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Converting a 6-cycle system into a Steiner triple system

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Dedicated to the memory of Anne Street

Abstract

Given a 6-cycle system, two ways of transforming it into a Steiner triple system have been previously considered: one can either inscribe into each 6-cycle two triangles, or one can squash each 6-cycle into two triangles. In this paper, we consider yet another way, which we call converting: delete two opposite edges of each 6-cycle and add two short diagonals to create two triangles. If, when doing this to every 6-cycle of a 6-cycle system, it results in a Steiner triple system, we call the latter a converted 6-cycle system. We prove that a converted 6-cycle system of order v exists if and only if $v \equiv 1$ or 9 (mod 12), $v \geq 13$. We also prove an analogous result for maximum packings of 6-cycles.

1 Introduction

A *Steiner triple system* of order v (STS(v)) is a pair $(V, \mathcal{B})$ where V is a finite set, and $\mathcal{B}$ is a collection of 3-element subsets of V called *triples* such that every 2-subset of V is contained in exactly one triple $B \in \mathcal{B}$. It is well known that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6) ([5]; cf., e.g., [2]).

Similarly, a 6-*cycle system* of order v is a pair $(V, \mathcal{C})$ where V is a finite set and $\mathcal{C}$ is a collection of 6-cycles with vertices in V such that every edge of the complete graph on V is contained in exactly one 6-cycle $C \in \mathcal{C}$. It is well known that a 6-cycle system of order v exists if and only if $v \equiv 1$ or 9 (mod 12) (cf. [8]). In other words, a Steiner triple system is a decomposition of the complete graph into triangles, and a 6-cycle system is a decomposition of the complete graph into hexagons (we will use the terms 6-cycle and hexagon interchangeably, and similarly for the terms triple and triangle).

In the literature, one can find two ways to convert a 6-cycle to two triangles. The first consists in "inscribing" two triangles in a hexagon. More precisely, given a 6-cycle (a, b, c, d, e, f), one obtains the two inscribed triangles (a, c, e) and (b, d, f). Another way to express this is to say that we join vertices at distance 2 (cf. [7]). The natural question that arises is: does there exist a 6-cycle system $(V, \mathcal{C})$ such that if we inscribe the two triangles in every 6-cycle of $\mathcal{C}$, the resulting set $\mathcal{B}$ of inscribed triples is the set of triples of an STS(v) $(V, \mathcal{B})$? The authors of [7] have answered this question in the affirmative: such a 2-*perfect* 6-cycle system exists for all $v \equiv 1$ or 9 (mod 12), $v > 9$ (for $v = 9$, such a system clearly cannot exist).

The other way to convert a 6-cycle into two triangles is to *squash* the 6-cycle by identifying its two opposite vertices, and renaming one of them with the other. More precisely, given the hexagon (a, b, c, d, e, f), we may identify a and d, or b and e, or c and f. The result is the *bowtie* $\{\{a, b, c\}, \{a, e, f\}\}$ or $\{\{b, c, d\}, \{d, e, f\}\}$, cf. Fig. 1 (and similarly, the bowtie $\{\{a, b, f\}, \{b, c, d\}\}$ or $\{\{a, e, f\}, \{c, d, e\}\}$, etc.).

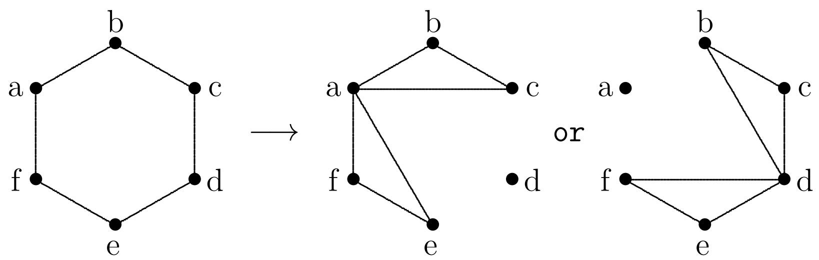

Figure 1

If there exists a 6-cycle system $(V, \mathcal{C})$ such that when each 6-cycle of $\mathcal{C}$ is squashed, the resulting collection of triples is the set of triples of a Steiner triple system $(V, \mathcal{B})$, then $(V, \mathcal{B})$ is said to be a *squashed* 6-cycle system.

It was shown in [6] that a squashed 6-cycle system of order v exists if and only if $v \equiv 1$ or 9 (mod 12), $v \geq 9$.

However, there is also a third way to transform a 6-cycle system into a Steiner triple system. Given the hexagon (a, b, c, d, e, f), we may delete its edges $\{a, f\}$ and $\{c, d\}$ and replace them with edges $\{a, c\}$ and $\{d, f\}$, thereby creating two triangles $\{a, b, c\}$ and $\{d, e, f\}$, as in Figure 2. These two triangles constitute a *converted*

6-cycle.

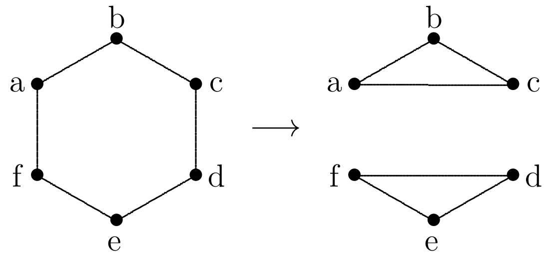

Figure 2

[Of course, we could also delete the two edges $\{b,c\},\{e,f\}$ and replace them with the edges $\{b,f\},\{c,e\}$, thus creating triangles $\{a,b,f\},\{c,d,e\}$, or delete the two edges $\{a,b\},\{d,e\}$ and replace them with edges $\{a,e\},\{b,d\}$, thus creating triangles $\{a,e,f\},\{b,c,d\}$.]

If there exists a 6-cycle system $(V,\mathcal{C})$ such that when each 6-cycle is converted, the collection of converted 6-cycles is a Steiner triple system $(V,\mathcal{B})$ then $(V,\mathcal{B})$ is said to be a *converted* 6-cycle system.

We prove in Section 2 that a converted 6-cycle system of order v exists if and only if $v \equiv 1$ or 9 (mod 12), $v \geq 13$.

When $v \equiv 3$ or 7 (mod 12), there exists no 6-cycle system of order v but there exists a maximum packing of 6-cycles whose leave is a triangle [4]. It is then possible to have a maximum packing with 6-cycles such that each of its 6-cycles can be converted, and the collection of these converted 6-cycles together with the leave constitutes the set of triples of a Steiner triple system $(V,\mathcal{B})$. In this case, we call $(V,\mathcal{B})$ a *converted* maximum packing of 6-cycles.

We prove in Section 3 that a converted maximum packing with 6-cycles exists if and only if $v \equiv 3$ or 7 (mod 12), $v \geq 15$.

The orders above exhaust all orders v for which there exists a Steiner triple system, namely $v \equiv 1$ or 3 (mod 6).

2 Converted 6-cycle systems

In order to prove our results, we need several auxiliary devices. One of these is a *group divisible design*. For a general definition of a group-divisble design (GDD), see, e.g., [1]. For our purposes we will only need GDDs with index $\lambda = 1$ and group- and block-sizes 3 and 4. In particular, we note that a GDD with t groups of size 3 and with blocks of size 3 and 4 exists for all $t \geq 3$ (cf. [1]). We denote such a GDD simply by GDD$(3t,\{3,4\})$.

If we now assign weight 4 to each point of such a GDD, we obtain a GDD with $12t$ points; each block of size 3 is replaced with a complete tripartite graph $K_{4,4,4}$, and each block of size 4 is replaced with a complete four-partite graph $K_{4,4,4,4}$. Two crucial lemmas follow.

Lemma 2.1 *There exists a decomposition of $K_{4,4,4}$ into 6-cycles which can be converted into a decomposition of $K_{4,4,4}$ into triangles.*

Proof.
The following eight 6-cycles constitute a decomposition of $K_{4,4,4}$ into 6-cycles (here $\{0,1,2,3\},\{4,5,6,7\},\{8,9,10,11\}$ are the partite sets of $K_{4,4,4}$):

$(0,4,8,1,5,9),(0,5,10,1,4,11),(0,6,9,1,7,8),(0,7,11,1,6,10),$
$(2,4,9,3,5,8),(2,5,11,3,4,10),(2,6,8,3,7,9),(2,7,10,3,6,11).$

If in any above-listed 6-cycle (a,b,c,d,e,f) the edges $\{a,f\},\{c,d\}$ are replaced with $\{a,c\},\{d,f\}$, then the resulting collection of converted triangles decomposes $K_{4,4,4}$. □

Lemma 2.2 *There exists a decomposition of $K_{4,4,4,4}$ into 6-cycles which can be converted into a decomposition of $K_{4,4,4,4}$ into triangles.*

Proof. Let Z_{16} be the vertex set of $K_{4,4,4,4}$, where $\{0,4,8,12\}$ (mod 16) are the partite sets (=groups). Then a collection of 16 six-cycles $(0,9,10,8,3,6)$ developed modulo 16 decomposes $K_{4,4,4,4}$. Replacing in $(0,9,10,8,3,6)$ the edges $\{0,6\},\{8,10\}$ with the edges $\{0,10\},\{6,8\}$ (mod 16) yields a set of triangles decomposing $K_{4,4,4,4}$. □

Example 2.3. A converted 6-cycle system of order 13 on $\{0,1,\dots,12\}$.

$(0,1,2,3,4,5),(0,3,6,1,4,7),(0,4,8,1,3,9),(0,10,5,1,11,6),(0,11,7,1,10,8),$
$(0,12,9,4,6,2),(1,12,5,7,8,9),(2,5,9,10,4,12),(2,8,3,5,6,7),$
$(2,10,7,9,11,4),(2,11,12,10,6,9),(3,7,12,8,5,11),(3,10,11,8,6,12).$

As earlier, in each 6-cycle above, replace the edges $\{a,f\},\{c,d\}$ with $\{a,c\},\{d,f\}$.

Example 2.4. A converted 6-cycle system of order 25 on Z_{25}.
$(0,12,1,5,13,7),(3,0,10,8,13,4)$ (mod 25).

With Examples 2.3, 2.4 and Lemmas 1.1, 1.2 in hand, we can now proceed to our first theorem.

Theorem 2.5 *Let $v \equiv 1$ (mod 12). Then there exists a converted 6-cycle system of order v.*

Proof. Let $v = 12t+1$. When $t = 1$ or $t = 2$, the theorem follows from Example 2.3 and Example 2.4, respectively, so assume $t \geq 3$. Let $V = X \times Y \cup \{\infty\}$ where X is a set such that $|X| = 12$, and $Y = \{y_1, y_2, \dots, y_t\}$, $X \cap Y = \emptyset$ and ∞ is disjoint from $X \cup Y$. Start with a GDD$(3t,\{3,4\})$ which exists, as already noted, for all $t \geq 3$, and weigh each of its points with weight 4; for each block of size 3 of this GDD, put on each resulting inflated block a copy of the converted 6-cycle system of $K_{4,4,4}$ from Lemma 2.1, and similarly, for each block of size 4 of this GDD, put on each resulting inflated block a copy of converted 6-cycle system of $K_{4,4,4,4}$ from Lemma 2.2. Finally, for each $i = 1, 2, \ldots, t$, put on the set $X \times \{y_i\} \cup \{\infty\}$ a copy of the converted 6-cycle system of order 13 from Example 2.3. This results in a converted 6-cycle system of order $12t + 1$ for each $t \geq 3$, and the proof is complete. □

We can now turn our attention to the case when $v \equiv 9 \pmod{12}$.

First of all, we observe that even though there are 640 nonisomorphic 6-cycle systems of order 9 [1], there exists no converted 6-cycle system of order 9. Indeed, if such a system existed, it would mean that we can partition the set of triples of the (unique) Steiner triple system of order 9 into 6 pairs of pairwise disjoint triples. It is easily seen, and well known, that this is impossible [3].

Example 2.6. A converted 6-cycle system of order 21.

Here we take as the set of elements $V = Z_7 \times \{1, 2, 3\}$, and as the set of converted 6-cycles

$$(0_1, 1_1, 3_1, 6_1, 0_2, 2_2), (0_1, 0_2, 4_2, 1_1, 6_2, 0_3), (0_1, 2_3, 2_2, 5_1, 6_3, 5_3),$$
$$(0_1, 3_3, 5_3, 1_2, 2_1, 6_3), (0_2, 1_2, 4_3, 1_3, 6_2, 5_3) \pmod{(7, 7, 7)}.$$

Example 2.7. A converted 6-cycle system of order 33.

Here we take as the set of elements $Z_{11} \times \{1, 2, 3\}$, and as the set of converted 6-cycles

$$(0_1, 1_1, 3_1, 0_2, 2_1, 1_2), (0_2, 0_1, 4_1, 1_1, 6_1, 9_2), (0_1, 2_2, 5_2, 1_1, 7_2, 0_3),$$
$$(0_1, 1_3, 1_2, 5_1, 0_3, 2_3), (0_1, 3_3, 2_3, 5_3, 1_1, 8_3), (0_1, 9_3, 4_2, 10_1, 4_3, 10_3)$$
$$(0_2, 4_2, 2_3, 6_3, 5_2, 10_2), (0_2, 6_3, 9_2, 0_3, 8_2, 7_3) \pmod{(11, 11, 11)}.$$

Example 2.8. A converted 6-cycle system of order 21 with a hole of size 9 (that is, a decomposition of $K_{21} \setminus K_9$ into 6-cycles such that if each of its 6-cycles is converted, the resulting set of triples decomposes $K_{21} \setminus K_9$).

Here the set of elements is $\{0, 1, \ldots, 20\}$, and the set of elements of the hole is $\{12, 13, \ldots, 20\}$. The set of converted 6-cycles is

(0, 1, 2, 3, 4, 5), (0, 12, 3, 1, 13, 4), (0, 13, 5, 1, 12, 6), (0, 14, 4, 1, 15, 3), (0, 15, 6, 1, 14, 7),
(0, 16, 7, 1, 17, 8), (0, 17, 9, 1, 16, 10), (0, 18, 8, 1, 19, 9), (0, 19, 10, 1, 18, 11),
(0, 20, 11, 3, 13, 2), (1, 20, 5, 2, 12, 11), (2, 14, 5, 3, 16, 6), (2, 15, 4, 6, 13, 9)
(2, 16, 8, 3, 14, 10), (2, 17, 6, 3, 19, 8), (2, 18, 10, 3, 17, 11), (2, 19, 7, 9, 12, 4)
(2, 20, 9, 4, 18, 7), (3, 18, 9, 5, 17, 7), (3, 20, 7, 5, 16, 9), (4, 16, 11, 5, 12, 8)
(4, 17, 10, 6, 14, 11), (4, 19, 6, 5, 15, 10), (4, 20, 8, 11, 13, 7), (5, 18, 6, 8, 13, 10)
(5, 19, 11, 9, 14, 8), (6, 7, 8, 9, 10, 11), (6, 20, 10, 7, 15, 9), (7, 12, 10, 8, 15, 11).

Theorem 2.9 *A converted* 6*-cycle system of order* v, $v \equiv 9 \pmod{12}$, *exists if and only if* $v \geq 21$.

Proof. Let $v = 12t + 9$. When $t = 1$ or $t = 2$, the theorem follows from Example 2.6 and Example 2.7, respectively. So let $t \geq 3$, and let $V = (X \times Y) \cup N$ where X

is a set such that $|X| = 12$, $Y = \{y_1, y_2, \ldots, y_t\}$ and N is a set such that $|N| = 9$, and X, Y, and N are pairwise disjoint. Put on the set $(X \times \{y_1\}) \cup N$ a copy of the converted 6-cycle system of order 21 from Example 2.6, and for each $i = 2, 3, \ldots, t$, put on the set $(X \times \{y_i\}) \cup N$ a copy of the converted 6-cycle system of order 21 with a hole of size 9 from Example 2.8 so that the hole aligns with N. The rest of the proof is the same as that of Theorem 2.5. □

Theorems 2.5 and 2.9 together yield the proof of the main result of this section.

Theorem 2.10 *A converted* 6*-cycle system of order* v *exists if and only if* $v \equiv 1$ *or* 9 (mod 12)*,* $v \geq 13$*.*

3 Converted 6-cycle packings

When $v \equiv 3$ or 7 (mod 12), we need to prove the existence of a maximum packing of 6-cycles such that each of its 6-cycles can be converted and the collection of converted 6-cycles together with the single triangle (the leave) forms a set of triples of a Steiner triple system of order v. We will abuse the language slightly and call such a Steiner triple system a *converted maximum packing.*

Consider first the case of $v \equiv 3$ (mod 12). Let $v = 12t + 3$. In order to facilitate recursion, we need to construct directly converted maximum packings of 6-cycles of order 15 and 27.

Example 3.1 A converted maximum packing of 6-cycles of order 15 on $\{0, 1, \ldots, 14\}$ with leave $\{9, 11, 13\}$:

$(0,1,2,3,4,5), (0,3,6,1,4,7), (0,4,8,1,3,9), (0,10,5,1,11,6), (0,11,7,1,10,8)$
$(0,12,9,2,4,13), (0,14,13,3,7,2), (1,12,13,2,5,9), (1,14,5,3,8,13)$
$(2,6,8,7,13,10), (2,11,12,7,9,8), (2,14,10,4,6,12), (3,11,14,9,6,10)$
$(3,12,10,9,4,14), (4,11,10,7,5,12), (5,11,8,14,7,6), (5,13,6,14,12,8).$

Before proceeding to the case of $v = 27$, we need several auxilliary devices. Let G_0 be the cocktail-party graph on 12 vertices, that is, the complete graph K_{12} from which a 1-factor I has been removed.

Lemma 3.2 *There exists a decomposition of* G_0 *into* 6*-cycles each of whose* 6*-cycles can be converted so that the collection of converted* 6*-cycles yields the triples of a maximum packing of* K_{12} *with triples.*

Proof. Let $V(G_0) = \{0, 1, \ldots, 11\}$, and let $I = \{\{2i, 2i+1\}, i = 0, 1, 2, \ldots, 5\}$. Converted 6-cycles:

$(0,2,4,1,3,5), (0,3,7,1,6,4), (0,6,10,5,11,7), (0,8,5,6,3,11),$
$(0,9,11,4,7,10), (1,2,11,6,9,5), (1,8,7,5,2,10), (1,9,10,4,8,11),$
$(2,6,8,3,4,9), (2,7,9,3,10,8).$ □

Let G_1 be the graph $K_{11} \setminus (C_3 \cup C_4)$, that is, the graph obtained from the complete graph K_{11} by removing from it a disjoint 3-cycle and 4-cycle.

Lemma 3.3 *There exists a decomposition of* G_1 *into* 6*-cycles each of whose* 6*-cycles can be converted so that the collection of converted* 6*-cycles yields the triples of a decomposition of* G_1 *into triples.*

Proof. Let $V(G_1) = \{0, 1, \ldots, 10\}$, and let $(0, 1, 2), (3, 4, 5, 6)$ be the two cycles deleted from K_{11}. Converted 6-cycles:

$(0, 3, 5, 1, 4, 6), (0, 4, 8, 1, 7, 5), (0, 7, 10, 1, 9, 8), (0, 9, 6, 1, 3, 10),$
$(2, 3, 8, 10, 4, 9), (2, 4, 7, 8, 5, 10), (2, 5, 9, 7, 6, 8), (2, 6, 10, 9, 3, 7).$

Lemma 3.4 *There exists a converted maximum packing of* 6*-cycles of order* 27*.*

Proof. Let $V = (X \times \{1, 2, 3\}) \cup \{a, b, c\}$ where $X = \{x_1, x_2, \ldots, x_8\}$. For each $i = 1, 2, 3$, consider the complete graph K_{11} on the set $(X \times \{i\}) \cup \{a, b, c\}$. Delete from it the cycles (a, b, c) and $(x_{1,i}, x_{2,i}, x_{3,i}, x_{4,i})$, thus obtaining the graph isomorphic to G_1. Put now on it a copy of the converted 6-cycle decomposition of the graph G_1. On the 12-element set $\cup_{i=1}^{3}\{x_{1,i}, x_{2,i}, x_{3,i}, x_{4,i}\}$ put a copy of the converted 6-cycle decomposition of the cocktail-party graph $K_{12} - I$ from Lemma 3.2 where $I = \cup_{i=1}^{3}\{\{x_{1,i}, x_{3,i}\}, \{x_{2,i}, x_{4,i}\}\}$. Finally, on each of the three sets

$\{x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}, x_{5,2}, x_{6,2}, x_{7,2}, x_{8,2}, x_{5,3}, x_{6,3}, x_{7,3}, x_{8,3}\},$
$\{x_{5,1}, x_{6,1}, x_{7,1}, x_{8,1}, x_{1,2}, x_{2,2}, x_{3,2}, x_{4,2}, x_{5,3}, x_{6,3}, x_{7,3}, x_{8,3}\},$
$\{x_{5,1}, x_{6,1}, x_{7,1}, x_{8,1}, x_{5,2}, x_{6,2}, x_{7,2}, x_{8,2}, x_{1,3}, x_{2,3}, x_{3,3}, x_{4,3}\},$

put a copy of a converted 6-cycle decomposition of $K_{4,4,4}$ from Example 2.1 (where obviously the second indices indicate the partite sets). It is easily verified directly that as a result we obtain a converted maximum packing of 6-cycles of order 27. □

Theorem 3.5 *Let* $v \equiv 3 \pmod{12}$, $v > 3$*. Then there exists a converted maximum packing of* 6*-cycles of order* v*.*

Proof. Let $v = 12t + 3$, and let $V = (X \times Y) \cup \{a, b, c\}$ where X is a set such that $|X| = 12$ and $Y = \{y_1, y_2, \ldots, y_t\}$. When $t = 1$ or $t = 2$, the statement follows from Example 3.1 and Lemma 3.4, respectively, so we may assume $t \geq 3$. For each $i = 1, 2 \ldots, t$, put on the set $(X \times \{y_i\}) \cup \{a, b, c\}$ a copy of the converted maximum packing of 6-cycles of order 15 from Example 3.1 in such a way that $\{a, b, c\}$ is the leave of the packing. The remainder of the proof is the same as that of Theorem 2.5. □

Finally, we are going to deal with the case when $v \equiv 7 \pmod{12}$.

Example 3.6. A converted maximum packing of 6-cycles of order 19 on $Z_{16} \cup \{16, 17, 18\}$ with leave $\{16, 17, 18\}$.

$(0, 1, 10, 8, 11, 6) \pmod{16}$, and

$(16, 0, 8, 17, 13, 5), (16, 9, 5, 17, 6, 2), (16, 10, 6, 18, 8, 4), (17, 14, 10, 18, 15, 11),$
$(16, 12, 4, 18, 5, 1), (16, 13, 1, 18, 3, 7), (16, 14, 2, 17, 15, 3), (16, 15, 7, 18, 14, 6),$
$(17, 1, 9, 18, 12, 0), (17, 4, 0, 18, 2, 10), (17, 7, 11, 18, 13, 9), (16, 11, 3, 17, 12, 8).$

Let G_2 be a graph isomorphic to $K_{15} \setminus K_7$, that is the complete graph K_{15} from which a complete subgraph K_7 has been deleted.

Lemma 3.7 *There exists a decomposition of G_2 into 6-cycles each of whose 6-cycles can be converted so that the collection of converted 6-cycles constitutes a decomposition of G_2 into triangles.*

Proof. Let $V(G_2) = \{0, 1, \ldots, 14\}$, and let the vertex set of the sub-K_7 be $\{0, 1, \ldots, 6\}$. The 14 converted 6-cycles are

$(0, 7, 9, 1, 8, 10), (0, 8, 12, 1, 11, 9), (0, 11, 14, 1, 13, 12), (0, 13, 10, 1, 7, 14),$
$(2, 7, 10, 3, 8, 11), (2, 8, 14, 3, 7, 13), (2, 9, 13, 3, 12, 10), (2, 12, 11, 4, 13, 14),$
$(3, 9, 14, 5, 13, 11), (4, 7, 12, 6, 13, 8), (4, 9, 8, 5, 12, 14), (4, 10, 11, 6, 9, 12),$
$(5, 7, 8, 6, 14, 10), (5, 9, 10, 6, 7, 11).$ □

Lemma 3.8 *There exists a converted maximum packing of 6-cycles of order 31.*

Proof. Let $V = (X \times \{1, 2, 3\}) \cup N$ where X is a set such that $|X| = 8$ and N is a set such that $|N| = 7$. Put on the set $(X \times \{1\}) \cup N$ a copy of a converted maximum packing of 6-cycles of order 15 from Example 3.1. For $i = 2, 3$, put on the set $(X \times \{i\}) \cup N$ a copy of a converted 6-cycle decomposition of the graph G_2 from Lemma 3.7 (so that the deleted K_7 is put on N). In addition, take a GDD(6; $\{3\}$) (also known as a Pasch configuration) with three groups of size 2 and give each of its 6 points weight 4. This inflation replaces each block of size 3 by a complete tripartite graph $K_{4,4,4}$. Now put on each inflated block a converted decomposition of $K_{4,4,4}$ into 6-cycles from Lemma 2.1. The union of the above constitutes a converted maximum packing of 6-cycles of order 31. □

Let G_3 be a graph isomorphic to $K_{19} \setminus K_7$, that is the complete graph K_{19} from which a complete subgraph K_7 has been deleted.

Lemma 3.9 *There exists a decomposition of G_3 into 6-cycles each of whose 6-cycles can be converted so that the collection of converted 6-cycles constitutes a decomposition of G_3 into triangles.*

Proof. Let $V(G_3) = \{0, 1, \ldots, 18\}$, and let the vertex set of the sub-K_7 be $\{12, 13, \ldots, 18\}$. The 25 converted 6-cycles are

$(0, 1, 2, 3, 4, 5), (0, 3, 12, 1, 11, 13), (0, 4, 13, 1, 8, 12), (0, 6, 14, 1, 3, 15),$
$(0, 7, 5, 3, 6, 2), (0, 8, 15, 1, 7, 14), (0, 9, 16, 1, 6, 17), (0, 10, 17, 1, 4, 18),$
$(0, 11, 18, 1, 5, 16), (2, 4, 12, 5, 11, 14), (2, 5, 13, 3, 7, 16), (2, 7, 15, 5, 10, 12),$
$(2, 8, 17, 9, 11, 15), (2, 9, 18, 3, 11, 17), (2, 10, 14, 3, 9, 13), (2, 11, 16, 8, 5, 18),$
$(3, 8, 14, 4, 7, 17), (3, 10, 18, 6, 4, 16), (4, 9, 14, 5, 6, 15), (4, 10, 15, 9, 5, 17),$
$(6, 7, 18, 8, 10, 16), (6, 8, 13, 7, 11, 12), (6, 9, 12, 7, 10, 13), (8, 4, 11, 10, 1, 9),$
$(8, 7, 9, 10, 6, 11).$ □

Theorem 3.10 *Let* $v \equiv 7 \pmod{12}$*. A converted maximum* 6*-cycle packing exists if and only if* $v \geq 19$*.*

Proof. Let $v = 12t + 7$. Clearly, there exists no converted 6-cycle system of order 7: the unique Steiner triple system of order 7 cannot contain two disjoint triples. When $t = 1$ or $t = 2$, the statement follows from Example 3.6 and Lemma 3.8, respectively. So let $t \geq 3$. Let $V = (X \times Y) \cup N$ where X is a set such that $|X| = 12$, $Y = \{y_1, y_2, \ldots, y_t\}$, and N is a set such that $|N| = 7$. Put on the set $(X \times \{y_1\}) \cup N$ a copy of a converted maximum packing of 6-cycles of order 19 from Example 3.6. For each $i = 2, 3, \ldots, t$, put on the set $(X \times \{y_i\}) \cup N$ a copy of the decomposition of G_3 into converted 6-cycles from Lemma 3.9 in such a way that the hole (i.e., the deleted K_7) aligns with N. The remainder of the proof is the same as that of Theorem 2.5. □

Combining Theorems 3.5 and 3.10 yields the main result of this section.

Theorem 3.11 *Let* $v \equiv 3$ *or* $7 \pmod{12}$*. A converted maximum* 6*-cycle packing of order* v *exists if and only if* $v \geq 15$*.*

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