots, n - n_1 - 1\} \text{ is the set of all vertices of degree} \\
&\qquad \text{two in interior edge of } S(T); \\
V_\Delta &= \{w_{s,l} \mid 3 \le s \le \Delta, 1 \le l \le n_s\} \text{ is the set of all vertices of degree at least} \\
&\qquad s \ (s \ge 3).
\end{aligned}
$$

See Figure 3 for an illustration.

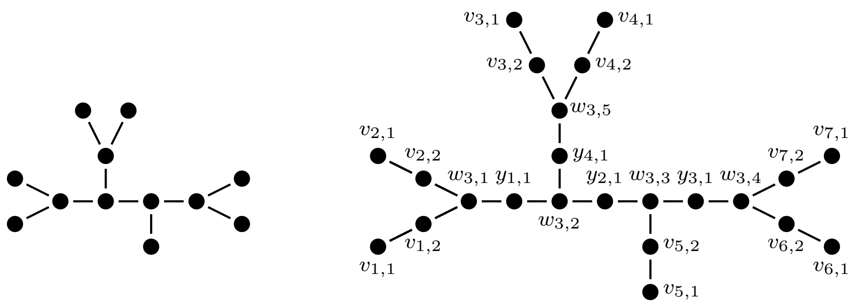

Figure 3: T with $n_2(T) = 0$ and $S(T)$.

Label all the edges of $S(T)$ by the following steps.

1. $\lambda(v_{i,1}v_{i,2}) = \begin{cases} i, & i = 1, 2, \ldots, t_2, \\ t_2, & i = t_2 + 1, \ldots, n_1; \end{cases}$
2. $\lambda\big(v_{i,2}w_{s,l}\big) = t_2$, where $v_{i,2} \sim w_{s,l}$;
3. $\lambda\big(y_{m,1}w_{s,l}\big) = t_2$, where $y_{m,1} \sim w_{s,l}$.

Label all the vertices of $S(T)$ by the following steps.

1. $\lambda(v_{i,1}) = \begin{cases} 1, & i = 1, 2, \ldots, t_2, \\ 2, 3, \ldots, (n_1 - t_2) + 1, & i = t_2 + 1, \ldots, n_i; \end{cases}$
2. $\lambda(v_{i,2}) = \begin{cases} (n_1 + 1) + i - (t_2 + i), & i = 1, 2, \ldots, t_2, \\ (n_1 + 1) + i - 2t_2, & i = t_2 + 1, \ldots, (n_1 - t_2); \end{cases}$
3. $\lambda(y_{m,1}) = 2n_1 + 1 + m - 2t_2$;
4. $\lambda(w_{s,l}) = (n_1 + 2) + s + |V_{s-1}| - st_2$ where $s = 3, 4, \ldots, \Delta$ and $l = 1, 2, \ldots, n_s$.

Under the labeling λ, the total weights of the vertices are described as follows.

- All vertices in V_1 admit consecutive weights from 2 to $2n_1 + 1$.

- All vertices in V_2 admit consecutive weights from $2n_1+2$ to $2n_1+2+\sum_{i=1}^{n_1} k_i + 1 + n_1$.
- All vertices $w_{s,l}$ have different weights from $st_2 + l$ where $l = 1, 2, \ldots, n_s$.

We would like to conclude this paper by providing the following algorithm to check whether a tree T has the total vertex irregularity strength equal to t_2.

Algorithm

1. Consider the tree T.
2. Consider $V'(T) = \{v_1, \ldots, v_{n_3+\cdots+n_\Delta}\}$, the set of vertices of degree at least 3.
3. For $i = 1, 2, \ldots, n_3 + \cdots + n_\Delta$,
   if v_i is not adjacent to a vertex of degree 2, then:
   a. T is not an $S(T_0; r_i, s_j)$ for any tree T_0.
   b. End algorithm.
4. T is an $S(T_0; r_i; s_j)$ for some T_0, and $\mathrm{tvs}(T) = t_2$.

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