AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **71(3)** (2018), Pages 381–393

Rainbow connectivity in some Cayley graphs*

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Abstract

The rainbow and strong rainbow connection numbers, as well as variants of these in which the edge colorings are required to be proper, are estimated and in some cases determined exactly for some Cayley graphs, including the Hamming graphs $H(n, q, k)$, excluding the cases (i) $k = n$ and (ii) $k < n$, k even and $q = 2$.

* This research was supported by NSF grant HRD-1247525.

1 Introduction

In this paper G is a finite connected graph with the edges colored. A subgraph H of G is *rainbow* with respect to the coloring if no color appears more than once on the edges of H. The graph G is *rainbow-connected* by the coloring if for any two vertices u, v of G there is a rainbow $u-v$ path in G. In this case, the coloring of edge set $E(G)$ of G is said to be a *rainbow connection coloring* of G. If for every $u, v \in V(G)$ (the vertex set of G), there is a rainbow u–v geodesic (a shortest u–v path) in G, then G is *strongly rainbow-connected* by the coloring and the coloring is a *strong rainbow connection coloring* of G.

The *rainbow connection number* of G, denoted $\mathrm{rc}(G)$, is the minimum number of colors needed for a rainbow connection coloring of G. The *strong rainbow connection number* of G, denoted $\mathrm{src}(G)$, is the minimum number of colors needed for a strong rainbow connection coloring of G. A (strong) rainbow connection coloring of G with $(\mathrm{s})\mathrm{rc}(G)$ colors is a *minimum* (strong) rainbow connection coloring of G.

Any coloring of the edges of K_n, the complete graph on n vertices, is a strong rainbow connection coloring of K_n. Hence for $n > 1$, $\mathrm{rc}(G) = \mathrm{src}(G) = 1$. This example serves to emphasize that rainbow connection colorings are not necessarily proper edge colorings: different edges incident to the same vertex may carry the same color.

We venture to define proper (strong) rainbow connection colorings in the same way as (strong) rainbow connection colorings were defined, with the additional requirement that the colorings be proper. Let the corresponding edge coloring parameters, the proper rainbow connection number and the proper strong rainbow connection number, be denoted prc and psrc, respectively. Clearly, for $n = 1, 2, \ldots$,

$$\mathrm{prc}(K_n) = \mathrm{psrc}(K_n) = \chi'(K_n),$$

where $\chi'(K_n)$ is the chromatic index of K_n.

For a connected simple graph G, we have

$$\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq \mathrm{src}(G) \leq \mathrm{psrc}(G) \leq |E(G)|, \tag{1}$$

$$\mathrm{rc}(G) \leq \mathrm{prc}(G), \tag{2}$$

and

$$\chi'(G) \leq \mathrm{prc}(G) \leq \mathrm{psrc}(G) \tag{3}$$

where $\mathrm{diam}(G)$ denotes the diameter of G.

Rainbow connectivity was introduced by Chartrand et al. in [1], where $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ are evaluated for G in various classes (quite spectacularly when G is a complete multipartite graph), and the question of possible ordered pairs $(\mathrm{rc}(G), \mathrm{src}(G))$ is addressed. Rainbow connections are studied in [3, 4], among which [4] is a survey of topics and problems. In particular, [4] contains a survey of recent results on rainbow connectivity of Cayley graphs and cartesian products. These results are essentially disjoint from results in our paper. We shall point out overlaps as they occur in the progress of our paper.

For our purposes, several of the results from [1] will be useful, which are collected in the following. (Here, C_n denotes the cycle on n vertices.)

Proposition 1.1 ([1]). *(a) $rc(G) = 2$ if and only if $src(G) = 2$;*

(b) $rc(G) = |E(G)|$ if and only if G is a tree;

(c) if $n \geq 4$, $rc(C_n) = src(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Corollary 1.2. *$src(G) = |E(G)|$ if and only if G is a tree.*

Proof. If G is a tree then $|E(G)| = \mathrm{rc}(G) \leq \mathrm{src}(G) \leq |E(G)|$ by Proposition 1.1(b) and inequalities (1), and hence $\mathrm{src}(G) = |E(G)|$.

Now suppose that G is not a tree. Then G contains a shortest cycle C. For any two vertices on C, the shortest path around C from one to the other is a shortest path in G between the two. It follows that for any geodesic in G containing at least an edge of C, there is a geodesic in G with the same end vertices whose intersection with C is a geodesic in C.

By Proposition 1.1 (c) and the fact that $\mathrm{src}(C_3) = \mathrm{src}(K_3) = 1$, we have that $\mathrm{src}(C) < |E(C)|$. Give C a minimum strong rainbow connection coloring and then color the remaining $|E(G)| - |E(C)|$ edges of G with that many different colors, all different from the colors on the edges of C. We then have a strong rainbow connection coloring of G with $\mathrm{src}(C) + |E(G)| - |E(C)| < |E(G)|$ colors, and hence $\mathrm{src}(G) < |E(G)|$. $\square$

The result of Corollary 1.2 is asserted in [1] without proof.

By Proposition 1.1 and inequalities (1)–(3), if G is a tree then $\mathrm{prc}(G) = \mathrm{psrc}(G) = |E(G)|$. The converse does not hold, since $\mathrm{prc}(K_3) = \mathrm{psrc}(K_3) = 3 = |E(K_3)|$. Presently we do not know of a characterization of connected graphs G such that $\mathrm{prc}(G) = |E(G)|$.

This is just one of numerous open questions bearing on the general problem of characterizing the quadruples (a, b, c, d) such that there exits a graph G with $a = \mathrm{rc}(G)$, $b = \mathrm{src}(G)$, $c = \mathrm{prc}(G)$, and $d = \mathrm{psrc}(G)$. Pursuit of this problem is not our aim here. We shall close this introduction with something that we have noticed, that refutes an obvious conjecture about the relation between rc and prc. The obvious conjecture: if $\mathrm{rc}(G) \geq \chi'(G)$, then $\mathrm{rc}(G) = \mathrm{prc}(G)$. A class of counterexamples is depicted in Figure 1.

Explanation: P_m is the path on m vertices. In any rainbow connection coloring of G, all $m-1$ edges of P_m are colored differently and with colors different from any of the colors on edges of the K_n incident to v. Therefore $\mathrm{src}(G) \geq \mathrm{rc}(G) \geq 1 + m - 1 = m$, and $\mathrm{src}(G) = m$ is achieved by coloring all edges of K_n with one color.

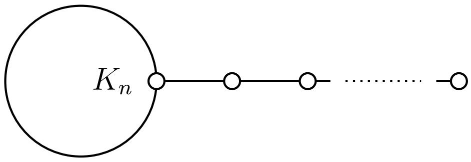

Figure 1: A graph G such that, if $m, n \geq 2$, then $\text{rc}(G) = \text{src}(G) = m$ and $\text{prc}(G) = \text{psrc}(G) = n + m - 2$. One end of a path of order m is identified to a vertex v of the complete graph of order n.

In any proper rainbow connection coloring of G there must be at least

$$\chi'(K_n) = \begin{cases} n-1, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

colors on K_n as well as $m-1$ colors on P_m different from the $n-1$ different colors on edges of K_n incident to v. Therefore $\text{psrc}(G) \geq \text{prc}(G) \geq n-1+m-1 = n+m-2$. We get $\text{psrc}(G) \leq n+m-2$ with a proper strong rainbow connection coloring with $\chi'(K_n)$ colors on K_n and $m-1$ colors on P_m, all different from those on K_n when n is even; when n is odd, use once on P_m the one color on K_n which is not on an edge incident to v.

We have

$$\begin{aligned} \text{psrc}(G) = \text{prc}(G) &= n + m - 2 \\ &\geq m = \text{src}(G) = \text{rc}(G), \end{aligned}$$

while $\chi'(G) = \Delta(G) = n$. Therefore, whenever $m > n > 2$,

$$\text{prc}(G) > \text{rc}(G) > \chi'(G).$$

2 Cubes and related graphs

The n-cube, Q_n, $n = 1, 2, \ldots$, has vertex set $V(Q_n) = \{0,1\}^n = \{\text{binary words of length } n\}$, with u, $v \in \{0,1\}^n$ adjacent if and only if u and v differ at exactly one position. Let the n positions in those binary words be numbered $1, \ldots, n$ and color $E(Q_n)$ as follows: if $uv \in E(Q_n)$ then color uv with the integer $i \in \{1, \ldots, n\}$ that is the number of the position at which u and v differ. This coloring and inequalities (1) show that $\text{diam}(Q_n) = n = \text{rc}(Q_n) = \text{src}(Q_n) = \text{prc}(Q_n) = \text{psrc}(Q_n)$ for all $n = 1, 2, \ldots$. This coloring also has the remarkable property that a path in Q_n is rainbow with respect to this coloring if and only if it is a geodesic path.

Remark. The fact that $\text{rc}(Q_n) = n$ appears in [4] (Corollary 2.67).

The coloring of Q_3 is illustrated in Figure 2. The illustration in Figure 2 shows a proper strong rainbow connection coloring of Q_3 with three colors that is essentially

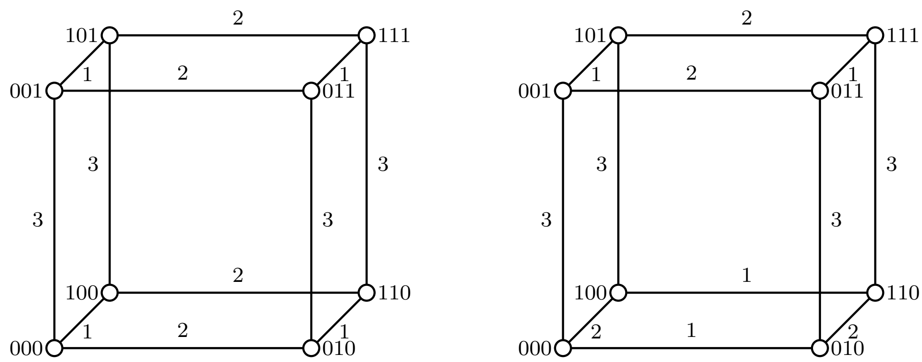

Figure 2: Two essentially different proper strong rainbow connection colorings of Q_3 with 3 colors.

different from the "coordinate-of-difference" coloring. You cannot get from one coloring to the other by renaming the colors and then applying an automorphism of Q_3. To verify this difference, observe that, for the second coloring, there are geodesics that are not rainbow colored.

As in [5], let $\square$ denote the direct Cartesian product of graphs. Since, for $n > 1$, $Q_n = Q_{n-1}\square K_2$, it is easy to see that the second coloring in Figure 2 can be extended to a proper strong rainbow connection coloring of Q_n with n colors which is essentially different from the "coordinate-of-difference" coloring.

But we digress! Our aim here is to give some answers to the question: what makes the cube Q_n, $n = 1, 2, \dots$, so wonderful with regard to rainbow connectivity? We depart from the obvious: $Q_n = \underbrace{K_2\square\cdots\square K_2}_{n \text{ times}}$ is a direct product of complete graphs. These are Cayley graphs.

Cayley graphs: Let $(A, +, 0)$ be a non-trivial finite abelian group. Suppose that $\emptyset \neq S \subseteq A\backslash\{0\}$ and $S = -S$. The *Cayley graph* $G = \mathrm{Cay}(A, S)$ is defined by: $V(G) = A$, and $a, b \in A$ are adjacent in G if and only if $b - a \in S$.

$\mathrm{Cay}(A, S)$, thus defined, is connected if and only if the subgroup of A generated by S is A itself. Some authors require this in the definition of Cayley graphs; we do not.

Cayley graphs can be associated with non-abelian groups and self-inverse subsets of them; for our purposes there is nothing to be gained (that we can see at this point) by thus enlarging our stock of Cayley graphs. All our groups here will be finite abelian groups.

Cayley digraphs can be defined similarly: given a non-trivial abelian group $(A, +, 0)$ and $\emptyset \neq S \subset A\backslash\{0\}$, the digraph $D = \mathrm{Cay}^*(A, S)$ has vertex set A, and *arc* set $\{(a, b) \in A^2 \mid b - a \in S\}$. Note that S is not required to be self-inverse. D is strongly connected if and only if every $a \in A$ is a sum of elements of S. In other words: the *semigroup* generated by S is A. We need the change in notation, using $*$, for a reason: if $S = -S$ then the digraph $D = \mathrm{Cay}^*(A, S)$ is obtained from the graph $G = \mathrm{Cay}(A, S)$ by replacing each edge of G by two arcs, one in each direction. In much of graph theory the graph G and the digraph $D = G^*$ are interchangeable, but this is not clear in questions about rainbow connectivity. There will be a question about this in the last section.

If $n > 1$ then K_n is a Cayley graph: take any abelian group $(A, +, 0)$ with $|A| = n$ and $S = A \setminus \{0\}$. The direct product of Cayley graphs is a Cayley graph: if $(B_i, +, 0)$ is an abelian group, $i = 1, \ldots, n$ (we omit indices on $+$ and 0), $\emptyset \neq S_i \subset B_i \setminus \{0\}$, $S_i = -S_i$, $i = 1, \ldots, n$ and $G_i = \mathrm{Cay}(B_i, S_i)$, then $G = G_1 \square \cdots \square G_n = \mathrm{Cay}(A, S)$ where $A = B_1 \times \cdots \times B_n$, a direct sum of finite abelian groups, and $S = \{(b_1, \ldots, b_n) \in A \mid b_i = 0$ for all but one value of i, and, for that value of i, $b_i \in S_i\}$.

We will have more on Cayley graphs in the next section. We conclude this section with an obvious result about direct products. In the following, for any graph H, the maximum degree of H is denoted $\Delta(H)$.

Theorem 2.1. *Suppose that $n > 1$, $G_1, \ldots, G_n$ are connected finite simple graphs, each with at least one edge, and $G = G_1 \square \cdots \square G_n$. Then for each $F \in \{rc, src, prc, psrc\}$,*

$$\sum_{i=1}^{n} \mathrm{diam}(G_i) \quad \leq F(G) \quad \leq \sum_{i=1}^{n} F(G_i). \tag{4}$$

Also,

$$\sum_{i=1}^{n} \Delta(G_i) \quad \leq \quad \mathrm{prc}(G). \tag{5}$$

Proof. The first inequality in (4) follows from inequalities (1) and (2), and the fact that $\mathrm{diam}(G) = \sum_{i=1}^{n} \mathrm{diam}(G_i)$. Inequality (5) follows from (3) and the fact that $\chi'(G) \geq \Delta(G) = \sum_{i=1}^{n} \Delta(G_i)$. What remains to be shown is the upper inequality in (4).

Let each G_i be edge-colored with $F(G_i)$ colors so that the coloring has the property (rainbow, strong rainbow, proper rainbow, or proper strong rainbow connection) indicated by F. Let the color sets on $E(G_i)$ and $E(G_j)$ be disjoint, whenever $i \neq j$. Now color $E(G)$ in the obvious way: if $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are adjacent in G then color uv with the color on the edge $u_i v_i \in E(G_i)$, where $i \in \{1, \ldots, n\}$ is the unique index such that $u_i \neq v_i$. The resulting coloring of G has the property indicated by F, and uses $\sum_{i=1}^{n} F(G_i)$ colors. Thus

$$F(G) \quad \leq \quad \sum_{i=1}^{n} F(G_i).$$

□

Remark. For $F = \mathrm{rc}$, the right-most inequality in (4) appears in [4] (Theorem 2.61), together with the remark that the equality holds if $\mathrm{rc}(G_i) = \mathrm{diam}(G_i)$ for each i. For every $F \in \{\mathrm{rc, src, prc, psrc}\}$, a similar condition for equality follows from (4).

Corollary 2.2. *Suppose that integers n, $p_1, \ldots, p_n > 1$ and $G = K_{p_1} \square \cdots \square K_{p_n}$. Then* $\mathrm{rc}(G) = \mathrm{src}(G) = n$ *and* $\sum_{i=1}^{n}(p_i - 1) \le \mathrm{prc}(G) \le \mathrm{psrc}(G) \le \sum_{i=1}^{n} \chi'(K_{p_i})$.

Proof. As previously noted,

$$\mathrm{diam}(K_{p_i}) = 1 = \mathrm{rc}(K_{p_i}) = \mathrm{src}(K_{p_i}),$$

$$\mathrm{psrc}(K_{p_i}) = \chi'(K_{p_i}) = \mathrm{prc}(K_{p_i}),$$

and, by definition, $\Delta(K_{p_i}) = p_i - 1$. The conclusions follow from Theorem 2.1. $\square$

Remark. The conclusion $\mathrm{rc}(G) = n$ in Corollary 2.2 appears in [4] (Corollary 2.67).

With G as in Corollary 2.2, note that the "coordinate-of-difference" coloring serves to verify that $\mathrm{rc}(G) = \mathrm{src}(G) = n$. Also, if all p_i are even, then $\chi'(K_{p_i}) = p_i - 1$, $i = 1, \ldots, n$. We have

$$\mathrm{prc}(G) = \mathrm{psrc}(G) = \sum_{i=1}^{n}(p_i - 1).$$

What happens when some of the p_i are odd? This is an open question.

Corollary 2.2 generalizes previously noted facts about

$$Q_n = K_2 \square \cdots \square K_2.$$

It may be noted that for all $p_1, \ldots, p_n > 1$ the "coordinate-of-difference" edge coloring of

$$G = K_{p_1} \square \cdots \square K_{p_n}$$

has the property that a path in G is rainbow if and only if it is a geodesic in G.

3 Hamming graphs

In this section, let n, q, and k be positive integers satisfying $n \ge 2$, $q \ge 2$ and $n \ge k$. The Hamming graph $H = H(n, q, k)$ is defined by $V(H) = A^n$, where A is a set (called an *alphabet*, in this context) with q elements (called *letters*), and u, $v \in A^n$ are adjacent in H if and only if they differ at exactly k positions. (In the language of coding theory, the *Hamming distance* between u and v is k.) The cubes $Q_n = H(n, 2, 1)$ are examples of Hamming graphs where $k = 1$. The elements of A^n will be denoted as *words* rather than as n-tuples: $a_1 \cdots a_n$ rather than $(a_1, \ldots, a_n)$.

The Hamming graph $H(n, q, k)$ is a Cayley graph: let A be an abelian group with $|A| = q$, let A^n be the n-fold direct sum of A, and let $S = \{a_1 \cdots a_n \in A^n \mid a_i \neq 0$ for exactly k values of $i\}$. Then $S = -S$ and $H(n, q, k) = \mathrm{Cay}(A^n, S)$. For more clarity, $H = H(n, q, k)$ is a Cayley graph for which $V(H)$ is a direct sum of finite abelian groups (all identical to A), but H is not a direct product of Cayley graphs unless $k = 1$, in which case $H = K_q \square \cdots \square K_q$. From Section 2, we know something about the rainbow connectivity of cubes.

For u, v words of the same length, let $h(u,v)$ denote the Hamming distance between u and v, that is, the number of positions at which u and v differ. Let the alphabet A be a finite abelian group. For $u \in A^n$, the *Hamming weight* of u is $wt(u) = h(u, 0^n)$, the number of positions in the word u where the letter is not 0, the identity element in A. Note that for all $u, v \in A^n$, $h(u,v) = wt(u - v)$. Also, u and v are adjacent in $H(n,q,k)$ ($q = |A|$) if and only $v = u + w$ for some $w \in A^n$ such that $wt(w) = k$. Therefore, adjacency is *translation invariant* in $H = H(n,q,k) = \mathrm{Cay}(A^n, S)$: if u, $v, x \in V(H)$ and u and v are adjacent in H, then so are $u + x$ and $v + x$.

Theorem 3.1. *$H = H(n,q,k)$ is connected if and only if either (i) $q > 2$ or (ii) $q = 2$, $k < n$, and k is odd.*

Proof. If n, q, k satisfy neither (i) nor (ii) then $q = 2$, so we can take $A = \{0, 1\}$, and either $k = n$ or $k < n$ and k is even. If $k = n$ then $H = H(n, 2, n)$ is a matching consisting of 2^{n-1} independent edges, so H is not connected, because $n > 1$. If k is even and u, $v \in A^n$ are adjacent then $wt(u) \equiv wt(v) \bmod 2$. Therefore there is no path in H from a vertex of even Hamming weight to one of odd Hamming weight, so H is not connected. This finishes the proof of the "only if" assertion.

Suppose that $q = 2$, $k < n$, and k is odd. When $k = 1$, $H = Q_n$, which we know to be connected, so we may assume that $3 \le k < n$. Because adjacency is translation invariant, to show that H is connected it suffices to show that there is a walk in H from $\mathbf{0} = 0^n$ to each vertex of Hamming weight 1. It will suffice to show that there is such a walk from $\mathbf{0}$ to $u = 10^{n-1}$. In other words, it suffices to show that 10^{n-1} is in the subgroup of A^n generated by $S = \{v \in A^n \mid wt(v) = k\}$.

For any two elements i, $j \in \{1, \ldots, n\}$, $i \neq j$, two k-subsets of $\{1, \ldots, n\}$ can be found with symmetric difference $\{i, j\}$. This implies that v, $w \in S$ can be found such that $v + w$ is the binary word with Hamming weight 2, with 1's in positions i and j. (Note that $A = \{0, 1\} \simeq \mathbb{Z}_2$, so $+$ and $-$ are the same in A and in A^n.) Therefore,

$$10^{n-1} = 1^k 0^{n-k} + \sum_{j=1}^{(k-1)/2} 0^{2j-1} 11 0^{n-2j-1}$$

is in the subgroup of A^n generated by S. That $H(n,q,k)$ is connected whenever $q > 2$ will follow from the following lemma. □

Let the distance in H between two vertices u, $v \in A^n = V(H)$ be denoted $\mathrm{dist}_H(u,v)$.

Lemma 3.2. *Let $H = H(n,q,k)$, $q > 2$, u, $v \in A^n = V(H)$, and $h(u,v) = r > 0$. If $r < k$ then $\mathrm{dist}_H(u,v) = 2$. Otherwise, $\mathrm{dist}_H(u,v) = \lceil r/k \rceil$.*

Proof. Let A be an abelian group, so that $H = \mathrm{Cay}(A^n, S)$, as above. Since adjacency is translation invariant, we may as well assume that $u = \mathbf{0}$, so v is a word of Hamming weight r. Without loss of generality, $v = a_1 \cdots a_r 0^{n-r}$, $a_1, \ldots, a_r \in A \backslash \{0\}$.

If $r < k$ then $\mathbf{0}$ and v are not adjacent in H, so $\mathrm{dist}_H(\mathbf{0}, v) \geq 2$. On the other hand, because $|A \backslash \{0\}| = q - 1 \geq 2$, it is straightforward to find $x = x_1 \cdots x_k 0^{n-k}$, $y = y_1 \cdots y_k 0^{n-k} \in S$ satisfying $x_i + y_i = a_i$, $i = 1, \ldots, r, x_i + y_i = 0$, $i = r+1, \ldots, k$. Thus $\mathrm{dist}_H(\mathbf{0}, v) = 2$.

Suppose $r \geq k$. Any sum of t words from $S = \{w \in A^n \mid wt(w) = k\}$ will have Hamming weight no greater than tk. Therefore, $\mathrm{dist}_H(\mathbf{0}, v) \geq \lceil r/k \rceil$. If k divides r, say $r = tk$, then clearly v is a sum of $t = r/k$ words from S. (Note, for future reference, that in every such sum the "supports" of the words—the sets of positions at which the words have non-zero entries—are pairwise disjoint.)

Suppose $r = tk + f$, $t \geq 1$, $1 \leq f < k$. Let x be a sum of $t - 1$ words from S with pairwise disjoint supports such that $wt(v - x) = k + f$. Without loss of generality, $v - x = a_1 \cdots a_{k+f} 0^{n-(k+f)}$. Again, because $q \geq 3$, there are $y_1, \ldots y_k$, $z_{f+1}, \ldots, z_{k+f} \in A \backslash \{0\}$, such that $y_i = a_i$, $i = 1, \ldots, f$, $z_i = a_i$, $i = k+1, \ldots, k+f$, and $y_i + z_i = a_i$, $f + 1 \leq i \leq k$. Let $y = y_1 \cdots y_k 0^{n-k}$, $z = 0^f z_{f+1} \cdots z_{f+k} 0^{n-(k+f)}$; then $v = x + y + z$ is a sum of $t - 1 + 2 = t + 1 = \lceil r/k \rceil$ words from S. (For future reference: although not pairwise disjoint, the supports of the words from S in that sum are all different.) Thus $\mathrm{dist}_H(\mathbf{0}, v) = \lceil r/k \rceil$. □

Corollary 3.3. *If* $q > 2$, $k < n$, *and* $H = H(n, q, k)$ *then* $\mathrm{diam}(H) = \left\lceil \frac{n}{k} \right\rceil$. *If* $q > 2$ *and* $n = k > 1$ *then* $\mathrm{diam}(H) = 2$.

For any Cayley graph $G = \mathrm{Cay}(A, S)$, there is a simple way to color the edges of G: partition $S = P_1 \cup \cdots \cup P_t$ into non-empty sets P_i satisfying $P_i = -P_i$, and then color each edge $uv \in E(G)$ with the index i such that $v - u \in P_i$. This type of coloring is called a *partition edge coloring* of G. It is straightforward to see that each partition edge-coloring of a Cayley graph is *translation invariant*: this means that for u, v, $w \in V(G) = A$, if $uv \in E(G)$ then the color on uv is the same as the color on $(u + w)(v + w)$.

Proposition 3.4. *If* $G = \mathrm{Cay}(A, S)$ *is a Cayley graph, then every translation invariant edge coloring of* G *is a partition edge coloring.*

Proof. If $\varphi : E(G) \to \{1, \ldots, t\}$ is a translation invariant edge coloring of G, then let

$$P_i = \{u \in S \mid \varphi(u0) = i\}.$$

Then $P_1 \ldots, P_t$ partition S, and for $u \in S$, $u0 = (u + 0)(u + (-u))$ so the edges $u0$ and $0(-u)$ have the same color; thus $-P_i = P_i$, $i = 1, \ldots, t$. Finally, φ is the partition edge coloring of G defined by $P_1, \ldots, P_t$. □

Translation invariance is probably not a striking property for an edge coloring of a Cayley graph. Partition edge colorings, on the other hand, are easy to define. By Proposition 3.4, a translation invariant edge coloring may be considered by way of a partition edge coloring. For proper edge colorings, translation invariant colorings are rare.

Proposition 3.5. *Suppose that* $G = Cay(A, S)$ *is a Cayley graph and* S *is partitioned into* $P_1 \cup \cdots \cup P_t$*, with each* P_i *satisfying* $P_i = -P_i$ *and* $P_i \neq \emptyset$*. The partition edge coloring defined by this partition is proper if and only if* $|P_i| = 1$ *for each* $i = 1, \ldots, t$*.*

Proof. If $u, v \in P_i$, $u \neq v$, then $u0$ and $v0$ are edges of G with the same color, and both are incident to the vertex 0. Therefore, if the partition coloring based on the partition is proper, then $|P_i| = 1$, $i = 1, \ldots, t$. The converse is straightforward. □

Corollary 3.6. *If an abelian group* A *contains an element that is not self-inverse, then there is no* $S \subseteq A \backslash \{0\}$ *satisfying* $S = -S$ *such that the Cayley graph* $\mathrm{Cay}(A, S)$ *is connected and has a proper translation invariant edge coloring.*

Proof. If there is an $S \subseteq A\backslash\{0\}$ satisfying $S = -S$ such that $\mathrm{Cay}(A, S)$ is connected and, for some partition $P_1, \ldots, P_t$ of S into non-empty sets satisfying $P_i = -P_i$, $i = 1, \ldots, t$, the partition edge coloring associated with the partition is proper, then, by Proposition 3.5, $|P_i| = 1$ for each $i = 1, \ldots, t$. Since $P_i = -P_i$, it must be that the unique element of each P_i is self-inverse. Since $S = P_1 \cup \cdots \cup P_t$ and the subgraph of A generated by S is all of A, because $\mathrm{Cay}(A, S)$ is connected, it must be that every element of A is self-inverse. □

Theorem 3.7. *Let* $H = H(n, q, k)$ *and suppose that* $q > 2$ *and* $k < n$*. Then*

$$\left\lceil \frac{n}{k} \right\rceil \leq rc(H) \leq \binom{n}{k}$$

and

$$(q-1)^k \binom{n}{k} \leq prc(H) \leq psrc(H) \leq \frac{1}{2} q^n (q-1)^k \binom{n}{k}.$$

Proof. All the inequalities except

$$\mathrm{rc}(H) \leq \binom{n}{k} \tag{6}$$

follow from inequalities (1) and (3), in conjunction with $\left\lceil \frac{n}{k} \right\rceil = \mathrm{diam}(H)$ (Corollary 3.3), $(q-1)^k \binom{n}{k} = \Delta(H) \leq \chi'(H)$, and $\frac{1}{2} q^n (q-1)^k \binom{n}{k} = |E(H)|$.

To prove (6) we consider H in its incarnation as a Cayley graph, $H = \mathrm{Cay}(A^n, S)$, as in the proofs of Theorem 3.1 and Lemma 3.2. With a variant of the standard notation, let $\binom{[n]}{k} = \{I \subseteq \{1, \ldots, n\} \mid |I| = k\}$, and for each $I \in \binom{[n]}{k}$, let $P_I = \{v = v_1 \cdots v_n \in S \mid I = \{i \mid v_i \neq 0\}\}$. That is, P_I is the set of words in H of Hamming weight k whose support is precisely I. Clearly $-P_I = P_I$ and the P_I, $I \in \binom{[n]}{k}$, partition S. The translation–invariant edge coloring defined by this partition uses $\binom{n}{k}$ colors. Therefore, to prove (6), it suffices to show that this coloring is a rainbow connection coloring of H. Because the coloring is translation invariant, as is adjacency in H, it suffices to show that for all $v \in A^n \backslash \{\mathbf{0}\}$ there is a rainbow path in H from $\mathbf{0}$ to v. Because of the way adjacency is defined in Cayley graphs, the proof boils down to showing that each v is a sum of words in S, no two in the same P_i — so, no two with the same support. The proof of Lemma 3.2 shows how to do this *except* in the case where $1 \leq r = wt(v) < k$. (In that case, in that proof, v is represented as the sum of $2 = \text{dist}_H(\mathbf{0}, v)$ words in S, each with the same support — so the 2 paths of length 2, each from $\mathbf{0}$ to v, defined by the sum are not rainbow. The edges involved have the same color.)

But under the assumptions that $q > 2$ and $k < n$, this situation can be remedied. Without loss of generality, let $v = v_1 \cdots v_r 0^{n-r}, v_i \neq 0$, $1 \leq i \leq r$, $1 \leq r \leq k-1 \leq n-2$. Then $v = x + y + z$ where

$$\begin{aligned} x &= x_1 \cdots x_k 0^{n-k}, \\ y &= 0 y_2 \cdots y_k y_{k+1} 0^{n-k-1} \\ z &= z_1 0 \cdots z_k z_{k+1} 0^{n-k-1}, \text{ and} \end{aligned}$$

the x_i, y_i, $z_i \in A$ are chosen to satisfy the equation $v = x + y + z$, with $0 \neq x_1, \ldots, x_k, y_2, \ldots, y_{k+1}, z_1, z_3, \ldots, z_{k+1}$. (When $r = 1, k = 2$, we have $x = x_1 x_2 0^{n-2}$, $y = 0 y_2 y_3 0^{n-3}$, $z = z_1 0 z_3 0^{n-3}$). Thus there is a rainbow path of length 3 from $\mathbf{0}$ to v, with respect to this partition coloring. □

When $k = 1$ the partition coloring of $H = H(n, q, 1)$ in the proof of Theorem 3.7 is the coordinate-of-difference coloring, and is a strong rainbow connection coloring. When $1 < k < n$, and $q > 2$, the coloring is *not* a strong rainbow connection coloring: the sum of two words in A^n each of Hamming weight k, with different supports, will have Hamming weight at least 2. Therefore, if $v \in A^n$ has $wt(v) = 1$, then v cannot be represented as a sum $x + y$, $x \in P_I$, $y \in P_J$, I, $J \in \binom{[n]}{k}$, $I \neq J$. Since $\text{dist}_H(\mathbf{0}, v) = 2$, by Lemma 3.2, we conclude from this that there is no rainbow (with respect to the partition coloring) geodesic in H from $\mathbf{0}$ to v.

Yet the partition coloring in the proof of Theorem 3.7 is very "close" to being a strong rainbow connection coloring. We claim—but we will not prove here—that for any $v \in A^n$ such that $wt(v) > 1$, there is a rainbow geodesic in H from $\mathbf{0}$ to v. (For $wt(v) \geq k$ the proof can be extracted from the proof of Lemma 3.2.)

4 Problems

Corollary 2.2 and Theorem 3.7 obviously leave a great many unanswered questions about all four rainbow connectivity parameters. We are especially intrigued by $H = H(n, 2, k)$, $1 < k < n$, k odd. We conjecture that

$$\text{rc}(H) \leq \text{prc}(H) = \binom{n}{k}.$$

The partition edge coloring of $H(n, q, k)$ used in the proof of Theorem 3.7 is, when $q = 2$, proper because each P_I is a singleton, whose only element is the characteristic vector of the set $I \in \binom{[n]}{k}$. To prove the conjecture, it suffices to prove that this coloring is a rainbow connection coloring, which seems a very do-able chore—in fact, quite a lot of the mopping is done in the proof of Theorem 3.1. But we are leaving this conjecture open because there is much that we would like to know about $H(n, 2, k)$, and hope to deal with this conjecture as part of an interesting whole at some point in the future.

When $1 < k < n$ and k is even, $H = H(n, 2, k)$ has two connected components; in one the vertices are the binary words of length n of even Hamming weight, and in the other the words have odd Hamming weight. These two graphs are isomorphic. Let $H_0 = H_0(n, 2, k)$ be the component of H in which the vertices have even Hamming weight. As in the cases when k is odd, $H_0 = \mathrm{Cay}(A, S)$ where A is the subgroup of $(\mathbb{Z}_2)^n$ whose elements are the vertices of H_0. H_0 has an obvious proper translation-invariant edge coloring with $\binom{n}{k}$ colors. Again, we conjecture that this coloring is a rainbow connection coloring, which would imply

$$\mathrm{rc}(H_0) \leq \mathrm{prc}(H_0) = \binom{n}{k} = \Delta(H_0).$$

Whether k be odd or even, are these canonical edge colorings of H or H_0 strong rainbow connection colorings? Recall that when $1 < k < n$ and $q > 2$, the analogous canonical partition edge-colorings of $H(n, q, k)$ with $\binom{n}{k}$ colors are almost but not quite strong rainbow connection colorings; the situation in the cases $q = 2$ is not so clear.

Strongly connected Cayley digraphs are briefly discussed in Section 2. For an instance of a rainbow connectivity problem in general strongly connected digraphs, see [2]. The definitions of rc and src are what one would expect: if D is a strongly connected digraph, a coloring of the arcs of D is a (strong) rainbow connection coloring if and only if for every ordered pair (u, v) of vertices of D there is a rainbow (shortest) directed path from u to v, with respect to the coloring, and (s)rc(D) is the smallest number of colors appearing in a (strong) rainbow connection arc-coloring of D. Because there are at least two different notions of proper arc coloring in digraphs, we will postpone discussion of proper (strong) rainbow connectivity.

Cayley digraphs $\mathrm{Cay}^*(A, S)$ are discussed in Section 2. As with Cayley graphs, the arcs of such a digraph can be colored by partitioning S; the partition sets P_i are not required to satisfy $P_i = -P_i$. In such a coloring, an arc (a, b) is colored with the i such that $b - a \in P_i$. As in the undirected case, such a coloring is translation invariant, since, for any $a, b, c \in A$, $(b + c) - (a + c) = b - a$.

As in Section 2, for any undirected graph G, let G^* denote the directed graph obtained from G by replacing each edge by two arcs, one in each direction. If G has a (strong) rainbow connection edge coloring, and the two arcs of G^* associated with an edge of G are both given the color of that edge, then the resulting arc-coloring of

G^* is a (strong) rainbow connection coloring. Therefore,

$$\mathrm{rc}(G^*) \leq \mathrm{rc}(G) \text{ and}$$
$$\mathrm{src}(G^*) \leq \mathrm{src}(G).$$

On slender evidence we suspect that when $H = H(n, q, k)$ is connected and $k > 1$, $\mathrm{rc}(H^*)$ may be significantly less than $\mathrm{rc}(H)$. One of us (Matzke) proposes, for n, $q > 2$, a partition arc-coloring of $H^* = H^*(n, q, 2) = \mathrm{Cay}^*(A^n, S)$ which, it is claimed, shows that $\mathrm{rc}(H^*) \leq n$; compare this with the conclusion in Theorem 3.7 that

$$\mathrm{rc}(H) \leq \binom{n}{2}.$$

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(Received 10 May 2017; revised 8 Nov 2017)