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On minimum cutsets in independent domination vertex-critical graphs

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In memory of Anne Penfold Street

Abstract

Let $\gamma_i(G)$ denote the independent domination number of G. A graph G is said to be k-γ_i-vertex-critical if $\gamma_i(G) = k$ and for each $x \in V(G)$, $\gamma_i(G-x) < k$. In this paper, we show that for any k-γ_i-vertex-critical graph H of order n with $k \geq 3$, there exists an n-connected k-γ_i-vertex-critical graph G_H containing H as an induced subgraph. Consequently, there are infinitely many non-isomorphic connected k-γ_i-vertex-critical graphs. We also establish a number of properties of connected 3-γ_i-vertex-critical graphs. In particular, we derive an upper bound on $\omega(G-S)$, the number of components of $G-S$ when G is a connected 3-γ_i-vertex-critical graph and S is a minimum cutset of G with $|S| \geq 3$.

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1 Introduction

All graphs in this paper are finite simple undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For a vertex u of G, the neighborhood of u in G, denoted by $N_G(u)$, is the set of all vertices of G that are adjacent to u. The closed neighborhood of u which is $N_G(u) \cup \{u\}$ is denoted by $N_G[u]$. For $S \subseteq V(G)$, $N_S(u) = N_G(u) \cap S$. For simplicity, if H is a subgraph of G, we write $N_H(u)$ instead of $N_{V(H)}(u)$. The degree of a vertex u in G, denoted by $\deg_G(u)$, is $|N_G(u)|$ while $\deg_S(u)$ denotes $|N_S(u)|$. Further, $\Delta(G)$ denotes $\max\{\deg_G(u)|u \in V(G)\}$.

A subset S of $V(G)$ is independent if no two vertices of S are adjacent. The number of components of G and the number of odd components of G are denoted by $\omega(G)$ and $\omega_0(G)$, respectively. A subset $S \subseteq V(G)$ is called a cutset if $\omega(G-S) > \omega(G)$. If $S = \{u\}$, then the vertex u is called a cutvertex and we shall write $\omega(G-u)$ instead of $\omega(G - \{u\})$.

A graph G is said to be k-factor-critical if $G - S$ has a perfect matching for every $S \subseteq V(G)$ with $|S| = k$. It is easy to see that $|V(G)| \equiv k(\mod 2)$. For $k = 1$ and $k = 2$, k-factor-critical graphs are also called factor-critical and bicritical, respectively. The concept of k-factor-critical graphs was introduced by Favaron [3] in 1996.

For subsets S and T of $V(G)$, S is called a dominating set of T, denoted by $S \succ T$, if each vertex of T either belongs to S or is adjacent to some vertex of S. For simplicity, we write $s \succ T$ if $S = \{s\}$ and $S \succ G$ if $T = V(G)$. The minimum cardinality of a dominating set of G is called the domination number of G and denoted by $\gamma(G)$. A dominating set S of G which is also an independent set is called an independent dominating set of G and is denoted by $S \succ_i G$. The independent domination number of G is the minimum cardinality of an independent dominating set of G and is denoted by $\gamma_i(G)$. It is easy to see that $\gamma(G) \leq \gamma_i(G)$ and if $\gamma(G) = 1$, then $\gamma_i(G) = 1$.

A graph G is said to be k-γ_i-vertex-critical if $\gamma_i(G) = k$ and for each $x \in V(G)$, $\gamma_i(G-x) < k$. In fact, it is easy to see that if G is k-γ_i-vertex-critical, then $\gamma_i(G - x) = k - 1$ for each $x \in V(G)$. Further, $|V(G)| \geq k$. The concept of k-γ_i-vertex-critical graphs was first introduced by Ao [1] in 1994. The problem that arises is that of characterizing connected k-γ_i-vertex-critical graphs. Ao [1] characterized the case $k = 1$ and $k = 2$. More specifically, she proved that the only 1-γ_i-vertex-critical graphs are K_1, and the only 2-γ_i-vertex-critical graphs are K_{2n} with a perfect matching deleted for some positive integer n. The following two simple results are useful in studying k-γ_i-vertex-critical graphs. In what follows, for a vertex x of a k-γ_i-vertex-critical graph G, we denote by I_x any minimum independent dominating set of $G - x$.

Lemma 1.1. *[1] Suppose G is a k-γ_i-vertex-critical graph for $k \geq 2$. Then for each $x \in V(G)$, $|I_x| = k - 1$.*

Lemma 1.2. *[5] Suppose G is a k-γ_i-vertex-critical graph for $k \geq 2$. Then for each $x \in V(G)$, $I_x \cap N_G[x] = \emptyset$.*

The following result follows directly from the definition.

Lemma 1.3. *Suppose G is a k-γ_i-vertex-critical graph for $k \geq 2$. For $x, y \in V(G)$ such that $x \neq y$, $I_x \neq I_y$.*

For $k \geq 3$, very few results on k-γ_i-vertex-critical graphs are known. In the next section, we establish that for $k \geq 3$, if H is a k-γ_i-vertex-critical graph on n vertices, then there exists an n-connected k-γ_i-vertex-critical graph on $kn+1$ vertices containing H as an induced subgraph. This suggests that characterizing connected k-γ_i-vertex-critical graphs for $k \geq 3$ is a very difficult task. The focus of this paper is the case $k = 3$.

We establish a number of properties of connected 3-γ_i-vertex-critical graphs. In Section 4, we derive an upper bound on the number of components $\omega(G - S)$ where G is a connected 3-γ_i-vertex-critical graph and S is a minimum cutset of G with $|S| \geq 3$. Section 3 provides some preliminary results that we make use of in our work.

We conclude this section by pointing out that critical concepts, in both edge-critical and vertex-critical graphs, are studied for various kinds of domination numbers such as ordinary domination number, connected domination number and total domination number. For more details of these, the reader is directed to the books by Haynes et al. [4] and Dehmer [2] and also references therein.

2 A family of connected k-γ_i-vertex-critical graphs

In this section, we provide a construction of a family of connected k-γ_i-vertex-critical graphs for $k \geq 3$. For a k-γ_i-vertex-critical graph H, we show that there are infinitely many connected k-γ_i-vertex-critical graphs containing H as an induced subgraph. Before presenting the construction, we make an observation that there are infinitely many k-γ_i-vertex-critical graphs. For positive integers $k \geq 3$ and n, $\overline{K}_{k-2} \cup (K_{2n}-$a perfect matching) is a simple example of k-γ_i-vertex-critical graph. Moreover, for positive integers m and n_i, $\bigcup_{i=1}^{m}(K_{2n_i}-$a perfect matching), and $K_1 \cup \bigcup_{i=1}^{m}(K_{2n_i}-$a perfect matching) are examples of k-γ_i-vertex-critical graphs when $k = 2m$ is even and $k = 2m+1$ is odd, respectively. For case $k = 3$, $K_1 \cup (K_{2n}-$ a perfect matching) is the only disconnected 3-γ_i-vertex-critical graphs. Some examples of connected 3-γ_i-vertex-critical graphs are $K_{3,3}$, C_7: a cycle of order 7 and the graphs shown in Figure 2.1 for any positive integers n and m. Note that "+" in our diagrams denotes the join and the dash line denotes a missing edge between vertices.

Our next result establishes a class of connected k-γ_i-vertex-critical graphs.

Theorem 2.1. *For $k \geq 3$, let H be a k-γ_i-vertex-critical graph of order n. Then there exists an n-connected k-γ_i-vertex-critical graph G_H such that H is an induced subgraph of G_H.*

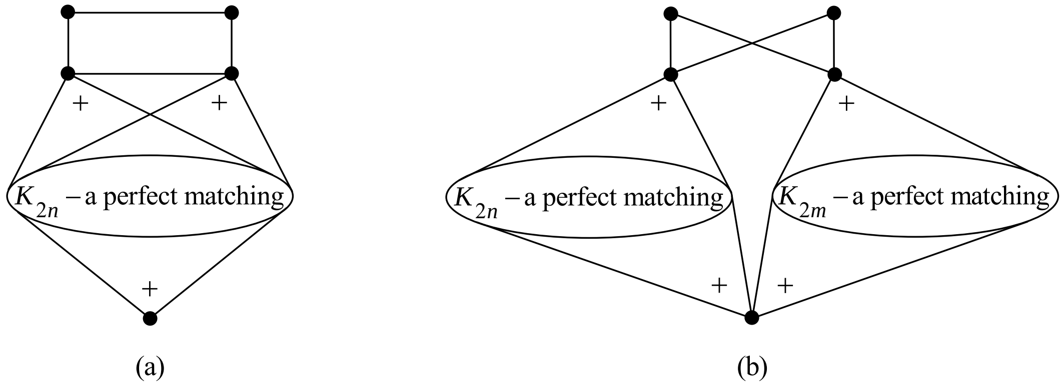

Figure 2.1: Connected 3-γ_i-vertex-critical graphs.

Proof. Put $V(H) = \{x_1, x_2, \ldots, x_n\}$. Now let G_H be a graph of order $kn + 1$ where $V(G) = \{u\} \cup \{x_1, x_2, \ldots, x_n\} \cup \bigcup_{j=1}^{k-1} Y_j$ where $Y_j = \{y_{j1}, y_{j2}, \ldots, y_{jn}\}$ and $E(G) = \{ux_i | 1 \leq i \leq n\} \cup E(H) \cup \bigcup_{j=1}^{k-1}\{x_i y_{jl} | 1 \leq i \leq n, 1 \leq l \leq n, i \neq l\} \cup \bigcup_{j=1}^{k-1}\{y_{jl}y_{jl'} | 1 \leq l \leq n, 1 \leq l' \leq n, l \neq l'\}$. Figure 2.2 illustrates our construction.

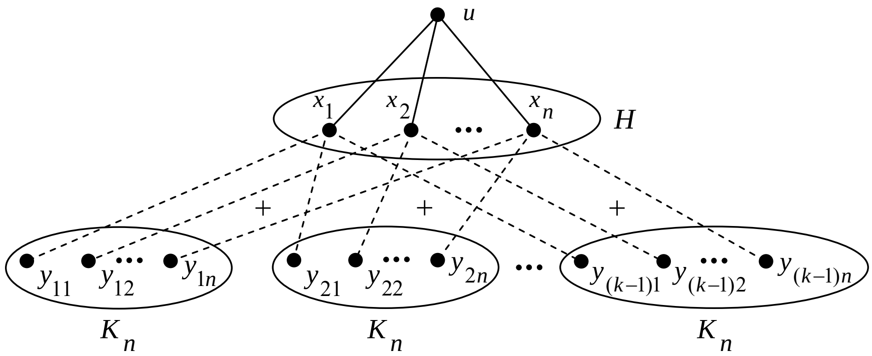

Figure 2.2: The graph G_H.

It is easy to see that H is an induced subgraph of G_H and G_H is n-connected. We only need to show that G_H is k-γ_i-vertex-critical. Let I be a minimum independent dominating set of H. Clearly, $I \succ_i G_H$. Then $\gamma_i(G_H) \leq k$. It is easy to see that no vertex of G_H dominates G_H, thus $\gamma_i(G_H) \geq 2$. Suppose there exists an independent dominating set I_1 of G_H where $|I_1| \leq k-1$. We first show that $u \notin I_1$. Suppose this is not the case. Since I_1 is independent, $(I_1 - \{u\}) \cap V(H) = \emptyset$. Thus $I_1 - \{u\} \subseteq \bigcup_{j=1}^{k-1} Y_j$. Since $|I_1 - \{u\}| \leq k-2$ and $G_H[\bigcup_{j=1}^{k-1} Y_j]$ consists of $k-1$ components, it follows that no vertex of I_1 dominates $Y_{j'}$, for some $1 \leq j' \leq k-1$, a contradiction. Hence, $u \notin I_1$ as required. Since $|I_1| \leq k-1$, $I_1 \nsubseteq V(H)$ otherwise $\gamma_i(H) < k$. Then there exists $w \in I_1 \cap (\bigcup_{j=1}^{k-1} Y_j)$. We may assume without loss of generality that $w = y_{11}$. Since $u \notin I_1$ and I_1 is independent, it follows that $I_1 \cap V(H) = \{x_1\}$ by our construction. Then $I_1 - \{x_1, y_{11}\} \subseteq \bigcup_{j=2}^{k-1} Y_j$. Because $|I_1 - \{x_1, y_{11}\}| \leq k-3$ and $G_H[\bigcup_{j=2}^{k-1} Y_j]$ consists of $k-2$ components, it follows that no vertex of I_1 dominates $Y_{j''}$ for some $2 \leq j'' \leq k-1$, again a contradiction. Hence, $\gamma_i(G_H) = k$.

We next show that G_H is k-γ_i-vertex-critical. It is easy to see that $I_u = \{y_{11}, y_{22}, \dots, y_{(k-1)(k-1)}\}$. Further, for $1 \le j \le k-1$, $1 \le l \le n$, $I_{y_{jl}} = \{x_l\} \cup \{y_{j'l} | 1 \le j' \le k-1, j' \ne j\}$. Since H is k-γ_i-vertex-critical, $|I_{x_i}| = k-1$ for $1 \le i \le n$ and it is easy to see that I_{x_i} dominates $G_H - x_i$. This proves that G_H is k-γ_i-vertex-critical and completes the proof of our theorem. ■

In view of Theorem 2.1, we may recursively construct a connected k-γ_i-vertex-critical graph for $k \ge 3$. Beginning with a k-γ_i-vertex-critical graph H of order n, put $G_1 = G_H$, $G_2 = G_{G_1}, G_3 = G_{G_2}, \dots, G_t = G_{G_{t-1}}, \dots$. Then $|V(G_t)| = k^t n + \frac{k^t-1}{k-1}$ and G_t is a $|V(G_{t-1})|$-connected k-γ_i-vertex-critical graph for any positive integer t. Further, each G_t contains H as an induced subgraph. By this recursive construction and examples of k-γ_i-vertex-critical graphs given at the beginning of this section, there are infinitely many non-isomorphic connected k-γ_i-vertex-critical graphs.

We next establish some matching properties of the graph G_H. For the rest of this section, F_Z denotes a perfect matching in $G_H[Z]$ where $Z \subseteq V(G_H)$.

Proposition 2.2. *For $k \ge 3$, let H be a k-γ_i-vertex-critical graph of order n and let G_H be the graph defined in the proof of Theorem 2.1. Then we have:*

1. *If H is $K_{1,s}$-free, then G_H is $K_{1,r}$-free where $r = \max\{s, k+1\}$.*
2. *If k and n are odd and $n \ge k+2$, then G_H is bicritical.*
3. *If either k or n is even, then G_H is factor-critical.*

Proof. (1) This follows immediately from the construction.

(2) Let w_1 and w_2 be distinct vertices of G_H. We need to show that $G_H - \{w_1, w_2\}$ has a perfect matching. We first suppose that $\{w_1, w_2\} \subseteq V(H)$. We may assume without loss of generality that $w_i = x_i$, for $1 \le i \le 2$. We now let

$$\begin{aligned} F &= \{ux_3\} \cup \{x_4y_{21}, x_5y_{31}, \dots, x_{k+1}y_{(k-1)1}\} \cup \{x_s y_{1(s+1)} | k+2 \le s \le n\} \cup \\ &\quad \bigcup_{l=2}^{k-1} F_{Y_l - \{y_{l1}\}} \cup F_{Y_1 - \{y_{1(s+1)} | k+2 \le s \le n\}} \end{aligned}$$

where our subscript is read modulo n. It is easy to see that F is a perfect matching in $G_H - \{w_1, w_2\}$. By similar arguments, it is not difficult to show that $G_H - \{w_1, w_2\}$ contains a perfect matching if $\{w_1, w_2\} \nsubseteq V(H)$. This proves (2).

(3) Let w be a vertex of G_H. We need to show that $G_H - w$ contains a perfect matching. We first suppose that $w = y_{11}$. If n is even, then

$$F_1 = \{ux_n\} \cup \{x_s y_{1(s+1)} | 1 \le s \le n-1\} \cup \bigcup_{l=2}^{k-1} F_{Y_l}$$

is a perfect matching in $G_H - w$. We now suppose that n is odd. Thus k is even by our hypothesis. Put

$$\begin{aligned} F_2 &= \{x_1u\} \cup \{x_2y_{31}, x_3y_{41}, \dots, x_{k-2}y_{(k-1)1}\} \cup \{x_s y_{2(s+1)} | k-1 \le s \le n\} \\ &\quad \cup F_{Y_1 - \{y_{11}\}} \cup F_{Y_2 - \{y_{2(s+1)} | k-1 \le s \le n\}} \cup \bigcup_{l=3}^{k-1} F_{Y_l - \{y_{l1}\}} \end{aligned}$$

where our subscript is read modulo n. It is easy to see that F_2 is a perfect matching in $G_H - y_{11}$. By similar arguments, it is not difficult to show that $G_H - w$ has a perfect matching if $w \notin \bigcup_{l=1}^{k-1} Y_l$. This proves (3) and completes the proof of our result. ■

Note that the lower bound on $n \geq k+2$ in the part 2 of the above result is sharp since the graph G_H, where H is $\overline{K}_k$, is not bicritical.

3 Some preliminary results

In this section, we establish some basic results that we make use of in establishing our results in the next section. Recall that, for a vertex x of a k-γ_i-vertex-critical graph G, I_x denotes any minimum independent dominating set of $G - x$. Our first result concerns a simple property of a graph with a cutset. It follows immediately from the fact that our cutset is minimum.

Lemma 3.1. *Let G be a connected graph and S a minimum cutset of G. Further, let C be a component of $G - S$. Then we have:*

1. *If there is a vertex $x \in V(C)$ such that x is not adjacent to some vertex of S, then $|V(C)| \geq 2$.*
2. *For each $u \in S$, $N_C(u) \neq \emptyset$.*

The following two results concern simple properties of connected 3-γ_i-vertex-critical graphs with a minimum cutset.

Lemma 3.2. *Let G be a connected 3-γ_i-vertex-critical graph and S a minimum cutset of G. If $\omega(G - S) \geq 4$, then*

1. *No vertex of $V(G)$ dominates S. Consequently, $\Delta(G[S]) \leq |S| - 2$ and $G - S$ has no singleton components.*
2. *$I_x \cap S \neq \emptyset$, for each $x \in V(G)$.*

Proof. (1) Suppose to the contrary that there is a vertex $y \in V(G)$ such that $y \succ S$. By Lemma 1.2, $I_y \cap S = \emptyset$. Thus $I_y \subseteq V(G) - S$. Since $|I_y| = 2$ and $\omega(G - S) \geq 4$, it follows that there is a vertex of $V(G) - S$ which is not dominated by I_y, a contradiction. This settles (1).

(2) It is easy to see that if $I_x \cap S = \emptyset$, for some $x \in V(G)$, then $I_x \subseteq V(G) - S$. Thus I_x does not dominate at least one component of $G - S$ since $|I_x| = 2$ and $\omega(G - S) \geq 4$. Hence, $I_x \cap S \neq \emptyset$ for each $x \in V(G)$. This settles (2) and completes the proof of our result. ■

Lemma 3.3. *Let G be a connected 3-γ_i-vertex-critical graph and S a cutset of G where $t = \omega(G - S) \geq 4$. Let $C_1, C_2, \ldots, C_t$ be the components of $G - S$. Suppose there exist $y_j \in V(C_j)$ and $y_{j'} \in V(C_{j'})$ for $1 \leq j \leq t, 1 \leq j' \leq t, j \neq j'$ such that $I_{y_j} \cap S = I_{y_{j'}} \cap S = \{u\}$ for some $u \in S$. Then*

1. $u \succ \bigcup_{i=1}^{t} V(C_i) - \{y_j, y_{j'}\}$*. Consequently,* $I_{y_j} = \{u, y_{j'}\}$ *and* $I_{y_{j'}} = \{u, y_j\}$*.*

2. $u \notin I_x$ *for any* $x \in \bigcup_{i=1}^{t} V(C_i) - \{y_j, y_{j'}\}$*.*

Proof. (1) Since $\{u\} = I_{y_j} \cap S = I_{y_{j'}} \cap S$, it follows by Lemma 1.2 that $uy_j, uy_{j'} \notin E(G)$. Put $\{z\} = I_{y_j} - \{u\}$ and $\{w\} = I_{y_{j'}} - \{u\}$. Then $uz, uw \notin E(G)$ since I_{y_j} and $I_{y_{j'}}$ are independent. It is easy to see that $z \in V(C_{j'})$ and $w \in V(C_j)$. Then $u \succ \bigcup_{i=1}^{t} V(C_i) - (V(C_{j'}) \cup \{y_j\})$ and $w = y_j$ since $u \in I_{y_j}$. Further, $u \succ \bigcup_{i=1}^{t} V(C_i) - (V(C_j) \cup \{y_{j'}\})$ and $z = y_{j'}$ since $u \in I_{y_{j'}}$. This settles (1).

(2) This follows by (1) and Lemma 1.2. This completes the proof of our lemma. ■

As a consequence of Lemmas 1.2 and 3.3, we have:

Corollary 3.4. *Let* G, S *and* $C_1, C_2, \dots, C_t$ *be defined as in Lemma 3.3. If there is* $\{w_1, w_2, \dots, w_r\} \subseteq \bigcup_{i=1}^{t} V(C_i)$*, where* $w_l \succ S - \{u\}$ *for some* $u \in S$ *and for* $1 \leq l \leq r$*, such that* $|\{w_1, w_2, \dots, w_r\} \cap V(C_j)| \leq 1$ *for* $1 \leq j \leq t$*, then* $r \leq 2$*.*

4 Results on minimum cutsets of connected 3-γ_i-vertex-critical graphs

In this section, we provide an upper bound on $\omega(G - S)$, where G is a connected 3-γ_i-vertex-critical graph and S is a minimum cutset of G. For $1 \leq |S| \leq 2$, Ruangthampisan and Ananchuen [5] showed that $\omega(G - S) \leq |S| + 1$:

Theorem 4.1. *[5] Let* G *be a connected* 3-γ_i*-vertex-critical graph and* S *a minimum cutset of* G*. Then*

$$\omega(G - S) \leq \begin{cases} 2, & \text{for } |S| = 1, \\ 3, & \text{for } |S| = 2. \end{cases}$$

We now establish that if $3 \leq |S| \leq 4$, then $\omega(G - S) \leq 3$ and if $|S| \geq 5$, then $\omega(G - S) \leq |S| - 1$ with some condition on S. We begin with some lemmas.

Lemma 4.2. *Let* G *be a connected* 3-γ_i*-vertex-critical graph and* S *a minimum cutset of* G*. If* $\Delta(G[S]) \leq 1$ *and* $t = \omega(G - S) \geq |S| \geq 5$*, then for each* $x \in V(G)$*,* $|I_x \cap S| = 1$*.*

Proof. Since $|I_x| = 2$, it is easy to see that the result holds for $\Delta(G[S]) = 0$. So we may now assume that $\Delta(G[S]) = 1$ and suppose to the contrary that there exists a vertex $u \in V(G)$ such that $|I_u \cap S| = 2$. Put $I_u \cap S = \{u_1, u_2\}$. Since $\Delta(G([S]) = 1$ and $|S| \geq 5$, it is easy to see that $u \in S$ and $|S| = 5$. Without loss of generality, we let $E(G[S]) = \{u_1u_3, u_2u_4\}$ where $\{u_3, u_4\} = S - \{u, u_1, u_2\}$. Thus u is not adjacent to any of vertex of $S - \{u\}$. Consequently, we have proven the following claim.

Claim 1. *For each* $x \in V(G) - \{u\}$*,* $|I_x \cap S| = 1$*.*

Consider $G - u_3$. Clearly, $u_1 \notin I_{u_3}$ since $u_1u_3 \in E(G)$. Further, $I_{u_3} \cap S \subseteq \{u, u_2, u_4\}$. Put $\{z\} = I_{u_3} - S$. Thus $zu_3 \notin E(G)$. Let $C_1, C_2, \ldots, C_t$ be the components of $G - S$. We may assume that $z \in V(C_1)$. We now establish the following claim.

Claim 2. *If $u_3 \in I_x \cap S$ for some $x \in V(C_i)$, $2 \le i \le t$ then $I_x - \{u_3\} \subseteq V(C_1) - \{z\}$. Further, $u_3 \succ \bigcup_{i=2}^{t} V(C_i) - \{x\}$ and thus $u_3 \notin I_y \cap S$ for each $y \in \bigcup_{i=2}^{t} V(C_i) - \{x\}$.*

Proof. Suppose $u_3 \in I_x \cap S$. Since $zu_3 \notin E(G)$, it follows that the only vertex of $I_x - \{u_3\}$ dominates $z \in V(C_1)$. By Claim 1, $I_x - \{u_3\} \subseteq V(C_1)$. If $I_x - \{u_3\} = \{z\}$, then no vertex of I_x is adjacent to the vertex of $I_{u_3} - \{z\}$, a contradiction. This proves that $I_x - \{u_3\} \subseteq V(C_1) - \{z\}$. Consequently, $u_3 \succ \bigcup_{i=2}^{t} V(C_i) - \{x\}$. It follows by Lemma 1.2 that $u_3 \notin I_y \cap S$ for each $y \in \bigcup_{i=2}^{t} V(C_i) - \{x\}$. This settles our claim. □

We now distinguish three cases according to $I_{u_3} \cap S$.

Case 1. $I_{u_3} \cap S = \{u\}$.

Then $uz \notin E(G)$ and $u \succ \bigcup_{i=2}^{t} V(C_i)$. For $2 \le i \le t$, choose $y_i \in N_{C_i}(u_4)$. Such a y_i exists by Lemma 3.1(2). Observe that $y_i \in N_{C_i}(u) \cap N_{C_i}(u_4)$. Then $I_{y_i} \cap S \subseteq \{u_1, u_2, u_3\}$ and $|I_{y_i} \cap S| = 1$ by Lemma 1.2 and Claim 1. Thus, by Lemma 3.3(2), $|\{y_i | I_{y_i} \cap S = \{u_1\}\}| \le 2$ and $|\{y_i | I_{y_i} \cap S = \{u_2\}\}| \le 2$. But, by Claim 2, $|\{y_i | I_{y_i} \cap S = \{u_3\}\}| \le 1$.

Case 1.1. $|\{y_i | I_{y_i} \cap S = \{u_3\}\}| = 0$.

Since $|\{y_2, y_3, \ldots, y_t\}| = t - 1 \ge 4$, it follows that $|\{y_i | I_{y_i} \cap S = \{u_1\}\}| = |\{y_i | I_{y_i} \cap S = \{u_2\}\}| = 2$ and $t = 5$. We may assume that $I_{y_2} \cap S = I_{y_3} \cap S = \{u_1\}$ and $I_{y_4} \cap S = I_{y_5} \cap S = \{u_2\}$. Then $u_1y_2, u_1y_3, u_2y_4, u_2y_5 \notin E(G)$. By Lemma 3.3(1), $u_1 \succ \bigcup_{i=1}^{5} V(C_i) - \{y_2, y_3\}$ and $u_2 \succ \bigcup_{i=1}^{5} V(C_i) - \{y_4, y_5\}$. Now, for $2 \le i \le t$, choose $w_i \in V(C_i) - \{y_i\}$. Then $w_i \in N_{C_i}(u) \cap N_{C_i}(u_1) \cap N_{C_i}(u_2)$. Such a w_i exists by Lemma 3.1(1) and the fact that $u_1y_2, u_1y_3, u_2y_4, u_2y_5 \notin E(G)$. Thus $I_{w_i} \cap S \subseteq \{u_3, u_4\}$. By Claims 1 and 2, $|\{w_i | I_{w_i} \cap S = \{u_3\}\}| \le 1$ and thus $|\{w_i | I_{w_i} \cap S = \{u_4\}\}| \ge 3$. But this contradicts Lemma 3.3(2). Hence, Case 1.1 cannot occur.

Case 1.2. $|\{y_i | I_{y_i} \cap S = \{u_3\}\}| = 1$.

Without loss of generality we may assume that $I_{y_2} \cap S = \{u_3\}$. Put $\{z_1\} = I_{y_2} - \{u_3\}$. By Claim 2, $z_1 \in V(C_1) - \{z\}$ and $u_3 \succ \bigcup_{i=2}^{t} V(C_i) - \{y_2\}$. Since $E(G[S]) = \{u_1u_3, u_2u_4\}$, z_1 is adjacent to every vertex of $\{u, u_2, u_4\}$. Thus $I_{z_1} \cap S \subseteq \{u_1, u_3\}$. If $I_{z_1} \cap S = \{u_3\}$, then the only vertex of $I_{z_1} - \{u_3\} \subseteq \bigcup_{i=1}^{t} V(C_i)$ is adjacent to $z \in V(C_1)$ and $y_2 \in V(C_2)$ since $u_3z, u_3y_2 \notin E(G)$. But this is not possible. Hence, $I_{z_1} \cap S = \{u_1\}$. It then follows by Lemma 3.3(2) that $|\{y_i | I_{y_i} \cap S = \{u_1\}\}| \le 1$ and $|\{y_i | I_{y_i} \cap S = \{u_2\}\}| \le 2$. Since $|\{y_2, y_3, \ldots, y_t\}| = t - 1 \ge 4$, it follows that $|\{y_i | I_{y_i} \cap S = \{u_1\}\}| = 1$ and $|\{y_i | I_{y_i} \cap S = \{u_2\}\}| = 2$. In fact, $t = 5$. We may assume without loss of generality that $I_{y_3} \cap S = \{u_1\}$ and $I_{y_4} \cap S = I_{y_5} \cap S = \{u_2\}$. By Lemma 3.3(1), $u_1 \succ \bigcup_{i=1}^{5} V(C_i) - \{z_1, y_3\}$ and $u_2 \succ \bigcup_{i=1}^{5} V(C_i) - \{y_4, y_5\}$. For $2 \le i \le 5$, choose $w_i \in V(C_i) - \{y_i\}$. Then $w_i \in N_{C_i}(u) \cap N_{C_i}(u_1) \cap N_{C_i}(u_2) \cap N_{C_i}(u_3)$. But this contradicts Corollary 3.4 since $|\{w_2, w_3, w_4, w_5\}| = 4$. Hence, Case 1.2 cannot occur and therefore Case 1 cannot occur.

Case 2. $I_{u_3} \cap S = \{u_2\}$.

By applying similar arguments as in the proof of Case 1, Case 2 cannot occur.

Case 3. $I_{u_3} \cap S = \{u_4\}$.

For $2 \leq i \leq t$, choose $y_i \in N_{C_i}(u_2)$. Then applying similar arguments as in the proof of Case 1, Case 3 cannot occur. This completes the proof of our result. ■

Lemma 4.3. *Let G be a connected 3-γ_i-vertex-critical graph and S a minimum cutset of G. Suppose $\Delta(G[S]) \leq 1$ and $t = \omega(G - S) \geq |S| \geq 5$. Let $C_1, C_2, \ldots, C_t$ be the components of $G - S$. Then for $x \in V(C_i)$, $1 \leq i \leq t$, $I_x - S \subseteq V(C_i) - \{x\}$.*

Proof. For $1 \leq i \leq t$, let $x \in V(C_i)$. Assume that $I_x = \{u, z\}$. By Lemma 4.2, we may assume that $u \in S$ and $z \notin S$. Clearly, $xu \notin E(G)$. Suppose to the contrary that $z \notin V(C_i)$. Then $z \in V(C_j)$ for some j, $j \neq i$. Because $\Delta(G[S]) \leq 1$, z dominates at least $|S| - 2$ vertices of S. Without loss of generality we may assume that $i = 1$ and $j = 2$. Since $I_x = \{u, z\}$, $u \succ \bigcup_{i=1}^{t} V(C_i) - (V(C_2) \cup \{x\})$. By Lemma 4.2, $|I_z \cap S| = 1$. We first show that $\{u\} \neq I_z \cap S$. Suppose this is not the case. Then $\{u\} = I_z \cap S$. By Lemma 3.3(1), $u \succ \bigcup_{i=1}^{t} V(C_i) - \{x, z\}$ and $I_z = \{u, x\}$. It follows by Lemma 1.2 that $I_u - S \subseteq \{x, z\}$. Put $\{w\} = I_u \cap S$. Then $wu \notin E(G)$. Since $I_x = \{u, z\}$, $wz \in E(G)$. Consequently, $I_u = \{w, x\}$. Then $wx \notin E(G)$. But this contradicts the fact that $I_z = \{u, x\}$ since $wu \notin E(G)$. This prove that $\{u\} \neq I_z \cap S$.

Put $I_z = \{u_1, y\}$ where $u_1 \in S - \{u\}$ and $y \notin S$. Then $zu_1 \notin E(G)$ but $uu_1 \in E(G)$ since $I_x = \{u, z\}$. Because $\Delta(G[S]) \leq 1$, no vertex of $\{u, u_1\}$ is adjacent to any vertex of $S - \{u, u_1\}$. Thus y dominates $S - \{u, u_1\}$. We next show that $y \in V(C_2)$. Suppose to the contrary that $y \notin V(C_2)$. Consider $G - y$. Since y dominates $S - \{u, u_1\}$, $I_y \cap S \subseteq \{u, u_1\}$ by Lemma 1.2. By Lemma 4.2, either $I_y \cap S = \{u\}$ or $I_y \cap S = \{u_1\}$. If $I_y \cap S = \{u\}$, then the only vertex of $I_y - \{u\} \subseteq \bigcup_{i=1}^{t} V(C_i)$ dominates $x \in V(C_1)$ and $z \in V(C_2)$ which is not possible. Hence, $I_y \cap S = \{u_1\}$. Since $I_z \cap S = \{u_1\}$, by Lemma 3.3(1), we have $u_1 \succ \bigcup_{i=1}^{t} V(C_i) - \{z, y\}$ and $I_y = \{u_1, z\}$. Thus z dominates $S - \{u, u_1\}$. It then follows by Lemma 1.2 that $I_{u_1} - S \subseteq \{z, y\}$ and $u \notin I_{u_1}$. But this contradicts the fact that I_{u_1} is independent since both z and y dominates $S - \{u, u_1\}$ and $|I_{u_1} \cap S| = 1$ by Lemma 4.2. This proves that $y \in V(C_2)$. It then follows that $u_1 \succ \bigcup_{i=1}^{t} V(C_i) - V(C_2)$.

Now choose $x_1 \in V(C_1) - \{x\}$. Such an x_1 exists by Lemma 3.1(1) since $xu \notin E(G)$. Further, for $3 \leq i \leq t$, choose $v_i \in V(C_i)$. Put $A = \{x_1, v_3, v_4, \ldots, v_t\}$. It is easy to see that if $a \in A$, $a \in N_{C_i}(u) \cap N_{C_i}(u_1)$. By Lemma 1.2, $I_a \cap \{u, u_1\} = \emptyset$ for each $a \in A$. Since $|A| = t - 1 \geq 4$ and $|S - \{u, u_1\}| = |S| - 2 < t - 1$, by Lemma 4.2 and Pigeonhole Principle, it follows that there is $u_2 \in S - \{u, u_1\}$ such that $\{u_2\} = I_{a_1} \cap S = I_{a_2} \cap S$ where $\{a_1, a_2\} \subseteq A$. By Lemma 3.3(1), $u_2 \succ \bigcup_{i=1}^{t} V(C_i) - \{a_1, a_2\}$, $I_{a_1} = \{u_2, a_2\}$ and $I_{a_2} = \{u_2, a_1\}$. Further, $I_{u_2} - S \subseteq \{a_1, a_2\}$. Put $\{u_3\} = I_{u_2} \cap S$ for some $u_3 \in S - \{u_2\}$. Then $u_2u_3 \notin E(G)$. Since $I_{a_1} = \{u_2, a_2\}$ and $I_{a_2} = \{u_1, a_1\}$, it follows that $u_3a_1, u_3a_2 \in E(G)$. But this contradicts the fact that I_{u_2} is independent. This completes the proof of our lemma. ■

Theorem 4.4. *Let G be a connected 3-γ_i-vertex-critical graph and S a minimum cutset of G. If $\Delta(G[S]) \leq 1$ and $|S| \geq 5$, then $\omega(G-S) \leq |S|-1$.*

Proof. Let $C_1, C_2, \ldots, C_t$ be the components of $G-S$, where $t = \omega(G-S)$. Suppose to the contrary that $t \geq |S|$. For $1 \leq i \leq t$, choose $x_i \in V(C_i)$. By Lemma 4.2, $|I_{x_i} \cap S| = 1$. Put $\{u_i\} = I_{x_i} \cap S$. It then follows by Lemma 4.3 that $u_i \succ \bigcup_{l=1}^{t} V(C_l) - V(C_i)$ and thus $u_i \neq u_j$ for $i \neq j$. Consequently, each vertex of $V(C_i)$ is adjacent to every vertex of $S - \{u_i\}$. Moreover, $I_{u_i} - S \subseteq V(C_i)$ by Lemma 1.2. But then I_{u_i} is not independent since $|I_{u_i} \cap S| = 1$ by Lemma 4.2, a contradiction. This settles our theorem. ■

Even though we do not give an upper bound on $\omega(G-S)$ when $\Delta(G[S]) \geq 2$ for $|S| \geq 5$, we can provide an upper bound on $\omega(G-S)$ for $3 \leq |S| \leq 4$. We now turn our attention to these cases.

Theorem 4.5. *Let G be a connected 3-γ_i-vertex-critical graph and S a minimum cutset of G. If $|S| = 3$, then $\omega(G-S) \leq 3$. Further, the bound is best possible.*

Proof. Let $S = \{u_1, u_2, u_3\}$ and let $C_1, C_2, \ldots, C_t$ be the components of $G-S$. Suppose to the contrary that $t \geq 4$. Consider $G - u_1$. By Lemma 3.2(2), we may assume that $u_2 \in I_{u_1} \cap S$. Put $\{z\} = I_{u_1} - \{u_2\}$. We first show that $z = u_3$. Suppose this is not the case. Then $z \in \bigcup_{i=1}^{t} V(C_i)$. We may assume that $z \in V(C_1)$. Then $u_2 \succ \bigcup_{i=2}^{t} V(C_i)$. For $2 \leq i \leq t$ choose $v_i \in N_{C_i}(u_3)$. Such a v_i exists by Lemma 3.1(2). Now $v_i \in N_{C_i}(u_2) \cap N_{C_i}(u_3)$. But this contradicts Corollary 3.4 since $t-1 \geq 3$. This proves that $z \notin \bigcup_{i=1}^{t} V(C_i)$. Hence, $z = u_3$ and thus $I_{u_1} = \{u_2, u_3\}$. For $1 \leq i \leq t$, choose $w_i \in N_{C_i}(u_1)$. Since $I_{u_1} = \{u_2, u_3\}$ and $|\{w_i | 1 \leq i \leq t\}| \geq 4$, it follows by Pigeonhole Principle that either u_2 or u_3 is adjacent to at least two vertices of $\{w_i | 1 \leq i \leq t\}$. We may assume without loss of generality that $w_1u_2, w_2u_2 \in E(G)$. Then, by Lemmas 1.2 and 3.2(2), $I_{w_1} \cap S = I_{w_2} \cap S = \{u_3\}$. Then $w_1u_3, w_2u_3 \notin E(G)$. By Lemma 3.3(1), $u_3 \succ \bigcup_{i=1}^{t} V(C_i) - \{w_1, w_2\}$. Consequently, $\{w_3, w_4\} \subseteq N_G(u_1) \cap N_G(u_3)$ and thus $I_{w_3} \cap S = I_{w_4} \cap S = \{u_2\}$. Then $w_3u_2, w_4u_2 \notin E(G)$. Again, by Lemma 3.3(1), $u_2 \succ \bigcup_{i=1}^{t} V(C_i) - \{w_3, w_4\}$. It then follows by Lemmas 3.1(2) and 3.2(1), that $t = 4$. By Lemma 3.1(1), $|V(C_i)| \geq 2$, for $1 \leq i \leq 4$. For $1 \leq i \leq 4$, we now choose $z_i \in V(C_i) - \{w_i\}$. Clearly, $z_i \succ S - \{u_1\}$. But this contradicts Corollary 3.4 since $|\{w_1, w_2, w_3, w_4\}| = 4$. This proves the first part of our theorem.

For a positive integer n, let G be the graph in Figure 4.1. It is easy to see that G is connected 3-γ_i-vertex-critical with $\{u_1, u_2, u_3\}$ a minimum cutset. Clearly, $\omega(G - \{u_1, u_2, u_3\}) = 3$. This shows that the bound in our theorem is best possible. ■

Theorem 4.6. *Let G be a connected 3-γ_i-vertex-critical graph and S a minimum cutset of G. If $|S| = 4$, then $\omega(G-S) \leq 3$. Further, the bound is best possible.*

Proof. Let $S = \{u_1, u_2, u_3, u_4\}$. Put $t = \omega(G-S)$. Let $C_1, C_2, \ldots, C_t$ be the components of $G-S$. Suppose to the contrary that $t \geq 4$.

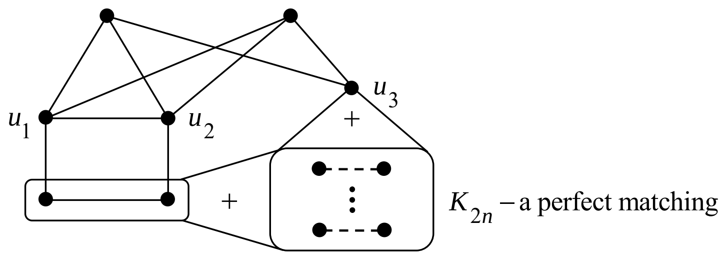

Figure 4.1: A 3-γ_i-vertex-critical graph with a minimum cutset of order 3.

Claim 1. *$G[S]$ contains an edge.*

Proof. Suppose this is not the case. Then S is independent. Thus $|I_{u_i} \cap S| = 1$ for $1 \leq i \leq 4$ by Lemma 3.2(2). We may assume that $I_{u_1} \cap S = \{u_2\}$. Put $I_{u_1} - \{u_2\} = \{z\}$. Assume that $z \in V(C_1)$. Then $u_2 \succ \bigcup_{i=2}^{t} V(C_i)$. By Lemma 1.2, $I_{u_2} - S \subseteq V(C_1)$. Then the only vertex of $I_{u_2} \cap S$ dominates $\bigcup_{i=2}^{t} V(C_i)$. Now let $w \in S - (I_{u_2} \cup \{u_2\})$. For $2 \leq i \leq t$, choose $y_i \in N_{C_i}(w)$. Such a y_i exists by Lemma 3.1(2). Observe that $|N_S(y_i)| \geq 3$. In fact, $N_S(y_i) = ((I_{u_2} \cap S) \cup \{w, u_2\})$ for $2 \leq i \leq t$ by Lemma 3.2(1). But this contradicts Corollary 3.4 since $|\{y_2, y_3, \ldots, y_t\}| = t - 1 \geq 3$. This settles our claim. □

We may now assume that $\deg_S(u_1) = \Delta(G[S])$. By Claim 1, $\deg_S(u_1) \geq 1$. Further, by Lemma 3.2(1), $\deg_S(u_1) \leq 2$. Thus $1 \leq \deg_S(u_1) \leq 2$. Let $\{u_2\} \subseteq N_S(u_1)$. Consider $G - u_1$. We may assume that $I_{u_1} \cap S = \{u_3\}$ by Lemmas 1.2 and 3.2(2). Put $\{z\} = I_{u_1} - \{u_3\}$. Then $u_1u_3, u_1z \notin E(G)$. We first show that $z \neq u_4$. Suppose this is not the case. Then $I_{u_1} = \{u_3, u_4\}$. Thus $u_1u_3, u_1u_4 \notin E(G)$ but either $u_2u_3 \in E(G)$ or $u_2u_4 \in E(G)$. Consequently, $\deg_S(u_2) \geq 2 > \deg_S(u_1)$. This contradicts the fact that $\deg_S(u_1) = \Delta(G[S])$. Hence, $z \neq u_4$. Assume that $z \in V(C_1)$. Then $u_3 \succ \bigcup_{i=2}^{t} V(C_i)$. For $2 \leq i \leq t$, choose $y_i \in N_{C_i}(u_4)$. Such a y_i exists be Lemma 3.1(2). Observe that $y_i \in N_{C_i}(u_3) \cap N_{C_i}(u_4)$ for $2 \leq i \leq t$. It follows by Lemma 1.2 that $I_{y_i} \cap S \subseteq \{u_1, u_2\}$ for $2 \leq i \leq t$. Since $u_1u_2 \in E(G)$, $|I_{y_i} \cap S| = 1$. By Lemma 3.3(2), $|\{y_i | I_{y_i} \cap S = \{u_1\}\}| \leq 2$ and $|\{y_i | I_{y_i} \cap S = \{u_2\}\}| \leq 2$. Because $zu_1 \notin E(G)$, $|\{y_i | I_{y_i} \cap S = \{u_1\}\}| \leq 1$ by Lemma 3.3(1). Consequently, $|\{y_i | I_{y_i} \cap S = \{u_1\}\}| = 1$, $|\{y_i | I_{y_i} \cap S = \{u_2\}\}| = 2$ and thus $t = 4$. We may assume that $I_{y_2} \cap S = \{u_1\}$, $I_{y_3} \cap S = I_{y_4} \cap S = \{u_2\}$. By Lemma 3.3(1), $u_2 \succ \bigcup_{i=1}^{4} V(C_i) - \{y_3, y_4\}$. Since $zu_1 \notin E(G)$, the only vertex of $I_{y_2} - \{u_1\} \subseteq V(C_1)$. Thus $u_1 \succ \bigcup_{i=2}^{4} V(C_i) - \{y_2\}$. By Lemma 3.1(1), $|V(C_i)| \geq 2$ for $2 \leq i \leq 4$ since $u_1y_2, u_2y_3, u_2y_4 \notin E(G)$. For $2 \leq i \leq 4$, we now choose $w_i \in V(C_i) - \{y_i\}$. Observe that $w_i \in N_{C_i}(u_1) \cap N_{C_i}(u_2) \cap N_{C_i}(u_3)$. But this contradicts Corollary 3.4 since $|\{w_2, w_3, w_4\}| = 3$. This proves the first part of our theorem.

We now show that the bound is best possible. Let G be a graph in Figure 4.2. It is easy to see that G is connected 3-γ_i-vertex-critical with $S = \{u_1, u_2, u_3, u_4\}$ is a minimum cutset. Clearly, $\omega(G - S) = 3$. ■

We conclude our paper by making the following conjecture.

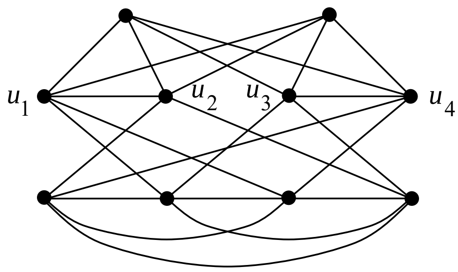

Figure 4.2: A 3-γ_i-vertex-critical graph with a minimum cutset of order 4.

Conjecture. *Let G be a connected 3-γ_i-vertex-critical graph and S a minimum cutset of G. If $|S| \geq 5$, then $\omega(G - S) \leq |S| - 1$.*

If the conjecture is true, then it follows by Theorems 4.1, 4.5 and 4.6 that every connected 3-γ_i-vertex-critical graph of even order contains a perfect matching.

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