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Signed edge domination numbers of complete tripartite graphs: part 2

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In memory of Anne Penfold Sreet

Abstract

The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having an end-vertex in common with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1, 1\}$. If $\sum_{x\in N[e]} f(x) \geq 1$ for each edge $e \in E(G)$, then f is called a signed edge dominating function of G. The signed edge domination number of G is the minimum weight of a signed edge dominating function of G. In this paper, we find the signed edge domination number of the complete tripartite graph $K_{m,n,p}$, where $1 \leq m \leq n$ and $p \geq m + n$. This completes the search for the signed edge domination numbers of the complete tripartite graphs.

1 Introduction

Let G be a simple non-empty graph with vertex set $V(G)$ and edge set $E(G)$. We use [4] for terminology and notation not defined here. Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \to \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{x\in S} f(x)$. If $S = N_G[e]$ for some $e \in E$, then we denote $f(S)$ by $f[e]$. The *weight* of vertex $v \in V(G)$ is defined by $f(v) = \sum_{e\in E(v)} f(e)$, where $E(v)$ is the set of all edges at vertex v. A function $f : E(G) \to \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G if $f[e] \geq 1$ for each edge $e \in E(G)$. The SEDF of a graph was first defined in [5]. The *weight* of f, denoted $w(f)$, is defined to be $w(f) = \sum_{e\in E(G)} f(e)$. *The signed edge domination number* (SEDN) $\gamma'_s(G)$ is defined as $\gamma'_s(G) = \min\{w(f) \mid f \text{ is an SEDF of } G\}$. An SEDF f is called a $\gamma'_s(G)$-function if $\omega(f) = \gamma'_s(G)$. In [5] it was conjectured that $\gamma'_s(G) \leq |V(G)| - 1$ for every graph G of order at least 2.

The signed edge domination numbers of the complete graph K_n and the complete bipartite graph $K_{m,n}$ were determined in [6] and [1], respectively. In [3], the signed edge domination number of $K_{m,n,p}$ was calculated when $1 \leq m \leq n \leq p \leq m+n$. For completeness, we state the main theorem of [3].

Theorem 1.1. Let m, n and p be positive integers and $m \leq n \leq p \leq m+n$. Let $(m, n, p) \notin \{(1, 1, 1), (2, 3, 5)\}$.

A. Let m, n and p be even.

1. If $m+n+p \equiv 0 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+n+p)/2$.
2. If $m+n+p \equiv 2 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+n+p+2)/2$.

B. Let m, n and p be odd.

1. If $m+n+p \equiv 1 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+n+p+1)/2$.
2. If $m+n+p \equiv 3 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+n+p+3)/2$.

C. Let m, n be odd and p be even or m, n be even and p be odd.

1. If $m+n \equiv 0 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+n)/2 + p + 1$.
2. If $m+n \equiv 2 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+n)/2 + p$.

D. Let m, p be odd and n be even or m, p be even and n be odd.

1. If $m+p \equiv 0 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+p)/2 + n + 1$.
2. If $m+p \equiv 2 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (m+p)/2 + n$.

E. Let n, p be odd and m be even or n, p be even and m be odd.

1. If $n+p \equiv 0 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (n+p)/2 + m + 1$.
2. If $n+p \equiv 2 \pmod 4$, then $\gamma'_s(K_{m,n,p}) = (n+p)/2 + m$.

In addition, $\gamma'_s(K_{1,1,1}) = 1$ and $\gamma'_s(K_{2,3,5}) = 5$.

In this paper, we find the signed edge domination number of the complete tripartite graph $K_{m,n,p}$, when $m, n \geq 1$ and $p \geq m+n$. In Section 2, we present some crucial results which will be employed in the rest of this paper. In Section 3, we prove that if f is a $\gamma'_s(K_{m,n,p})$-function, then $f(w) \geq -1$ for every vertex w in the largest partite set of $K_{m,n,p}$. In Section 4, we present general constructions for the SEDFs of $K_{m,n,p}$ with minimum weight. In Section 5, we calculate the signed edge domination numbers of $K_{1,n,p}$ and $K_{2,2,p}$, where p is even. These cases do not follow the constructions given in Section 4. In addition, we notice that $\gamma'_s(K_{1,n,n+3}) = 2n+3$ when n is odd. So there is an infinite family of graphs which achieve the upper bound given in Xu's conjecture (see [5]). The main theorem of this paper is presented in Section 6.

2 Preliminary results

Let f be a SEDF of G. An edge $e \in E(G)$ is called a *negative edge* (*positive edge*) if $f(e) = -1$ ($f(e) = 1$). Let uv be an edge of G and suppose that x, y are the number of negative edges at vertices u and v, respectively. Then

$$f[uv] = \deg(u) + \deg(v) - 2x - 2y - f(uv) \geq 1. \tag{1}$$

Hence, if $e = uv$, then $f(u) + f(v) \geq 0$ for every edge $e \in E(G)$. In addition, if $f(u) + f(v) = 0$ or 1, then $f(e) = -1$.

The following result can be found in [2]. Since there are typographical errors in the proof given in [2] we modify the proof and present it here.

Lemma 2.1. Let G be a graph, $u, v \in V(G)$ and $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Let f be a SEDF of G. Then there exists a SEDF of G, say g, with $w(g) = w(f)$ such that the difference between the number of negative edges at u and at v is at most 1.

Proof. Let $V(G) = \{v_1 = u, v_2 = v, v_3, \ldots, v_n\}$ and let x_i, $1 \leq i \leq n$, be the number of edges e at v_i with $f(e) = -1$.

If $x_1 \leq x_2 - 2$, then there exists a vertex v_ℓ such that $f(v_1v_\ell) = 1$ and $f(v_2v_\ell) = -1$ for some $\ell \in \{3, 4, \ldots, n\}$. Define $g : E(G) \to \{-1, 1\}$ by $g(v_1v_\ell) = -1$, $g(v_2v_\ell) = 1$ and $g(e) = f(e)$ for $e \in E(G) \setminus \{v_1v_\ell, v_2v_\ell\}$. Let y_i, $1 \leq i \leq n$, be the number of edges e at v_i with $g(e) = -1$. Obviously, $y_1 = x_1 + 1$, $y_2 = x_2 - 1$ and $y_i = x_i$ for $3 \leq i \leq n$. In addition, $w(g) = w(f)$. We prove that g is a SEDF of G.

If $v_1v_j \in E(G)$ and $j \notin \{1, 2, \ell\}$

$$\begin{aligned}
g[v_1v_j] &= \deg(v_1) + \deg(v_j) - 2y_1 - 2y_j - g(v_1v_j) \\
&= \deg(v_1) + \deg(v_j) - 2x_1 - 2 - 2x_j - f(v_1v_j) \\
&\geq \deg(v_2) + \deg(v_j) - 2x_2 + 2 - 2x_j - f(v_1v_j) \\
&\geq \deg(v_2) + \deg(v_j) - 2x_2 - 2x_j - f(v_2v_j) \\
&\geq 1 \qquad \text{by (1)},
\end{aligned}$$

$$\begin{aligned}
g[v_2v_j] &= \deg(v_2) + \deg(v_j) - 2y_2 - 2y_j - g(v_2v_j) \\
&= \deg(v_2) + \deg(v_j) - 2x_2 + 2 - 2x_j - f(v_2v_j) \\
&\geq 3 \qquad \text{by (1)}.
\end{aligned}$$

Similarly, for $e \in E(G) \setminus \{v_1v_j, v_2v_j \mid j \notin \{1, 2, \ell\}\}$, we obtain $g[e] = f[e] \geq 1$. In addition, if $v_1v_2 \in E(G)$, then

$$\begin{aligned}
g[v_1v_2] &= \deg(v_1) + \deg(v_2) - 2y_1 - 2y_2 - g(v_1v_2) \\
&= \deg(v_1) + \deg(v_2) - 2x_1 - 2x_2 - f(v_1v_2) \\
&\geq 1 \qquad \text{by (1)}.
\end{aligned}$$

Hence, g is a SEDF of G. If g satisfies the required condition, the proof is complete. Otherwise, by repeating this process we can obtain the required function. □

Corollary 2.2. Let G be a complete multipartite graph. There exists a $\gamma_s'(G)$-function f such that the difference between the number of negative edges at every two vertices in the same partite set is at most 1.

3 SEDFs of complete tripartite graphs with vertices of negative weight

Consider the complete tripartite graph $K_{m,n,p}$ with partite sets U, V and W. Throughout this section we assume $|U| = m$, $|V| = n$ and $|W| = p$, where $1 \leq m \leq n \leq p$. In this section, we study $\gamma'_s(K_{m,n,p})$-functions with the property that the difference between the number of negative edges at every two vertices in the same partite set is at most 1 (see Corollary 2.2).

Lemma 3.1. Let m, n, p be all even or all odd and $1 \leq m \leq n \leq p$. If f is a $\gamma'_s(K_{m,n,p})$-function such that the difference between the number of negative edges at every two vertices in the same partite set is at most 1, then $f(w) \geq 0$ for every vertex of $w \in W$.

Proof. The proof is by contradiction. Assume $f(w) = -2k$, where $1 \leq k \leq (m+n)/2$, for some $w \in W$, and $f(w') \geq -2k$ for all $w' \in W$. Then there are $(m+n+2k)/2$ negative edges and $(m+n-2k)/2$ positive edges at w. Therefore the weight of $(m+n+2k)/2$ vertices in $U \cup V$ must be at least $2k$ and the weight of $(m+n-2k)/2$ vertices in $U \cup V$ must be at least $2k+2$. Since $m \leq n$ and f is a $\gamma'_s(K_{m,n,p})$-function, we can assume $f(u) = 2k$ for every $u \in U$. So there are $(n+p-2k)/2$ negative edges at every vertex $u \in U$. Let $U \cup V_1$, where $V_1 \subseteq V$, consist of vertices of weight $2k$ and let $V_2 = V \setminus V_1$ consist of vertices of weight $2k+2$. Since $(m+n+2k)/2 > m$ and f is a $\gamma'_s(K_{m,n,p})$-function, it follows that there is a vertex $v \in V_1$ of weight $2k$. Indeed, $|V_1| = (n-m+2k)/2$. Therefore there are $(m+p-2k)/2$ negative edges at v. Let $W_1 \subseteq W$ consist of vertices of weight $-2k$. Since every vertex in V_1 must be joined to every vertex in W_1 with a negative edge by (1), it follows that $|W_1| \leq (m+p-2k)/2$. Hence, $|W \setminus W_1| \geq (p-m+2k)/2$ and every vertex in this set has weight $-2k+2$. Note that $(n+p-2k)/2 - (m+p-2k)/2 = (n-m)/2$. Let W_2 be a subset of $W \setminus W_1$ with $(n-m)/2$ vertices and the edges between W_2 and U are all negative edges. Let $W_3 = W \setminus (W_1 \cup W_2)$. Then

$$|W_3| \geq p - [(m+p-2k)/2 + (n-m)/2] = (p-n+2k)/2 \geq 1.$$

Now let $w' \in W_3$. Then the edges between w' and $U \cup V_1$ are all positive edges. Therefor the number of negative edges at w' is at most $(n+m-2k)/2$. On the other hand, for every vertex $w \in W_1$ there are $(n+m+2k)/2$ negative edges. Since $k \geq 1$, the difference between the number of negative edges at w and at w' is $2k \geq 2$, which is a contradiction. ☐

The proof of the following result is similar to the proof of Lemma 3.1.

Lemma 3.2. Let m, n be even and p be odd or m, n be odd and p be even, where $1 \leq m \leq n < p$. If f is a $\gamma'_s(K_{m,n,p})$-function such that the difference between the number of negative edges at every two vertices in the same partite set is at most 1, then $f(w) \geq 0$ for every vertex of $w \in W$.

Proof. The proof is by contradiction. Let m, n be even and p be odd. (The case m, n odd and p even is similar.) Assume $f(w) = -2k$, where $1 \le k \le (m+n)/2$, for some $w \in W$, and $f(w') \ge -2k$ for all $w' \in W$. Then there are $(m+n+2k)/2$ negative edges and $(m+n-2k)/2$ positive edges at w. Therefore the weight of $(m+n+2k)/2$ vertices in $U \cup V$ must be at least $2k+1$ and the weight of $(m+n-2k)/2$ vertices in $U \cup V$ must be at least $2k+3$. Since $m \le n$ and f is a $\gamma'_s(K_{m,n,p})$-function, we can assume $f(u) = 2k+1$ for every $u \in U$. So there are $(n+p-2k-1)/2$ negative edges at every vertex $u \in U$. Let $U \cup V_1$, where $V_1 \subseteq V$, consist of vertices of weight $2k+1$ and let $V_2 = V \setminus V_1$ consist of vertices of weight $2k+3$. Since $(m+n+2k)/2 > m$ and f is a $\gamma'_s(K_{m,n,p})$-function, it follows that there is a vertex $v \in V_1$ of weight $2k+1$. Indeed, $|V_1| = (n-m+2k)/2$. Therefore there are $(m+p-2k-1)/2$ negative edges at v. Let $W_1 \subseteq W$ consist of vertices of weight $-2k$. Since every vertex in V_1 must be joined to every vertex in W_1 with a negative edge by (1), it follows that $|W_1| \le (m+p-2k-1)/2$. Hence, $|W \setminus W_1| \ge (p-m+2k+1)/2$ and every vertex in this set has weight $-2k+2$. Note that $(n+p-2k-1)/2-(m+p-2k-1)/2 = (n-m)/2$. Let W_2 be a subset of $W \setminus W_1$ with $(n-m)/2$ vertices and the edges between W_2 and U are all negative edges. Let $W_3 = W \setminus (W_1 \cup W_2)$. Then

$$|W_3| \ge p - [(m+p-2k-1)/2 + (n-m)/2] = (p-n+2k+1)/2 \ge 1.$$

Now let $w' \in W_3$. Then the edges between w' and $U \cup V_1$ are all positive edges. Therefor the number of negative edges at w' is at most $(n+m-2k)/2$. On the other hand, for every vertex $w \in W_1$ there are $(n+m+2k)/2$ negative edges. Since $k \ge 1$, the difference between the number of negative edges at w and at w' is $2k \ge 2$, which is a contradiction. ☐

Lemma 3.3. Let m be odd and n, p be even or m, p be even and n be odd, where $1 \le m < n \le p$. If f is a $\gamma'_s(K_{m,n,p})$-function such that the difference between the number of negative edges at every two vertices in the same partite set is at most 1, then $f(w) \ge -1$ for every vertex of $w \in W$.

Proof. The proof is by contradiction. Let m be odd and n, p be even. (The case m, p even and n odd is similar.) Assume $f(w) = -2k-1$, where $1 \le k \le (m+n-1)/2$, for some $w \in W$, and $f(w') \ge -2k-1$ for all $w' \in W$. Then there are $(m+n+2k+1)/2$ negative edges and $(m+n-2k-1)/2$ positive edges at w. Therefore the weight of $(m+n+2k+1)/2$ vertices in $U \cup V$ must be at least $2k+1$ and the weight of $(m+n-2k-1)/2$ vertices in $U \cup V$ must be at least $2k+3$. Since $m < n$ and f is a $\gamma'_s(K_{m,n,p})$-function, we can assume $f(u) = 2k+2$ for every vertex $u \in U$. So there are $(n+p-2k-2)/2$ negative edges at every vertex $u \in U$. Let $V_1 \subset V$ consist of vertices of weight $2k+1$ and let $V_2 = V \setminus V_1$ consist of vertices of weight $2k+3$. Since $(m+n+2k+1)/2 > m$ and f is a $\gamma'_s(K_{m,n,p})$-function, it follows that there is a vertex $v \in V_1$ of weight $2k+1$. Indeed, $|V_1| = (n-m+2k+1)/2$. Therefore there are $(m+p-2k-1)/2$ negative edges at v. Let $W_1 \subseteq W$ consist of vertices of weight $-2k-1$. Since every vertex in V_1 must be joined to every vertex in W_1 with a negative edge by (1), it follows that $|W_1| \le (m+p-2k-1)/2$. Hence, $|W \setminus W_1| \ge (p-m+2k+1)/2$ and every vertex in this set has weight $-2k+1$.

Note that $(n+p-2k-2)/2-(m+p-2k-1)/2=(n-m-1)/2$. Let W_2 be a subset of $W \setminus W_1$ with $(n-m-1)/2$ vertices and the edges between W_2 and U are all negative edges. Let $W_3 = W \setminus (W_1 \cup W_2)$. Then

$$|W_3|d \geq p - [(m+p-2k-1)/2 + (n-m-1)/2] = (p-n+2k+2)/2 \geq 2.$$

Now let $w' \in W_3$. Then the edges between w' and $U \cup V_1$ are all positive edges. Therefor the number of negative edges at w' is at most $(n+m-2k-1)/2$. On the other hand, for every vertex $w \in W_1$ there are $(n+m+2k+1)/2$ negative edges. Since $k \geq 1$, the difference between the number of negative edges at w and at w' is $2k+1 \geq 3$, which is a contradiction. □

The proof of the following result is similar to the proof of Lemma 3.3.

Lemma 3.4. Let m, p be odd and n be even or m be even and n, p be odd, where $1 \leq m < n \leq p$. If f is a $\gamma'_s(K_{m,n,p})$-function such that the difference between the number of negative edges at every two vertices in the same partite set is at most 1, then $f(w) \geq -1$ for every vertex of $w \in W$.

Proof. The proof is by contradiction. Let m, p be odd and n be even. (The case m even and n, p odd is similar.) Assume $f(w) = -2k-1$, where $1 \leq k \leq (m+n-1)/2$, for some $w \in W$, and $f(w') \geq -2k-1$ for all $w' \in W$. Then there are $(m+n+2k+1)/2$ negative edges and $(m+n-2k-1)/2$ positive edges at w. Therefore the weight of $(m+n+2k+1)/2$ vertices in $U \cup V$ must be at least $2k+1$ and the weight of $(m+n-2k-1)/2$ vertices in $U \cup V$ must be at least $2k+3$. Since $m < n$ and f is a $\gamma'_s(K_{m,n,p})$-function, we can assume $f(u) = 2k+1$ for every vertex $u \in U$. So there are $(n+p-2k-1)/2$ negative edges at every vertex $u \in U$. Let $V_1 \subset V$ consist of vertices of weight $2k+2$ and let $V_2 = V \setminus V_1$ consist of vertices of weight $2k+4$. Since $(m+n+2k+1)/2 > m$ and f is a $\gamma'_s(K_{m,n,p})$-function, it follows that there is a vertex $v \in V_1$ of weight $2k+2$. Indeed, $|V_1| = (n-m+2k+1)/2$. Therefore there are $(m+p-2k-2)/2$ negative edges at v. Let $W_1 \subseteq W$ consist of vertices of weight $-2k-1$. Since every vertex in V_1 must be joined to every vertex in W_1 with a negative edge by (1), it follows that $|W_1| \leq (m+p-2k-2)/2$. Hence, $|W \setminus W_1| \geq (p-m+2k+2)/2$ and every vertex in this set has weight $-2k+1$.

Note that $(n+p-2k-1)/2-(m+p-2k-2)/2=(n-m+1)/2$. Let W_2 be a subset of $W \setminus W_1$ with $(n-m+1)/2$ vertices and the edges between W_2 and U are all negative edges. Let $W_3 = W \setminus (W_1 \cup W_2)$. Then

$$|W_3| \geq p - [(m+p-2k-2)/2 + (n-m+1)/2] = (p-n+2k+1)/2 \geq 1.$$

Now let $w' \in W_3$. Then the edges between w' and $U \cup V_1$ are all positive edges. Therefor the number of negative edges at w' is at most $(n+m-2k-1)/2$. On the other hand, for every vertex $w \in W_1$ there are $(n+m+2k+1)/2$ negative edges. Since $k \geq 1$, the difference between the number of negative edges at w and at w' is $2k+1 \geq 3$, which is a contradiction. □

4 The SEDN of $K_{m,n,p}$

Consider the complete tripartite graph $K_{m,n,p}$ whose partite sets are U, V and W. Throughout this section we assume $|U| = m$, $|V| = n$ and $|W| = p$, where m, n and p are positive integers, $m \leq n$ and $p \geq m+n$. In this section we compute the signed edge domination number of $K_{m,n,p}$, where $m \geq 2$ and $(m, n) \neq (2, 2)$ if p is odd.

Proposition 4.1. Let m, n and p be even and $p \geq m+n$. Then $\gamma_s'(K_{m,n,p}) = m+n$.

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W. By assumption

$$m(n+p-2)/2 + n(m+p-2)/2 - p(m+n)/2 = mn - m - n$$

is even. First we label $(mn - m - n)/2$ edges between U and V with -1 in the following way. Label an edge uv, where $u \in U$ and $v \in V$, with -1 if

1. the total number of negative edges between U and V is less than $(mn-m-n)/2$,
2. the number of negative edges at u is less than $(n+p-2)/2$,
3. the number of negative edges at v is less than $(m+p-2)/2$,
4. the number of negative edges at u is less than or equal to the number of negative edges at u' for every $u' \in (U \setminus \{u\})$, and
5. the number of negative edges at v is less than or equal to the number of negative edges at v' for every $v' \in (V \setminus \{v\})$.

Then we label $p(m+n)/2$ edges between $U \cup V$ and W with -1 in a similar fashion described above. An edge rw, where $r \in (U \cup V)$ and $w \in W$ is labelled by -1 if

1. the number of negative edges between $U \cup V$ and W is less than $p(m+n)/2$,
2. the number of negative edges at r is less than $(n+p-2)/2$ if $r \in U$,
3. the number of negative edges at r is less than $(m+p-2)/2$ if $r \in V$,
4. the number of negative edges at w is less than $(m+n)/2$,
5. the number of negative edges at r is less than or equal to the number of negative edges at u for every $u \in (U \setminus \{r\})$ if $r \in U$,
6. the number of negative edges at r is less than or equal to the number of negative edges at v for every $v \in (V \setminus \{r\})$ if $r \in V$,
7. the number of negative edges at w is less than or equal to the number of negative edges at w' for every $w' \in (W \setminus \{w\})$.

Then there are exactly $((p+n)/2)-1$, $((p+m)/2)-1$ and $(m+n)/2$ negative edges at every vertex in U, V and W, respectively. Label the remaining edges of $K_{m,n,p}$ by $+1$. Then all vertices in $U \cup V$ have weight 2 and all the vertices in W have weight zero. Hence, this labeling defines a SEDF f of $K_{m,n,p}$ by (1), and $\omega(f) = m+n$. Note that since the weight of every vertex of W is zero, no vertices in $U \cup V$ can have weight zero by (1). Now by Lemma 3.1 and the facts that $f(W) = 0$ and $f(r) = 2$ for every $r \in U \cup V$, it follows that $\gamma'_s(K_{m,n,p}) = m+n$. □

Proposition 4.2. Let m, n and p be odd, $m, n \geq 3$ and $p \geq m+n+1$. Then $\gamma'_s(K_{m,n,p}) = m+n+1$.

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W. By assumption,

$$m(n+p-2)/2 + n(m+p-2)/2 - p(m+n)/2 = mn - m - n$$

is odd. In addition, $(n+p-2)/2$, $(m+p-2)/2$ and $(m+n)/2$ are all odd, or two are even and one is odd. Hence, there is no graph whose m vertices have degree $(n+p-2)/2$, n vertices have degree $(n+p-2)/2$ and p vertices have degree $(m+n)/2$. On the other hand,

$$\begin{aligned}&m(n+p-2)/2 + n(m+p-2)/2 - (p-1)(m+n)/2\\ &-(m+n-2)/2 = mn - m - n + 1,\end{aligned}$$

is an even number. We label $(mn-m-n+1)/2$ edges between U and V and $(p-1)(m+n)/2 + (m+n-2)/2$ edges between $U \cup V$ and W with -1 in a similar fashion described in Proposition 4.1. Then there are $(n+p-2)/2$ negative edges at each vertex of U, $(m+p-2)/2$ negative edges at each vertex of V and $(m+n)/2$ negative edges at each vertex of W except one vertex which is incident with $(m+n-2)/2$ negative edges. We label the remaining edges of $K_{m,n,p}$ with $+1$. Then the weight of vertices in $U \cup V$ are all 2 and the weight of vertices in W are all zero except one vertex of W whose weight is 2. Hence, this labeling defines a SEDF f of $K_{m,n,p}$ by (1), and $\omega(f) = m+n+1$.

Note that since the weight of every vertex of W is zero, no vertices in $U \cup V$ can have weight zero by (1). Now by Lemma 3.1 and the facts that $f(W) = 2$ and $f(r) = 2$ for every $r \in U \cup V$, it follows that $\gamma'_s(K_{m,n,p}) = m+n+1$. □

Proposition 4.3. Let $m, n \geq 3$ be odd, p be even and $p \geq m+n$. Then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m+3n+2}{2}$ if $m+n \equiv 0 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m+3n}{2}$ if $m+n \equiv 2 \pmod 4$.

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W.

Case 1. $m+n \equiv 0 \pmod 4$.

By assumption,

$$\begin{aligned}&(m-1)(n+p-3)/2 + (n+p-5)/2 + n(m+p-3)/2\\ &-p(m+n)/2 = mn - 1 - (3m+3n)/2\end{aligned}$$

is even. Label $(1/2)[mn-1-(3m+3n)/2]$ edges between U and V and $p(m+n)/2$ edges between $U \cup V$ and W with -1 in a similar fashion described in the proof of Proposition 4.1. Then every vertex in U is incident with $(n+p-3)/2$ negative edges except one vertex which is incident with $(n+p-5)/2$ negative edges. Every vertices in V is incident with $(m+p-3)/2$ negative edges and every vertex in W is incident with $(m+n)/2$ negative edges. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in $U \cup V$ are all 3 except one vertex of U whose weight is 5, and the weight of vertices in W are all zero. Hence, this labeling defines a SEDF f with $w(f) = (3m + 3n + 2)/2$.

Note that since the weight of every vertex of W is zero, no vertices in $U \cup V$ can have weight one by (1). Now by Lemma 3.2 and the facts that $f(W) = 0$ and $f(r) = 3$ for every $r \in U \cup V$ except one vertex which has weight 5, it follows that $\gamma'_s(K_{m,n,p}) = (3m + 3n + 2)/2$.

Case 2. $m + n \equiv 2 \pmod 4$.
By assumption,

$$m(p + n - 3)/2 + n(m + p - 3)/2 - p(m + n)/2 = mn - (3m + 3n)/2$$

is even. Label $(1/2)[mn - (3m + 3n)/2]$ edges between U and V and $p(m + n)/2$ edges between $U \cup V$ and W with -1 in a similar fashion described in the proof of Proposition 4.1. Then every vertex in U is incident with $(n+p-3)/2$ negative edges, every vertex in V is incident with $(m + p - 3)/2$ negative edges and every vertex in W is incident with $(m + n)/2$ negative edges. Label the remaining edges of $K_{n,m,p}$ with $+1$. Then the weight of vertices in $U \cup V$ are all 3 and the weight of vertices in W are all zero. Hence, this labeling defines a SEDF f with $w(f) = (3m + 3n)/2$.

Note that since the weight of every vertex of W is zero, no vertices in $U \cup V$ can have weight one by (1). Now by Lemma 3.2 and the facts that $f(W) = 0$ and $f(r) = 3$ for every $r \in U \cup V$, it follows that $\gamma'_s(K_{m,n,p}) = (3m + 3n)/2$. □

Proposition 4.4. Let m and n be even, $(m, n) \neq (2, 2)$, p be odd and $p \geq m+n+1$. Then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 3n}{2}$ if $m + n \equiv 0 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 3n + 2}{2}$ if $m + n \equiv 2 \pmod 4$.

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W.
Case 1. $m + n \equiv 0 \pmod 4$.
By assumption,

$$m(n + p - 3)/2 + n(m + p - 3)/2 - p(m + n)/2 = mn - (3m + 3n)/2$$

is even. Label $(1/2)[mn-3(m+n)/2]$ edges between U and V and $p(m+n)/2$ edges between $U \cup V$ and W with -1 as described in the proof of Proposition 4.1. Then every vertex in U is incident with $(n + p - 3)/2$ negative edges, every vertex in V is incident with $(m+p-3)/2$ negative edges and every vertex in W is incident with $(m+n)/2$ negative edges. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in $U \cup V$ are all 3 and the weight of vertices in W are all zero. Hence, this labeling defines a SEDF f with $w(f) = (3m+3n)/2$.

Note that since the weight of every vertex of W is zero, no vertices in $U \cup V$ can have weight one by (1). Now by Lemma 3.2 and the facts that $f(W) = 0$ and $f(r) = 3$ for every $r \in U \cup V$, it follows that $\gamma'_s(K_{m,n,p}) = (3m+3n)/2$.

Case 2. $m+n \equiv 2 \pmod 4$.
By assumption,

$$\begin{aligned}&(m-1)(n+p-3)/2 + (n+p-5)/2 + n(m+p-3)/2\\ &-p(m+n)/2 = mn-1-(3m+3n)/2\end{aligned}$$

is even. Label $(1/2)[mn-1-3(m+n)/2]$ edges between U and V and $p(m+n)/2$ edges between $U \cup V$ and W with -1 as described in the proof of Proposition 4.1. Then every vertex in U is incident with $(n+p-3)/2$ negative edges except one vertex which is incident with $(n+p-5)/2$ negative edges, every vertex in V is incident with $(m+p-3)/2$ negative edges, and every vertex in W is incident with $(m+n)/2$ negative edges. Label the remaining edges of $K_{n,m,p}$ with $+1$. Then the weight of vertices in $U \cup V$ are all 3 except one vertex whose weight is 5 and the weight of vertices in W are all zero. Hence, this labeling defines a SEDF f with $w(f) = (3m+3n+2)/2$.

Note that since the weight of every vertex of W is zero, no vertices in $U \cup V$ can have weight one by (1). Now by Lemma 3.2 and the facts that $f(W) = 0$, $f(u) = 3$ for every vertex u in U except one vertex whose weight is 5, and $f(v) = 3$ for every $v \in V$, it follows that $\gamma'_s(K_{m,n,p}) = (3m+3n+2)/2$. □

Proposition 4.5. Let m be odd, n, p be even, $3 \le m < n$ and $p \ge m+n+1$. Then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m+2n+1}{2}$ if $m \equiv 1 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m+2n-1}{2}$ if $m \equiv 3 \pmod 4$

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W.
Case 1. $m \equiv 1 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n+p-2)/2 + ((n-m-1)/2)(m+p-1)/2\\ &+((n+m+1)/2)(m+p-3)/2 - (p/2)(m+n+1)/2\\ &-(p/2)(m+n-1)/2 = mn-n-(3m+1)/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1| = (n-m-1)/2$ and $|V_2| = (n+m+1)/2$. Also partition W into W_1 and W_2 with $|W_1| = |W_2|$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn-n-(3m+1)/2]$ edges between U and V and $(p/2)(m+n+1)/2 + (p/2)(m+n-1)/2$ edges between $U \cup V$ and W

with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-2)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-1)/2$ and $(m+p-3)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 2, the weight of vertices in V_1, V_2 are all $1, 3$, respectively, and the weight of vertices in W_1 are all -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f) = (3m + 2n + 1)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U \cup V$ can have weight zero or -1 by (1). Now by Lemma 3.3 and the facts that $f(W) = 0$, $f(u) = 2$ for every $u \in U$, $f(v) = 1$ for every $v \in V_1$ and $f(v) = 3$ for every $v \in V_2$, it follows that $\gamma'_s(K_{m,n,p}) = (3m + 2n + 1)/2$.

Case 2. $m \equiv 3 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n+p-2)/2 + ((n-m+1)/2)(m+p-1)/2\\&+((n+m-1)/2)(m+p-3)/2 - (p/2)(m+n+1)/2\\&-(p/2)(m+n-1)/2 = mn - n - (3m-1)/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1| = (n-m+1)/2$ and $|V_2| = (n+m-1)/2$. Also partition, W into W_1 and W_2 with $|W_1| = |W_2|$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn - n - (3m-1)/2]$ edges between U and V and $(p/2)(m+n+1)/2 + (p/2)(m+n-1)/2$ edges between $U \cup V$ and W with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-2)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-1)/2$ and $(m+p-3)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 2, the weight of vertices in V_1, V_2 are all $1, 3$, respectively, and the weight of vertices in W_1 are all -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f) = (3m + 2n - 1)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U \cup V$ can have weight zero or -1 by (1). Now by Lemma 3.3 and the facts that $f(W) = 0$, $f(u) = 2$ for every $u \in U$, $f(v) = 1$ for every $v \in V_1$ and $f(v) = 3$ for every $v \in V_2$, it follows that $\gamma'_s(K_{m,n,p}) = (3m + 2n - 1)/2$. □

Proposition 4.6. Let m, p be even, n be odd, $m < n$ and $p \geq m + n + 1$. Then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{2m + 3n + 1}{2}$ if $n \equiv 1 \pmod 4$,

2. $\gamma'_s(K_{m,n,p}) = \dfrac{2m + 3n - 1}{2}$ if $n \equiv 3 \pmod 4$.

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W.

Case 1. $n \equiv 1 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n+p-1)/2+((n-m-1)/2)(m+p-2)/2\\&+((n+m+1)/2)(m+p-4)/2-(p/2)(m+n+1)/2\\&-(p/2)(m+n-1)/2=mn-m-(3n+1)/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1|=(n-m-1)/2$ and $|V_2|=(n+m+1)/2$. Also partition, W into W_1 and W_2 with $|W_1|=|W_2|$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn-n-(3n+1)/2]$ edges between U and V and $(p/2)(m+n+1)/2+(p/2)(m+n-1)/2$ edges between $U\cup V$ and W with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-1)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-2)/2$ and $(m+p-4)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 1, the weight of vertices in V_1, V_2 are all $2, 4$, respectively, the weight of vertices in W_1 are all -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f)=(2m+3n+1)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U\cup V$ can have weight zero or -1 by (1). Now by Lemma 3.3 and the facts that $f(W)=0$, $f(u)=1$ for every $u\in U$, $f(v)=2$ for every $v\in V_1$ and $f(v)=4$ for every $v\in V_2$, it follows that $\gamma'_s(K_{m,n,p})=(2m+3n+1)/2$.

Case 2. $n \equiv 3 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n+p-1)/2+((n-m-1)/2)(m+p-2)/2\\&+((n+m+1)/2)(m+p-4)/2-(p/2)(m+n+1)/2\\&-(p/2)(m+n-1)/2=mn-m-(3n-1)/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1|=(n-m+1)/2$ and $|V_2|=(n+m-1)/2$. Also partition, W into W_1 and W_2 with $|W_1|=|W_2|$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn-n-(3n+1)/2]$ edges between U and V and $(p/2)(m+n+1)/2+(p/2)(m+n-1)/2$ edges between $U\cup V$ and W with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-1)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-2)/2$ and $(m+p-4)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 1, the weight of vertices in V_1, V_2 are all $2, 4$, respectively, and the weight of vertices in W_1 are all -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f)=(2m+3n-1)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U\cup V$ can have weight zero or -1 by (1). Now by Lemma 3.3 and the facts that $f(W)=0$, $f(u)=1$ for every $u\in U$, $f(v)=2$ for every $v\in V_1$ and $f(v)=4$ for every $v\in V_2$, it follows that $\gamma'_s(K_{m,n,p})=(2m+3n-1)/2$. □

Proposition 4.7. Let m, p be odd, n be even, $3 \leq m < n$, and $p \geq m + n$. Then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{2m+3n}{2}$ if $n \equiv 0 \pmod 4$,

2. $\gamma'_s(K_{m,n,p}) = \dfrac{2m+3n-2}{2}$ if $n \equiv 2 \pmod 4$,

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W.
Case 1. $n \equiv 0 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n+p-1)/2 + ((n-m-1)/2)(m+p-2)/2\\&+((n+m+1)/2)(m+p-4)/2 - ((p+1)/2)(m+n+1)/2\\&-((p-1)/2)(m+n-1)/2 = mn - m - (3n+2)/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1| = (n-m-1)/2$ and $|V_2| = (n+m+1)/2$. Also partition W into W_1 and W_2 with $|W_1| = |W_2|+1$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn - m - (3n+2)/2]$ edges between U and V and $((p+1)/2)(m+n+1)/2 + ((p-1)/2)(m+n-1)/2$ edges between $U \cup V$ and W with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-1)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-2)/2$ and $(m+p-4)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 1, the weight of vertices in V_1, V_2 are all $2, 4$, respectively, and the weight of vertices in W_1 are -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f) = (2m+3n)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U \cup V$ can have weight zero or -1 by (1). Now by Lemma 3.4 and the facts that $f(W) = -1$, $f(u) = 1$ for every $u \in U$, $f(v) = 2$ for every $v \in V_1$ and $f(v) = 4$ for every $v \in V_2$, it follows that $\gamma'_s(K_{m,n,p}) = (2m+3n)/2$.

Case 2. $n \equiv 2 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n+p-1)/2 + ((n-m+1)/2)(m+p-2)/2\\&+((n+m-1)/2)(m+p-4)/2 - ((p+1)/2)(m+n+1)/2\\&-((p-1)/2)(m+n-1)/2 = mn - m - 3n/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1| = (n-m+1)/2$ and $|V_2| = (n+m-1)/2$. Also partition W into W_1 and W_2 with $|W_1| = |W_2|+1$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn - m - 3n/2]$ edges between U and V and $((p+1)/2)(m+n+1)/2 + ((p-1)/2)(m+n-1)/2$ edges between $U \cup V$ and W with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-1)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-2)/2$ and $(m+p-4)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 1, the weight of vertices in V_1, V_2 are all $2, 4$, respectively, and the weight of vertices in W_1 are all -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f) = (2m + 3n - 2)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U \cup V$ can have weight zero or -1 by (1). Now by Lemma 3.4 and the facts that $f(W) = -1$, $f(u) = 1$ for every $u \in U$, $f(v) = 2$ for every $v \in V_1$ and $f(v) = 4$ for every $v \in V_2$, it follows that $\gamma'_s(K_{m,n,p}) = (2m + 3n - 2)/2$. □

Proposition 4.8. Let m be even, n, p be odd, $m < n$, and $p \geq m + n$. Then there is an SEDF f of $K_{m,n,p}$ with

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 2n}{2}$ if $m \equiv 0 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 2n - 2}{2}$ if $m \equiv 2 \pmod 4$.

Proof. Consider the graph $K_{m,n,p}$ whose partite sets are U, V and W.
Case 1. $m \equiv 0 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n + p - 2)/2 + ((n - m + 1)/2)(m + p - 1)/2\\&+((n + m - 1)/2)(m + p - 3)/2 - ((p - 1)/2)(m + n + 1)/2\\&-((p + 1)/2)(m + n - 1)/2 = mn - n - (3m - 2)/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1| = (n-m+1)/2$ and $|V_2| = (n+m-1)/2$. Also partition W into W_1 and W_2 with $|W_1| = |W_2|-1$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn - n - (3m - 2)/2]$ edges between U and V and $((p-1)/2)(m+n+1)/2 + ((p+1)/2)(m+n-1)/2$ edges between $U \cup V$ and W with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-2)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-1)/2$ and $(m+p-3)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 2, the weight of vertices in V_1, V_2 are all $1, 3$, respectively, and the weight of vertices in W_1 are all -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f) = (3m + 2n)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U \cup V$ can have weight zero or -1 by (1). Now by Lemma 3.4 and the facts that $f(W) = 1$, $f(u) = 2$ for every $u \in U$, $f(v) = 1$ for every $v \in V_1$ and $f(v) = 3$ for every $v \in V_2$, it follows that $\gamma'_s(K_{m,n,p}) = (3m + 2n)/2$.

Case 2. $m \equiv 2 \pmod 4$.
By assumption,

$$\begin{aligned}&m(n + p - 2)/2 + ((n - m + 1)/2)(m + p - 1)/2\\&+((n + m - 1)/2)(m + p - 3)/2 - ((p + 1)/2)(m + n + 1)/2\\&-((p - 1)/2)(m + n - 1)/2 = mn - n - (3m)/2\end{aligned}$$

is even. Partition V into V_1 and V_2 with $|V_1| = (n-m+1)/2$ and $|V_2| = (n+m-1)/2$. Also partition W into W_1 and W_2 with $|W_1| = |W_2|+1$. In a similar fashion described in the proof of Proposition 4.1, label $(1/2)[mn - n - 3m/2]$ edges between U and V and $((p+1)/2)(m+n+1)/2 + ((p-1)/2)(m+n-1)/2$ edges between $U \cup V$ and W with -1 such that the edges between V_1 and W_1 are all negative edges. In addition, every vertex in U is incident with $(n+p-2)/2$ negative edges, every vertex in V_1, V_2 is incident with $(m+p-1)/2$ and $(m+p-3)/2$ negative edges, respectively, and every vertex in W_1, W_2 is incident with $(m+n+1)/2$ and $(m+n-1)/2$ negative edges, respectively. Label the remaining edges of $K_{n,m,p}$ by $+1$. Then the weight of vertices in U are all 2, the weight of vertices in V_1, V_2 are all $1, 3$, respectively, and the weight of vertices in W_1 are all -1 and in W_2 are $+1$. Hence, this labeling defines a SEDF f with $w(f) = (3m + 2n - 2)/2$.

Note that since the weight of some vertices in W is -1, no vertices in $U \cup V$ can have weight zero or -1 by (1). Now by Lemma 3.4 and the facts that $f(W) = -1$, $f(u) = 2$ for every $u \in U$, $f(v) = 1$ for every $v \in V_1$ and $f(v) = 3$ for every $v \in V_2$, it follows that $\gamma'_s(K_{m,n,p}) = (3m + 2n - 2)/2$. □

5 The SEDNs of $K_{1,n,p}$ and $K_{2,2,p}$

The constructions given in Section 4 work if the sum of the desired negative edges at vertices in U and at vertices in V is not less than the desired negative edges at vertices in W. In this section we calculate the signed edge domination numbers of $K_{1,n,p}$ and $K_{2,2,p}$ which are not covered by constructions given in Section 4.

Lemma 5.1. Let $n \geq 1$ and $p \geq n + 2$.

1. If n, p are odd, then $\gamma'_s(K_{1,n,p}) = n + 2$.
2. If n, p are even, then $\gamma'_s(K_{1,n,p}) = n + 2$.
3. If n is even and $p \geq n + 3$ is odd, then $\gamma'_s(K_{1,n,p}) = 2n + 1$.
4. If n is odd and $p \geq n + 3$ is even, then $\gamma'_s(K_{1,n,p}) = 2n + 3$.

Proof. Consider the graph $K_{1,n,p}$ whose partite sets are U, V and W.
Case 1. n and p are odd.
Since $mn-m-n = n-1-n = -1$ when $m = 1$, the construction given in Proposition 4.2 does not work. On the other hand,

$$m(n + p - 2)/2 + n(m + p - 2)/2 - (p - 1)(m + n)/2 - (m + n - 2)/2 = 0$$

when $m = 1$. So we can label $(n+p-2)/2 + n(1+p-2)/2$ edges between $U \cup V$ and W such that the vertex in U is incident with $(n+p-2)/2$ negative edges, each vertex in V is incident with $(p-1)/2$ negative edges and all the vertices in W are incident with $(n + 1)/2$ negative edges except one vertex which is incident with $(n - 1)/2$ negative edges. We label the remaining edges with $+1$. This yields a signed edge dominating function of weight $n+2$. It is easy to see that this is the minimum weight of a SEDF of $K_{1,n,p}$ when n, p are odd.

Case 2. n and p are even.
Since $mn - n - (3m+1)/2 = -2$ when $m = 1$, the construction given in Proposition 4.5 does not work. On the other hand,

$$\begin{aligned}&m(n+p-2)/2 + ((n-m+1)/2)(m+p-1)/2\\&+((n+m-1)/2)(m+p-3)/2 - (p/2)(m+n+1)/2\\&-((p-2)/2)(m+n-1)/2 - (m+n-3)/2 = mn - n - (3m-3)/2 = 0\end{aligned}$$

when $m = 1$. So we can label $m(n+p-2)/2 + ((n-m+1)/2)(m+p-1)/2$ $+ ((n+m-1)/2)(m+p-3)/2$ edges between $U \cup V$ and W such that the vertex in U is incident with $(n+p-2)/2$ negative edges, half of the vertices in V are incident with $p/2$ negative edges and the other half are incident with $(p-2)/2$ negative edges, $p/2$ vertices in W are incident with $(n+2)/2$, $(p-2)/2$ vertices are incident with $n/2$ and one vertex is incident with $(n-2)/2$ negative edges. We label the remaining edges with $+1$. This yields a signed edge dominating function of weight $n+2$. It is easy to see that this is the minimum weight of a SEDF of $K_{1,n,p}$ when n, p are even.

Case 3. n is even and $p \geq n+3$ is odd.
If $m = 1$, then $mn - m - 3n/2 < 0$, so the constructions given in Proposition 4.7 do not work. On the other hand,

$$\begin{aligned}&m(n+p-1)/2 + ((n-m+1)/2)(m+p-2)/2\\&+((n+m-1)/2)(m+p-4)/2 - ((p-n-1)/2)(m+n+1)/2\\&-((p+n+1)/2)(m+n-1)/2 = mn - m - n + 1 = 0\end{aligned}$$

when $m = 1$. Place $m(n+p-1)/2 + ((n-m+1)/2)(m+p-2)/2 + ((n+m-1)/2)(m+p-4)/2$ negative edges between $U \cup V$ and W such that the vertex in U is incident with $(n+p-1)/2$ negative edges, $(n-m+1)/2$ vertices in V are incident with $(m+p-2)/2$ negative edges, $(n+m-1)/2$ vertices are incident with $(m+p-4)/2$ negative edges, $(p-n-1)/2$ vertices of W are incident with (m+n+1)/2 negative edges and $(p+n+1)/2$ vertices of W are incident with (m+n-1)/2 negative edges. Label the remaining edges with $+1$. This yields a signed edge dominating function of weight $2n+m$ which is $2n+1$ when $m = 1$. It is an easy to see that $\gamma(K_{1,n,p}) = 2n+1$.

Case 4. n is odd and $p \geq n+3$ is even.
If $m = 1$, then $mn - (3m+3n)/2 < 0$, so the constructions given in Proposition 4.3 do not work. On the other hand,

$$\begin{aligned}&m(n+p-3)/2 + n(m+p-3)/2 - (p-(n+3)/2)(m+n)/2\\&-((n+3)/2)(m+n-2)/2 = (2mn-3m-2n+3)/2 = 0\end{aligned}$$

when $m = 1$. So we can find a signed edge dominating function of weight $(1/2)[3 + 3n + 2(n+3)/2] = 2n+3$. It is easy to verify that $\gamma'_s(K_{1,n,p}) = 2n+3$. □

In [5] it was conjectured that $\gamma'_s(G) \leq |V(G)| - 1$ for every graph G of order at least 2. Note that if n is odd, then $\gamma'_s(K_{1,n,n+3}) = 2n + 3$ by Lemma 5.1, Part 4. Hence, the graph $K_{1,n,n+3}$ achieves the upper bound in this conjecture.

Lemma 5.2. Let $p \geq 5$ be odd. Then $\gamma'_s(K_{2,2,p}) = 8$.

Proof. Consider the graph $K_{2,2,p}$ with partite sets U, V and W. Partition W into W_1, W_2 and W_3 such that $|W_1| = |W_2|$ and $|W_3| = 1$. Label the edges between U and W_1 and between V and W_2 with -1 and the remaining edges with $+1$. Then the weight of vertices in $W_1 \cup W_2$ are zero and the weight of the vertex in W_3 is 4. The weight of the vertices in $U \cup V$ are all 3. This leads to $\gamma'_s(K_{2,2,p}) = 8$. ☐

6 Main Theorem

Let m, n, p be positive integers, $m \leq n$ and $p \geq m + n$. In this section we state the Main Theorem of this paper, which consists of putting together the several lemmas and propositions that are proved earlier. This result together with the main result of [3] provide the signed edge domination number of $K_{m,n,p}$ for all positive integers m, n and p.

Main Theorem Let m, n and p be positive integers, $m \leq n$ and $p \geq m + n$. Let $m \geq 2$ and if p is odd, $(m, n) \neq (2, 2)$.

A. If m, n and p are even, then $\gamma'_s(K_{m,n,p}) = m + n$.

B. If m, n and p are odd and $m, n \geq 3$, then $\gamma'_s(K_{m,n,p}) = m + n + 1$.

C. If $m, n \geq 3$ are odd and p is even, then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 3n + 2}{2}$ if $m + n \equiv 0 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 3n}{2}$ if $m + n \equiv 2 \pmod 4$.

D. If m, n are even, p is odd and $(m, n) \neq (2, 2)$, then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 3n}{2}$ if $m + n \equiv 0 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 3n + 2}{2}$ if $m + n \equiv 2 \pmod 4$.

E. If m is odd, n, p are even and $3 \leq m < n$, then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 2n + 1}{2}$ if $m \equiv 1 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m + 2n - 1}{2}$ if $m \equiv 3 \pmod 4$.

F. If m, p are even, n is odd and $m < n$, then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{2m+3n+1}{2}$ if $n \equiv 1 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{2m+3n-1}{2}$ if $n \equiv 3 \pmod 4$.

G. If m, p are odd, n is even and $3 \le m < n$, then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{2m+3n}{2}$ if $n \equiv 0 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{2m+3n-2}{2}$ if $n \equiv 2 \pmod 4$.

H. If m is even, n, p are odd and $m < n$, then

1. $\gamma'_s(K_{m,n,p}) = \dfrac{3m+2n}{2}$ if $m \equiv 0 \pmod 4$,
2. $\gamma'_s(K_{m,n,p}) = \dfrac{3m+2n-2}{2}$ if $m \equiv 2 \pmod 4$.

In addition, $\gamma'_s(K_{1,n,p}) = n+2$ if n, p are both odd or both even, $\gamma'_s(K_{1,n,p}) = 2n+1$ if n is even and $p \ge n+3$ is odd, $\gamma'_s(K_{1,n,p}) = 2n+3$ if n is odd and $p \ge n+3$ is even, and $\gamma'_s(K_{2,2,p}) = 8$ if p is odd.

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