AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **72(1)** (2018), Pages 1–18

Asymptotic estimates for the number of permutations without short cycles

Robertas Petuchovas

Department of Mathematics and Informatics
Vilnius University, Vilnius
Lithuania

Abstract

We explore the probability $\kappa(n, r)$ that a permutation sampled uniformly at random from the symmetric group S_n has only cycles of length greater than r, where $1 \le r < n$. As the order of the group increases, new asymptotic formulas valid in the specified regions are obtained using the saddle-point method in combination with ideas originated in analytic number theory. Some results for permutations without long cycles are employed.

1 Introduction

Results

Denote by S_n the symmetric group on a finite set of n symbols. Let $\sigma \in \mathrm{S}_n$ and $k_j(\sigma)$ be the number of cycles of length j in σ. The general task is to find a new asymptotic formula for

$$\kappa(n, r) = \frac{1}{n!} \left| \{ \sigma \in \mathrm{S}_n : k_j(\sigma) = 0 \ \forall j \in \overline{1, r} \} \right|$$

if $1 \le r < n$ and $n \to \infty$. To begin with, we have a classic example of derangements

$$\kappa(n, 1) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} = \mathrm{e}^{-1} + O\left(\frac{1}{n!}\right)$$

([10, p. 135] or [4, p. 107]), and the trivial case $\kappa(n, r) = 1/n$ if $n/2 \le r < n$. Nevertheless, a general formula is complex. Formally, for $1 \le r < n$, we have

$$\kappa(n, r) = \sum_{\substack{s_1 + 2s_2 + \ldots + ns_n = n \\ s_j = 0, j \le r}} \prod_{j=1}^{n} \frac{1}{j^{s_j} s_j!}$$

([4, p. 80]), where the summation is over the vectors $(s_1, s_2, \dots, s_n) \in \mathbb{N}_0^n$ with $s_j = 0$ for $j \le r$. The formula can be rewritten in terms of independent Poisson random variables Z_j, $1 \le j \le n$, such that $\mathbb{E}Z_j = 1/j$. Namely,

$$\kappa(n,r) = \exp\Big\{ \sum_{j=r+1}^{n} \frac{1}{j} \Big\} \Pr\Big(\sum_{j=r+1}^{n} jZ_j = n \Big). \tag{1.1}$$

We need the following information to address previous results. Buchstab's function $\omega(v)$ [12, p. 399] is the unique continuous solution to the equation

$$(v\omega(v))' = \omega(v-1) \tag{1.2}$$

for $v > 2$ with the initial condition $\omega(v) = 1/v$ if $1 \le v \le 2$. Moreover, $1/2 \le \omega(v) \le 1$ if $v \ge 1$ [12, p. 400]. Dickman's function $\varrho(v)$ is a unique continuous solution to the equation

$$v\varrho'(v) + \varrho(v-1) = 0 \tag{1.3}$$

for $v > 1$ with the initial condition $\varrho(v) = 1$ if $0 \le v \le 1$. Furthermore, $\varrho(v) \le 1/\Gamma(v)$ for $v \ge 1$ [12, p. 366], where Γ is the gamma function. The function $R(v)$, introduced in [13, p. 6], is described in Definition 3.3 and Lemma 3.4 below. It comes from the evaluation of $\omega(v) - \mathrm{e}^{-\gamma}$, where γ is the Euler-Mascheroni constant. For the purpose of this paper, it is sufficient to know that

$$R(v) = \varrho(v) \exp\left\{ -\frac{\pi^2 v}{2(\log v)^2} + O\left(\frac{v \log\log(v+2)}{(\log v)^3} \right) \right\} \tag{1.4}$$

if $v \ge v_0$, where v_0 is some sufficiently large constant, and $R(v) = O(1)$ if $1 \le v < v_0$ (Lemma 3.5). As the ratio n/r will appear frequently in the sequel, hereafter we denote

$$u = n/r.$$

At this point, the results obtained so far can be mentioned.

The total variation distance for permutations was estimated by Arratia and Tavare in their notable work [2] of 1992. From the estimate it follows that

$$\left| \kappa(n,r) - \mathrm{e}^{-H_r} \right| \le \sqrt{2\pi \lfloor u \rfloor} \frac{2^{\lfloor u \rfloor - 1}}{(\lfloor u \rfloor - 1)!} + \frac{1}{\lfloor u \rfloor !} + 3\left(\frac{\mathrm{e}}{u} \right)^u \tag{1.5}$$

$$= \exp\Big\{ -u \log u + u \log(2\mathrm{e}) + O\left(\log(u+2)\right) \Big\} \tag{1.6}$$

if $1 \le r < n$ and $H_r = \sum_{j=1}^{r} 1/j$. Later, in 2002 Manstavičius [7] showed that

$$\kappa(n,r) = \mathrm{e}^{-H_r + \gamma} \omega(u) + O\left(\frac{1}{r^2} \right) \tag{1.7}$$

if $1 \le r < n$. His result was not noticed and, therefore, similar results relying on the same proof technique were presented by other authors, namely, in [3] and [5]. A recurrence relation

$$\kappa(n,r) = \frac{1}{n} + \frac{1}{n} \sum_{r<j<n-r} \kappa(j,r),$$

and also the induction principle and approximation of sums by integrals, were applied in [3], [5] and [7] to get (1.7). Moreover, none of the aforementioned authors noticed that there was a result by Hildebrand and Tenenbaum

$$\omega(u) = \mathrm{e}^{-\gamma} + O(R(u)) \tag{1.8}$$

if $u \geq 1$ (Lemma 3.4 below), which combined with the estimates (1.4),

$$\varrho(u) = \exp\left\{-u\log u - u\log\log(u+2) + u\left(1 + O\left(\frac{\log\log(u+2)}{\log(u+2)}\right)\right)\right\} \tag{1.9}$$

(Corollary 2.3 in [6]), and (1.6), gives

$$\kappa(n,r) = \mathrm{e}^{-H_r+\gamma}\omega(u) + O\left(\exp\left\{ - u\log u + u\log(2\mathrm{e}) + O\left(\log u\right)\right\}\right) \tag{1.10}$$

for $u \geq 1$. Therefore, in the case of $(\log\log n)^{-1}\log n = o(u)$, the result (1.5) combined with (1.8) is stronger than those mentioned after it, but in the case of $u \leq (\log\log n)^{-1}\log n$, Manstavičius' result (1.7) provides a sharper order of the error term.

The recent result by Weingartner [14] supplements the previous ones:

Proposition 1.1. *Let $u = n/r$. For $1 \leq r \leq n/\log n$, we have*

$$\kappa(n,r) = \mathrm{e}^{-H_r} + O\left(\frac{(u/\mathrm{e})^{-u}}{r^2}\right). \tag{1.11}$$

If $r \geq 3$, we can replace e *by 1 in the error term.*

To improve on (1.11) in this paper, one makes use of Cauchy's integral representation

$$\Pr\Big(\sum_{j=r+1}^{n} jZ_j = n\Big) = \frac{1}{2\pi i}\int_{|z|=\beta} \exp\Big\{\sum_{j=r+1}^{n}\frac{z^j-1}{j}\Big\}\frac{dz}{z^{n+1}},$$

where $\beta > 0$ is to be chosen. Firstly, after some transformations of the integral, the residue theorem is applied to obtain the main asymptotic term, and secondly, the remaining integral is evaluated using different saddle-point approximations. This idea has already been adopted in Manstavičius' and the author's recent work [8]. We have proven the following theorem:

Theorem 1.2. *If $\sqrt{n\log n} \leq r < n$ and $u = n/r$, then*

$$\kappa(n,r) = \mathrm{e}^{-H_r+\gamma}\omega(u) + O\left(\frac{R(u)u^{3/2}\log^2(u+1)}{r^2}\right). \tag{1.12}$$

Although not effective, the error estimate is the sharpest in the region $\sqrt{n\log n} \leq r < n$ as far as the author knows. This theorem is useful for solving a more general problem, which is presented in the next section.

The paper provides new asymptotic estimates for $\kappa(n,r)$ when it is approximated by e^{-H_r} and $2 \leq r \leq n(\log\log n)/\log n$.

Stressing the dependence on a parameter v in an estimate, we will write $O_v(\cdot)$. The results in this paper are in the following four propositions.

Theorem 1.3. *Let ε and δ be arbitrary but fixed positive numbers and $u = n/r$. We have*

$$\kappa(n,r) = \mathrm{e}^{-H_r} + O_{\varepsilon,\delta}\left(\frac{\varrho(u)}{r}\exp\left\{-\frac{2u\,(1-\delta)}{\pi^2(\log(1+u))^2}\right\}\right) \tag{1.13}$$

if $(\log n)^{3+\varepsilon} \le r < n$ *and*

$$\kappa(n,r) = \mathrm{e}^{-H_r} + O\left(\frac{\varrho(u)u^{u/r}}{r}\right) \tag{1.14}$$

if $\log n \le r < (\log n)^{3+\varepsilon}$.

Recalling (1.9) one can verify that Theorem 1.3 presents a sharper order of the error term if $\log n \le r \le n(\log\log n)/\log n$. Moreover, the proof of Theorem 1.3 can give an asymptotic formula in the whole region $1 \le r < n$, but the precision of it for $2 \le r \le \log n$ does not satisfy us as much as that given in Corollary 1.5, which follows from the next theorem.

Theorem 1.4. *Let*

$$\nu(n,r) = \frac{1}{n!}\left|\{\sigma \in \mathrm{S}_n :\ k_j(\sigma) = 0\ \forall j \in \overline{r+1,n}\}\right|,$$

and ε be an arbitrary but fixed positive number; then

$$\kappa(n,r) = \mathrm{e}^{-H_r} + O_{\varepsilon}\left(\frac{\nu(n,r)}{r}\exp\left\{-\frac{u^{1-4/r}(1-\varepsilon)}{4\pi^2(\log(u+1))^2}\right\}\right) \tag{1.15}$$

if $5 \le r < n$ *and*

$$\kappa(n,r) = \mathrm{e}^{-H_r} + O\left(\nu(n,r)n^{5/2}\right) \tag{1.16}$$

if $2 \le r < 5$.

The proof of the proposition applies Theorem 2 and Corollary 5 of [9]. Theorem 1.4 can be useful in formulas where both probabilities $\nu(n,r)$ and $\kappa(n,r)$ are involved. This is shown in the next section.

Using Theorem 1 of [9], we deduce Lemma 3.8, which combined with Theorem 1.4 gives the following corollary:

Corollary 1.5. *For $2 \le r \le \log n$, we have*

$$\kappa(n,r) = \mathrm{e}^{-H_r} + O\left(\exp\left\{-\frac{n}{r}\log\frac{n}{\mathrm{e}} + \frac{n}{\log n} + \frac{3n}{(\log n)^2}\right\}\right).$$

The corollary improves on the orders of the previous asymptotic formulas error estimates in its region of validity.

Finally, it is important to mention that Theorems 1.3 and 1.4 follow from the following effective inequality:

Theorem 1.6. *For all* $1 \le r < n/2$ *and* $\alpha > 1$*, we have*

$$
\begin{aligned}
|\kappa(n,r) - \mathrm{e}^{-H_r}| \le{}& \frac{\pi \mathrm{e}^4 \alpha^{2r-n+3/2}}{n^2(\alpha-1)^2} \exp\left\{\sum_{j=1}^{r} \frac{\alpha^j - 2}{j} + E(r,\alpha)\right\} \\
&+ \frac{4\mathrm{e}\alpha^{2r-n+2}}{\pi n^2 r(\alpha-1)^3} \exp\left\{-\frac{\alpha(\alpha^r-1)}{2r(\alpha-1)} - H_r\right\}
\end{aligned}
$$

where

$$
E(r,\alpha) = -\frac{2}{r}\frac{\alpha^{r+1}}{\alpha-1}\left(\frac{\pi^{-2}}{1+(r\alpha-r)^2} - \alpha^{-r/2}\right)_+ + \min\{2r\log\alpha, 2\log(er)\}
$$

and $(a)_+ = \max\{a, 0\}$ *if* $a \in \mathbf{R}$.

To obtain Theorems 1.3 and 1.4, we have chosen different functions $\alpha = \alpha(n,r)$. However, for both theorems we have $u^{1/r} \le \alpha \le u^{2/r}$. It can be guessed that an appropriate choice of α leads to an improvement of the effective inequality (1.5). Yet the author failed to find some pragmatic expression.

Analytic results obtained for permutations usually can be compared to these obtained in number theory. In our case, one can have in mind the analysis of the number of natural numbers missing small prime factors. Theorem 1.3 is comparable to Corollary 7.4 given in [12, p. 417] while Theorem 1.4 is comparable to Theorem 1 given on page 397 of the same book. Information about parallelism between theories can be found in [1] or partially in [14].

2 Motivation

The density $\kappa(n,r)$, as well as $\nu(n,r)$, is important in answering the following question: How many permutations with a given cycle structure pattern are in a symmetric group? It can clearly be seen in the case when the pattern restrictions are defined for "short" cycles.

For the demonstration of this, let us take $\bar{k}_r(\sigma) = (k_1(\sigma), \ldots, k_r(\sigma))$ (a truncated structure vector of σ) and vector $\bar{Z}_r = (Z_1, \ldots, Z_r)$ with the coordinates of the independent Poisson random variables such that $\mathbb{E}Z_j = 1/j$. For $1 \le r \le n$, we have

$$
\begin{aligned}
d_{TV}(n,r) :={}& \sup_{A \subset \mathbb{N}_0^n} \left| \frac{\#\{\sigma :\ \bar{k}_r(\sigma) \in A\}}{n!} - \Pr(\bar{Z}_r \in A) \right| \\
={}& \frac{1}{2}\sum_{m=0}^{\infty} \nu(m,r)\left|\kappa(n-m,r) - \mathrm{e}^{-H_r}\right|
\end{aligned}
$$

([8]). It is clear that $d_{TV}(n,r)$ is the total variation distance between the distributions of random vectors $\bar{k}_r$ and $\bar{Z}_r$.

Evidently, Theorem 1.4 provides an estimate for the distance only through the function $\nu(n,r)$. Applying it for $5 \le r < n$, we get

$$
d_{TV}(n,r) \ll \frac{1}{r}\sum_{m=0}^{n-r-1} \nu(m,r)\nu(n-m,r) + \frac{\mathrm{e}^{-H_r}}{2}\sum_{m=n-r}^{\infty} \nu(m,r) + \frac{1}{2}\nu(n,r).
$$

Thus, one can try to improve the order of (1.5) relying only on the results for density $\nu(n,r)$. Moreover, an improvement of the factor $u^{3/2}\log^2(u+1)$ in (1.12) would cause an improvement on the same factor in the formula

$$d_{TV}(n,r) = H(u)\left(1 + O\left(\frac{u^{3/2}\log^2(u+1)}{r}\right)\right)$$

if $\sqrt{n\log n} \le r \le n$, where

$$H(u) = \frac{1}{2}\int_0^\infty \left|\omega(u-v) - \mathrm{e}^{-\gamma}\right|\varrho(v)dv + \frac{\varrho(u)}{2}$$

and $\omega(v) = 0$ if $v < 1$ ([8]).

The next two sections are dedicated to the auxiliary propositions and the proofs of the new theorems.

3 Auxiliary Lemmas

There are eight lemmas in this section; six of them introduce some functions and their estimates needed in the proofs of Theorem 1.3, Theorem 1.4, and Corollary 1.5. Two other lemmas (Lemma 3.4 and Lemma 3.5) have already been discussed in the introduction. We will use the following notation

$$I(s) = \int_0^s \frac{\mathrm{e}^t - 1}{t}dt,$$

$$T(s) = \int_0^s \frac{\mathrm{e}^t - 1}{t}\left(\frac{t\mathrm{e}^{t/r}}{r(\mathrm{e}^{t/r}-1)} - 1\right)dt.$$

Therefore,

$$\begin{aligned} I(s) + T(s) &= \int_0^{s/r}\sum_{j=1}^r \mathrm{e}^{jt}dt \\ &= \sum_{j=1}^r \frac{\mathrm{e}^{js/r}-1}{j}. \end{aligned} \tag{3.1}$$

Later on the function $\xi(v)$, $v \ge 1$, is defined as in the next lemma.

Lemma 3.1. *For $v > 1$, define $\xi = \xi(v)$ as the nonzero solution to the equation*

$$\mathrm{e}^\xi = 1 + v\xi$$

and put $\xi(1) = 0$. If $v > 1$, then $\log v < \xi < 2\log v$,

$$\xi = \log v + \log\log(v+2) + O\left(\frac{\log\log(v+2)}{\log(v+2)}\right) \tag{3.2}$$

and

$$\xi' := \xi'(v) = \frac{1}{v}\frac{\xi}{\xi - 1 + 1/v} = \frac{1}{v}\exp\left\{O\left(\frac{1}{\log(v+1)}\right)\right\}. \tag{3.3}$$

Proof. This is Lemma 6 in [9]. □

Lemma 3.2. *Let $\varrho(v)$, $v \geq 1$, be the Dickman function* (1.3). *For $v \geq 1$, we have*

$$\varrho(v) = \sqrt{\frac{\xi'(v)}{2\pi}} \exp\left\{\gamma - v\xi(v) + I(\xi(v))\right\}\left(1 + O\left(\frac{1}{v}\right)\right).$$

Proof. This is a known result; it can be found in [12] on page 374. □

Definition 3.3. *If $v_0 > 1$ is a sufficiently large constant and $v \geq v_0$, we let*

$$R(v) = \left|\frac{\exp\left\{-v\zeta_0(v) - I(\zeta_0(v))\right\}}{\zeta_0(v)\sqrt{2\pi v(1 - 1/\zeta_0(v))}}\right|$$

where ζ_0 is the unique solution to the equation $\mathrm{e}^{\zeta} = 1 - v\zeta$ that satisfies inequality $|\zeta_0(v) - \xi(v) + i\pi| \leq \pi$. If $1 \leq v < v_0$, we set $R(v) = O(1)$. For the original definition, see [13].

Lemma 3.4. *Let $\omega(v)$, $v \geq 1$, be the Buchstab function* (1.2). *Then, for $v \geq 1$, we have*

$$\omega(v) - \mathrm{e}^{-\gamma} = -2\mathrm{e}^{-\gamma}R(v)\left(\cos\vartheta(v) + O\left(1/v\right)\right)$$

where $R(v)$ and $\vartheta(v)$ are real valued differentiable functions. Moreover, $R(v)$ is decreasing for sufficiently large v and

$$R(v) = \varrho(v)\exp\left\{\frac{-\pi^2 v}{2\xi(v)^2} + O\left(\frac{v}{\xi(v)^3}\right)\right\}.$$

Proof. This is Lemma 4 in [13]. □

Lemma 3.5. *For a sufficiently large constant v_0, we have*

$$R(v) = \varrho(v)\exp\left\{\frac{-\pi^2 v}{2(\log v)^2} + O\left(\frac{v\log\log(v+2)}{(\log v)^3}\right)\right\}$$

if $v \geq v_0$, and $R(v) = O(1)$ if $1 \leq v < v_0$.

Proof. This follows from Lemmas 3.1 and 3.4. □

Lemma 3.6. *Let $1 \leq r \leq n$. Denote by x the positive solution to the equation*

$$\sum_{j=1}^{r} x^j = n,$$

and let $u = n/r$. We have

$$\mathrm{e}^{(\log u)/r} \leq x \leq \mathrm{e}^{(2\log u)/r}.$$

Moreover,

$$x = \exp\left\{\frac{\log(u \cdot \min\{r, \log u\})}{r}\right\}\left(1 + O\left(\frac{1}{r}\right)\right) \tag{3.4}$$

if $u \geq 3$.

Proof. Bounds for x follow from inequalities

$$x^{r/2} \le \sqrt[r]{x^1 x^2 \dots x^r} \le u = \frac{1}{r}\sum_{j=1}^{r} x^j \le x^r,$$

and the asymptotic formula is taken from Lemma 9 in [9]. □

Recall that

$$\nu(n,r) = \frac{1}{n!}\left|\{\sigma \in \mathrm{S}_n :\ k_j(\sigma) = 0\ \forall j \in \overline{r+1,n}\}\right|.$$

Lemma 3.7. *Let x be the positive solution to the equation $\sum_{j=1}^{r} x^j = n$, and $\lambda(x) = \sum_{j=1}^{r} jx^j$. For $1 \le r \le n$, we have*

$$\nu(n,r) = \frac{\exp\left\{\sum_{j=1}^{r} x^j/j\right\}}{x^n\sqrt{2\pi\lambda(x)}}\left(1 + O\left(\frac{r}{n}\right)\right)$$

and

$$\frac{r^2}{2} < \lambda(x) \le rn.$$

Proof. This is Corollary 5 from [9], except the effective bounds for $\lambda(x)$, which trivially follow from inequalities

$$\sum_{j=1}^{r} j \le \sum_{j=1}^{r} jx^j \le r\sum_{j=1}^{r} x^j.$$

□

Lemma 3.8. *For $1 \le r \le \log n$, we have*

$$\nu(n,r) \ll \exp\left\{-\frac{n\log n}{r} + \frac{n}{r} + \frac{n}{\log n} + \frac{3n}{(\log n)^2} - \frac{2n}{(\log n)^3} + \log\frac{\log n}{r}\right\}.$$

Proof. We apply Theorem 1 from [9],

$$\begin{aligned}\nu(n,r) &= \frac{1}{\sqrt{2\pi nr}}\exp\left\{-\frac{n\log n}{r} + \frac{n}{r} + \sum_{N=1}^{r} d_{rN} n^{(r-N)/r}\right\}\left(1 + O\left(n^{-1/r}\right)\right)\\ &\ll \exp\left\{-\frac{n\log n}{r} + \frac{n}{r} + \sum_{N=1}^{r} d_{rN} n^{(r-N)/r} - \log r\right\},\end{aligned}$$

where $d_{rr} = -(1/r)\sum_{j=2}^{r} 1/j$ and

$$d_{rN} = \frac{\Gamma(N + N/r)}{(r-N)\Gamma(N+1)\Gamma(1+N/r)} \le \frac{1}{r-N}$$

if $0 < N/r < 1$. Therefore,

$$\nu(n,r) \ll \exp\left\{-\frac{n\log n}{r} + \frac{n}{r} + \sum_{N=1}^{r-1}\frac{n^{N/r}}{N} - \log r\right\},$$

and the proposition follows from an estimate

$$\begin{aligned}\sum_{N=1}^{r-1}\frac{n^{N/r}}{N} &\le \int_1^r \frac{\mathrm{e}^{(t\log n)/r}}{t}dt \le \int_1^{\log n}\frac{\mathrm{e}^t-1}{t}dt + \log\log n \\ &= \frac{\mathrm{e}^t-t-1}{t}\Big|_1^{\log n} + \frac{\mathrm{e}^t-t^2/2-t-1}{t^2}\Big|_1^{\log n} + 2\int_1^{\log n}\frac{\mathrm{e}^t-t^2/2-t-1}{t^3}dt \\ &\quad + \log\log n \\ &\le \frac{n}{\log n} + \frac{n}{(\log n)^2} + 2(\log n - 1)\frac{n}{(\log n)^3} + \log\log n.\end{aligned}$$

□

Lemma 3.9. *For $0 < y < 2\pi r$, we have*

$$T(y) = \int_0^y \frac{\mathrm{e}^t-1}{t}\left(\frac{t}{r}\frac{\mathrm{e}^{\frac{t}{r}}}{\mathrm{e}^{\frac{t}{r}}-1} - 1\right)dt \le \frac{\mathrm{e}^y}{2r} + \frac{y\mathrm{e}^y}{12r^2}.$$

Proof. A well-known theory of Bernoulli numbers $\{B_n\}$, $n \ge 0$, [11, p. 142] gives us the series

$$\frac{t\mathrm{e}^t}{\mathrm{e}^t-1} = t + \sum_{n=0}^{\infty}\frac{B_n(-t)^n}{n!} = 1 + \frac{t}{2} + 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\zeta(2k)}{(2\pi)^{2k}}t^{2k} \tag{3.5}$$

converging for $|t| < 2\pi$. Here $\zeta(2k) = \sum_{m\ge1} m^{-2k} \le \zeta(2) = \pi^2/6$. Hence, if $0 < t < 2\pi$,

$$\frac{t\mathrm{e}^t}{\mathrm{e}^t-1} = 1 + \frac{t}{2} + 2\frac{\zeta(2)}{(2\pi)^2}t^2 + \ldots < 1 + \frac{t}{2} + 2\frac{\zeta(2)}{(2\pi)^2}t^2 = 1 + \frac{t}{2} + \frac{t^2}{12}$$

and

$$T(y) \le \frac{1}{2r}\int_0^y(\mathrm{e}^t-1)dt + \frac{1}{12r^2}\int_0^y t(\mathrm{e}^t-1)dt.$$

□

4 The proofs of Theorem 1.3 and Theorem 1.4

The next lemma is an essential part of the proofs. We use the following notation

$$(a)_+ = \max\{a, 0\}$$

if $a \in \mathbf{R}$.

Lemma 4.1. *If $1/r \le |t| \le \pi$, $r \in \mathbf{N}$, and $\alpha > 1$, then*

$$\sum_{j=1}^{r} \frac{1-\alpha^j \cos(tj)}{j} \le \sum_{j=1}^{r} \frac{\alpha^j - 1}{j} - \frac{2}{r}\frac{\alpha^{r+1}}{\alpha-1}\left(\frac{\pi^{-2}}{1+(r\alpha-r)^2} - \alpha^{-r/2}\right)_+ + \log\frac{|\alpha - \mathrm{e}^{it}|}{\alpha-1} + \min\{2r\log\alpha,\ 2\log(\mathrm{e}r)\} + 4, \tag{4.1}$$

and

$$\sum_{j=1}^{r} \frac{1-\alpha^j \cos(tj)}{j} \le -\frac{1}{2r}\frac{\alpha(\alpha^r-1)}{\alpha-1} + 1 \tag{4.2}$$

if $|t| \le 1/r$.

Proof. Note that

$$2t^2/\pi^2 \le 1 - \cos t \le t^2/2 \tag{4.3}$$

if $|t| \le \pi$. An estimate (4.2) is not difficult to obtain. If $|t| \le 1/r$, we have

$$\begin{aligned}\sum_{j=1}^{r} \frac{1-\alpha^j \cos(tj)}{j} &\le -\sum_{j=1}^{r}\frac{\alpha^j-1}{j} + \sum_{j=1}^{r}\frac{\alpha^j\,(1-\cos(j/r))}{j}\\ &\le -\sum_{j=1}^{r}\frac{\alpha^j-1}{j} + \frac{1}{2r^2}\sum_{j=1}^{r} j\alpha^j\\ &\le -\frac{1}{2r}\sum_{j=1}^{r}\alpha^j + 1.\end{aligned}$$

To obtain an estimate (4.1), we use the technique found in [12, p. 407]. Notice that

$$\sum_{j=1}^{r} \frac{1-\alpha^j \cos(tj)}{j} = \sum_{j=1}^{r}\frac{\alpha^j-1}{j} - \sum_{j=1}^{r}\frac{\alpha^j\,(1+\cos(tj))-2}{j}. \tag{4.4}$$

Thus, we evaluate only the second sum

$$
\begin{aligned}
&\sum_{j=1}^{r} \frac{\alpha^j\left(1+\cos(tj)\right)-2}{j}\\
&\geq \sum_{j=1}^{[r/2]} \frac{\cos(tj)-1}{j} + \frac{1}{r}\sum_{j=[r/2]+1}^{r} \alpha^j\left(1+\cos(tj)\right) - \sum_{j=[r/2]+1}^{r} \frac{2}{j}\\
&\geq \sum_{j=1}^{r} \frac{\cos(tj)-1}{j} + \frac{1}{r}\Re \sum_{j=[r/2]+1}^{r} \alpha^j\left(1+\mathrm{e}^{itj}\right) - 4. \qquad (4.5)
\end{aligned}
$$

To obtain -4 we used inequalities $\log(1+r) \leq \sum_{j=1}^{r} 1/j \leq 1+\log r$. Let us turn our attention to the most difficult part, namely, to the sum

$$
\begin{aligned}
&\Re \sum_{j=[r/2]+1}^{r} \alpha^j\left(1+\mathrm{e}^{itj}\right)\\
&= \frac{\alpha^{r+1}-\alpha^{[r/2]+1}}{\alpha-1} + \Re\frac{(\alpha\mathrm{e}^{it})^{r+1}-(\alpha\mathrm{e}^{it})^{[r/2]+1}}{\alpha\mathrm{e}^{it}-1}\\
&= \frac{\alpha^{r+1}}{\alpha-1}\left(1-\alpha^{[r/2]-r} + \Re\frac{\mathrm{e}^{it(r+1)}-\alpha^{[r/2]-r}\mathrm{e}^{it([r/2]+1)}}{\alpha\mathrm{e}^{it}-1}(\alpha-1)\right)\\
&\geq \frac{\alpha^{r+1}}{\alpha-1}\left(1-2\alpha^{[r/2]-r} + \Re\frac{\mathrm{e}^{it(r+1)}(\alpha-1)}{\alpha\mathrm{e}^{it}-1}\right)_+\\
&\geq \frac{\alpha^{r+1}}{\alpha-1}\left(1-2\alpha^{-r/2} - \left|\frac{\alpha-1}{\alpha\mathrm{e}^{it}-1}\right|\right)_+. \qquad (4.6)
\end{aligned}
$$

In the last steps, we used a relation $\Re z \geq -|z|$, $z \in \mathbf{C}$. Applying

$$
1-\frac{p}{\sqrt{p^2+v^2}} \geq \frac{1}{2}\frac{v^2}{p^2+v^2}, \quad p>0, \quad v \in \mathbf{R},
$$

with $p=\alpha-1$ and $v=\sqrt{2\alpha(1-\cos t)}$ and recalling (4.3), we get that

$$
\begin{aligned}
1-\left|\frac{\alpha-1}{\alpha\mathrm{e}^{it}-1}\right| &\geq \frac{1}{2}\frac{2\alpha(1-\cos t)}{(\alpha-1)^2+2\alpha(1-\cos t)}\\
&\geq \frac{(1-\cos(1/r))}{(\alpha-1)^2+2(1-\cos(1/r))}\\
&\geq \frac{2}{\pi^2}\frac{1}{(r\alpha-r)^2+1}.
\end{aligned}
$$

Applying the latter lower estimate for (4.6), in conjunction with (4.5), we obtain

$$
\begin{aligned}
&\sum_{j=1}^{r} \frac{\alpha^j\left(1+\cos(tj)\right)-2}{j}\\
&\geq \sum_{j=1}^{r} \frac{\cos(tj)-1}{j} + \frac{2}{r}\frac{\alpha^{r+1}}{\alpha-1}\left(\frac{\pi^{-2}}{1+(r\alpha-r)^2} - \alpha^{-r/2}\right)_+ - 4. \qquad (4.7)
\end{aligned}
$$

It is necessary to make a specific estimate for the remaining sum. We have

$$\begin{aligned}\sum_{j=1}^{r}\frac{1-\cos(tj)}{j} &\le \Re\sum_{j=1}^{\infty}\frac{1-\mathrm{e}^{itj}}{j}\alpha^{-j}+\sum_{j=1}^{r}\frac{1-\cos(tj)}{j}(1-\alpha^{-j})\\ &\le \log\frac{|1-\alpha^{-1}\mathrm{e}^{it}|}{1-\alpha^{-1}}+2\sum_{j=1}^{r}\frac{1-\alpha^{-j}}{j}\\ &\le \log\frac{|\alpha-\mathrm{e}^{it}|}{\alpha-1}+\min\{2r\log\alpha,\ 2\log(\mathrm{e}r)\}. \qquad (4.8)\end{aligned}$$

The reasons for obtaining this estimate will reveal in the proofs of Theorems 1.3 and 1.4. Combining (4.8) with (4.7) and (4.4) we arrive at the assertion.

□

The common part of the proofs (Theorem 1.6). We set the proofs for the case $r < n/2$. The case $n/2 \le r \le n$ is an easy exercise. For $0 < \beta < 1$, we have

$$\begin{aligned}\Pr\Big(\sum_{j=r+1}^{n} jZ_j = n\Big) &= \frac{1}{2\pi i}\int_{|z|=\beta}\exp\left\{\sum_{j=r+1}^{n}\frac{z^j-1}{j}\right\}\frac{dz}{z^{n+1}}\\ &= \frac{\mathrm{e}^{-H_n}}{2\pi i}\int_{|z|=\beta}\exp\left\{\sum_{j=1}^{r}\frac{1-z^j}{j}\right\}\frac{dz}{(1-z)z^{n+1}},\end{aligned}$$

because, if we add $\sum_{j=n+1}^{\infty} z^j/j$ to $\sum_{j=r+1}^{n} z^j/j$, n-th Taylor coefficient of the integrand function does not change. Notice that the resulting integrand function has two singularities at $z = 1$ and $z = 0$. Let $s = -r\log z$ and $u = n/r$, then

$$\begin{aligned}\Pr\Big(\sum_{j=r+1}^{n} jZ_j = n\Big) &= \frac{\mathrm{e}^{-H_n}}{2\pi i}\int_{-r\log\beta-ir\pi}^{-r\log\beta+ir\pi}\exp\left\{\sum_{j=1}^{r}\frac{1-\mathrm{e}^{-js/r}}{j}\right\}\frac{\mathrm{e}^{us+s/r}}{r(\mathrm{e}^{s/r}-1)}ds\\ &=: \frac{\mathrm{e}^{-H_n}}{2\pi i}\int_{-r\log\beta-ir\pi}^{-r\log\beta+ir\pi} L(s)\mathrm{e}^{us}ds.\end{aligned}$$

The obtained integrand function has only one singularity at $s = 0$. Furthermore, $\mathrm{Res}_{s=0}\, L(s)\mathrm{e}^{us} = 1$. Let $\delta = \delta_1 \cup \delta_2 \cup \delta_3 \cup \delta_4$ and $\alpha > 1$ where

$$\delta_1 = \{-r\log\beta + it : t \in [-r\pi, r\pi]\}, \quad \delta_2 = \{\sigma + ir\pi : \sigma \in [-r\log\alpha, -r\log\beta]\},$$

$$\delta_3 = \{-r\log\alpha + it : t \in [-r\pi, r\pi]\}, \quad \delta_4 = \{\sigma - ir\pi : \sigma \in [-r\log\alpha, -r\log\beta]\}.$$

We apply the residue theorem to get

$$\begin{aligned}&\Pr\Big(\sum_{j=r+1}^{n} jZ_j = n\Big)\\ &= \frac{\mathrm{e}^{-H_n}}{2\pi i}\left(\int_{\delta} L(s)\mathrm{e}^{us}ds - \left(\int_{\delta_2} L(s)\mathrm{e}^{us}ds + \int_{\delta_4} L(s)\mathrm{e}^{us}ds\right) - \int_{\delta_3} L(s)\mathrm{e}^{us}ds\right)\\ &= \mathrm{e}^{-H_n}\left(1 - 0 - \frac{1}{2\pi i}\int_{\delta_3} L(s)\mathrm{e}^{us}ds\right)\\ &=: \mathrm{e}^{-H_n}(1+R). \qquad (4.9)\end{aligned}$$

At this point, it remains to estimate R. Firstly, we will apply integration by parts twice. Note that $L'(s) = -\frac{\mathrm{e}^{-s}}{r(\mathrm{e}^{s/r}-1)}L(s)$.

$$\begin{aligned} R &= \frac{1}{2\pi i u}\int_{-r\log\alpha-ir\pi}^{-r\log\alpha+ir\pi} L(s)d\mathrm{e}^{us} \\ &= -\frac{1}{2\pi i u}\int_{-r\log\alpha-ir\pi}^{-r\log\alpha+ir\pi} \mathrm{e}^{us}L'(s)ds \\ &= -\frac{1}{2\pi i u(u-1)}\int_{-r\log\alpha-ir\pi}^{-r\log\alpha+ir\pi} \mathrm{e}^{s}L'(s)d\mathrm{e}^{(u-1)s} \\ &= \frac{1}{2\pi i u(u-1)}\int_{-r\log\alpha-ir\pi}^{-r\log\alpha+ir\pi} \mathrm{e}^{(u-1)s}\left(\mathrm{e}^{s}L'(s)\right)' ds \\ &= \frac{1}{2\pi i n(n-r)}\int_{-r\log\alpha-ir\pi}^{-r\log\alpha+ir\pi} \frac{\mathrm{e}^{s/r}+\mathrm{e}^{-s}}{(\mathrm{e}^{s/r}-1)^2}L(s)\mathrm{e}^{(u-1)s}ds. \end{aligned} \tag{4.10}$$

Secondly, using abbreviations

$$E(r,\alpha) := -\frac{2}{r}\frac{\alpha^{r+1}}{\alpha-1}\left(\frac{\pi^{-2}}{1+(r\alpha-r)^2}-\alpha^{-r/2}\right)_+ + \min\{2r\log\alpha, 2\log(\mathrm{e}r)\} \tag{4.11}$$

and

$$J(\tau,\alpha) := \frac{1}{(\alpha-1)^2+\alpha\left(2\tau/(\pi r)\right)^2},\ \tau\in\mathbf{R},$$

we apply (4.3) and Lemma 4.1 to (4.10):

$$\begin{aligned} |R| &\le \frac{4\alpha^{2r-n+2}}{\pi n^2 r}\int_0^{r\pi}\frac{1}{\left|\mathrm{e}^{i\tau/r}-\alpha\right|^3}\exp\left\{\sum_{j=1}^{r}\frac{1-\alpha^j\cos(\tau j/r)}{j}\right\}d\tau \\ &\le \frac{4\mathrm{e}^4\alpha^{2r-n+2}}{\pi n^2 r(\alpha-1)}\exp\left\{\sum_{j=1}^{r}\frac{\alpha^j-1}{j}+E(r,\alpha)\right\}\int_1^{r\pi}J(\tau,\alpha)d\tau \\ &\quad + \frac{4\mathrm{e}\alpha^{2r-n+2}}{\pi n^2 r}\exp\left\{-\frac{\alpha(\alpha^r-1)}{2r(\alpha-1)}\right\}\int_0^1 J(\tau,\alpha)^{3/2}d\tau \end{aligned}$$

due to $\left|\mathrm{e}^{s/r}+\mathrm{e}^{-s}\right|\le 2\alpha^r$ and the additional factor $1/|\mathrm{e}^{i\tau/r}-\alpha|$ from $|L(s)|$. Since

$$\int_1^{r\pi}J(\tau,\alpha)d\tau = \frac{\pi r}{2\sqrt{\alpha}(\alpha-1)}\arctan\left(\frac{2\sqrt{\alpha}\tau}{\pi r(\alpha-1)}\right)\Big|_1^{r\pi} \le \frac{\pi^2 r}{4\sqrt{\alpha}(\alpha-1)}$$

and $\int_0^1 J(\tau,\alpha)^{3/2}d\tau \le (\alpha-1)^{-3}$, we obtain

$$|R| \le \frac{\pi\mathrm{e}^4\alpha^{2r-n+3/2}}{n^2(\alpha-1)^2}\exp\left\{\sum_{j=1}^{r}\frac{\alpha^j-1}{j}+E(r,\alpha)\right\} + \frac{4\mathrm{e}\alpha^{2r-n+2}}{\pi n^2 r(\alpha-1)^3}\exp\left\{-\frac{\alpha(\alpha^r-1)}{2r(\alpha-1)}\right\}. \tag{4.12}$$

□

Hereafter, we divide the argument into two parts. In the first part we take $\alpha = \mathrm{e}^{\xi/r}$ (Theorem 1.3) and in the second $\alpha = x$ (Theorem 1.4), where $\xi = \xi(n/r)$ is defined in Lemma 3.1, and $x = x(n,r)$ in Lemma 3.6.

Proof of Theorem 1.3. Let $\alpha = \mathrm{e}^{\xi/r}$. Recall that $\log n \le r < n/2$ and $\log u \le \xi \le 2\log u$. Note, referring to the equation $\mathrm{e}^{\xi} = 1 + u\xi$, we have $\xi > 1$. We will need the following estimates

$$I(\xi) = \int_0^{\xi} \frac{1}{t} d(\mathrm{e}^t - t - 1) \ge \frac{\mathrm{e}^t - t - 1}{t}\Big|_0^{\xi} = u - 1,$$

$$0 \le T(\xi) \le \frac{1 + u\xi}{2r} + \frac{2r(1 + u\xi)}{12r^2} \le \frac{2u\xi}{3r} + \frac{1}{r} \tag{4.13}$$

(this follows from Lemma 3.9; note $\xi \le 2r$), and

$$\begin{aligned} &E(r, \mathrm{e}^{\xi/r}) - \min\{2\xi, 2\log(\mathrm{e}r)\} \\ &= -\frac{2}{r}\frac{\mathrm{e}^{\xi+\xi/r}}{\mathrm{e}^{\xi/r} - 1}\left(\frac{\pi^{-2}}{1 + (r\mathrm{e}^{\xi/r} - r)^2} - \mathrm{e}^{-\xi/2}\right)_+ \\ &= \frac{-2\mathrm{e}^{\xi}\pi^{-2}}{\xi + \xi(r\mathrm{e}^{\xi/r} - r)^2}\frac{\xi\mathrm{e}^{\xi/r}}{r(\mathrm{e}^{\xi/r} - 1)}\left(1 - \frac{\pi^2 + \pi^2(r\mathrm{e}^{\xi/r} - r)^2}{\mathrm{e}^{\xi/2}}\right)_+ . \end{aligned}$$

Now, applying the estimates $(\xi/r) < 2$, $(\xi/r)\mathrm{e}^{\xi/r}/(\mathrm{e}^{\xi/r} - 1) = 1 + O_+(\xi/r)$ (see (3.5)), and

$$\begin{aligned} r\mathrm{e}^{\xi/r} - r &= \frac{\xi}{1!} + \frac{\xi^2}{2!r} + \frac{\xi^3}{3!r^2} + \dots \\ &= \xi(1 + O_+(\xi/r)) \end{aligned}$$

(O_+ is the Big-O notation where $+$ indicates that the estimated quantity is greater than 0), we obtain

$$\begin{aligned} &E(r, \mathrm{e}^{\xi/r}) - \min\{2\xi, 2\log(\mathrm{e}r)\} \\ &= \frac{-2(u + 1/\xi)\pi^{-2}}{1 + \xi^2(1 + O_+(\xi/r))}\frac{\xi\mathrm{e}^{\xi/r}}{r(\mathrm{e}^{\xi/r} - 1)}\left(1 + O\left(\frac{(\log u)^{3/2}}{\sqrt{u}}\right)\right) \qquad (4.14) \\ &= \frac{-2(u + 1/\xi)\pi^{-2}}{1 + \xi^2(1 + O_+(\xi/r))}\left(1 + O\left(\frac{\log u}{r} + \frac{(\log u)^{3/2}}{\sqrt{u}}\right)\right). \qquad (4.15) \end{aligned}$$

We see that $I(\xi) + T(\xi) + E(r, \mathrm{e}^{\xi/r}) \gtrsim u$ if $u \to \infty$. Recalling (3.1) and applying the latter estimate to (4.12), we conclude that

$$\begin{aligned} R &\ll \frac{\mathrm{e}^{2\xi - u\xi}}{(u\xi)^2}\exp\left\{\sum_{j=1}^{r}\frac{\mathrm{e}^{j\xi/r} - 1}{j} + E(r, \mathrm{e}^{\xi/r})\right\} + \frac{\mathrm{e}^{2\xi - u\xi}}{u^2\xi^3}\exp\left\{-\frac{u\xi}{2r}\frac{\mathrm{e}^{\xi/r}}{\mathrm{e}^{\xi/r} - 1}\right\} \\ &\ll \exp\left\{-u\xi + I(\xi) + T(\xi) + E(r, \mathrm{e}^{\xi/r})\right\}. \qquad (4.16) \end{aligned}$$

To obtain (1.14), we first observe that combining (1.1), (4.9) and (4.16) we get

$$\kappa(n, r) = \mathrm{e}^{-H_r} + O\left(\frac{1}{r}\exp\left\{-u\xi + I(\xi) + T(\xi) + E(r, \mathrm{e}^{\xi/r})\right\}\right).$$

We apply Lemma 3.2 and formula (3.3) here to obtain

$$\kappa(n,r) = \mathrm{e}^{-H_r} + O\left(\frac{\varrho(u)}{r}\exp\left\{T(\xi) + E(r,\mathrm{e}^{\xi/r}) + \log u\right\}\right). \tag{4.17}$$

It is left to evaluate $T(\xi) + E(r,\mathrm{e}^{\xi/r}) + \log u$. For that we use (4.13), (4.14)–(4.15) and then we apply (3.2). Recalling that $\log n \le r \le (\log n)^{3+\varepsilon}$, we have

$$\begin{aligned}
&T(\xi) + E(r,\mathrm{e}^{\xi/r}) + \log u \\
&\le \frac{2u\xi}{3r} + \frac{1}{r} + 2\xi - \frac{2(u+1/\xi)\pi^{-2}}{1+\xi^2(1+O_+(\xi/r))}\left(1 + O\left(\frac{\log u}{r} + \frac{(\log u)^{3/2}}{\sqrt{u}}\right)\right) + \log u \\
&= \frac{2u\xi}{3r}\left(1 - \frac{r}{\xi^3}\cdot\frac{3}{(\pi/\xi)^2 + \pi^2(1+O_+(\xi/r))}\left(1+O\left(\frac{\log u}{r}+\frac{(\log u)^{3/2}}{\sqrt{u}}\right)\right)+O\left(\frac{r}{u}\right)\right) \\
&\le \frac{u\log u}{r}
\end{aligned} \tag{4.18}$$

if u is sufficiently large, because $(\log u)/r$ (in the error estimate) came from the factor in (4.14) that is greater than 1. When u is bounded, equation (1.14) is trivial. Applying (4.18) to (4.17), we prove (1.14).

Now we turn to assertion (1.13), recalling that $(\log n)^{3+\varepsilon} \le r < n$, $\varepsilon > 0$, and $\delta > 0$. Let $n_0 = n_0(\varepsilon,\delta)$ and $u_0 = u_0(\varepsilon,\delta)$ be such sufficiently large positive constants, depending on parameters ε and δ, that if $n \ge n_0$ and $u \ge u_0$, using (4.15), (4.13) and Lemma 3.1, we obtain

$$\begin{aligned}
&E(r,\mathrm{e}^{\xi/r}) + 2\xi + T(\xi) \\
&= -\frac{2u}{\pi^2\xi^2}\left(1 + O\left(\frac{\log u}{r} + \frac{1}{(\log u)^2}\right)\right) + \min\{2\xi, 2\log(\mathrm{e}r)\} + 2\xi + T(\xi) \\
&\le -\frac{2u}{\pi^2\xi^2}\left(1 + O\left(\frac{\log u}{r} + \frac{1}{(\log u)^2}\right)\right) + 4\xi + \frac{2u\xi}{3r} + \frac{1}{r} \\
&= -\frac{2u}{\pi^2\xi^2}\left(1 + O\left(\frac{1}{(\log u)^2} + \frac{(\log u)^3}{r}\right)\right) \\
&= -\frac{2u}{\pi^2(\log u)^2}\left(1 + O\left(\frac{\log\log(u+2)}{\log u} + \frac{1}{(\log n)^{\varepsilon}}\right)\right) \\
&\le -\frac{2u}{\pi^2(\log(1+u))^2}\left(1-\delta\right).
\end{aligned}$$

Combining the latter estimate with (4.16), (4.9), and Lemma 3.2 we prove assertion (1.13) for the case $n \ge n_0$ and $u \ge u_0$. In case when $n \le n_0$ or $u \le u_0$, estimate (1.13) follows trivially from (4.9), (4.15), (4.16) and Lemma 3.2. □

Proof of Theorem 1.4. Let $\alpha = x$. Recall that $u > 2$ and $r \ge 5$. Throughout the proof, we apply inequalities $\mathrm{e}^{(\log u)/r} \le x \le \mathrm{e}^{(2\log u)/r}$ and equation

$$\frac{x^{r+1} - x}{x - 1} = n,$$

which come from Lemma 3.6. Recall (4.11). The proof starts with the following observation:

$$\begin{aligned} E(r,x) - \min\{2r\log x, 2\log(\mathrm{e}r)\} &\leq -2u\left(\frac{\pi^{-2}}{1+(rx-r)^2} - x^{-r/2}\right)_+ \\ &\leq \frac{-u^{1-4/r}}{4\pi^2(\log u)^2}\left(1 - \frac{\pi^2}{x^{r/2}} - \frac{\pi^2(rx-r)^2}{x^{r/2}}\right)_+ \\ &\leq \frac{-u^{1-4/r}}{4\pi^2(\log u)^2}\left(1 - \frac{\pi^2}{\sqrt{u}} - \frac{\pi^2 x^{3r/2+2}}{u^2}\right)_+ \end{aligned}$$

where the second inequality is obtained using estimate $rx - r \leq r\mathrm{e}^{(2\log u)/r} - r \leq 2(\log u)u^{2/r}$ and the third—inequalities $x \geq \mathrm{e}^{(\log u)/r}$ and $(x-1)/x^r \leq x/n$. Now we apply the latter estimate and $x^2/(nx-n)^2 \ll 1$ to (4.12) and thus obtain

$$\begin{aligned} R &\ll x^{2r-n}\exp\left\{\sum_{j=1}^{r}\frac{x^j-1}{j} - \frac{u^{1-4/r}}{4\pi^2(\log u)^2}\left(1 - \frac{\pi^2}{\sqrt{u}} - \frac{\pi^2 x^{3r/2+2}}{u^2}\right)_+\right\} \\ &\quad + \frac{x^{2r-n}}{\log u}\exp\left\{-\frac{u}{2}\right\}, \end{aligned}$$

because $x(x^r-1)/(2rx-2r) = u/2$ (see Lemma 3.6). Applying Lemma 3.7, it follows that

$$R \ll \sqrt{u}x^{2r}\nu(n,r)\exp\left\{-\frac{u^{1-4/r}}{4\pi^2(\log u)^2}\left(1 - \frac{\pi^2}{\sqrt{u}} - \frac{\pi^2 x^{3r/2+2}}{u^2}\right)_+\right\} \tag{4.19}$$

$$\ll \nu(n,r)\exp\left\{-\frac{u^{1-4/r}}{4\pi^2(\log u)^2}\left(1 - \frac{\pi^2}{\sqrt{u}} - \frac{\pi^2 x^{3r/2+2}}{u^2}\right)_+ + 5\log u\right\}. \tag{4.20}$$

Now, formula (1.16) follows simply from (4.19) if u is bounded (see Lemma 3.6). If u is sufficiently large or $u \to \infty$ we apply (3.4):

$$\begin{aligned} R &\ll \sqrt{u}x^{2r}\nu(n,r) \\ &= \sqrt{u}\left(\exp\{(\log n)/r\}\left(1 + O(1/r)\right)\right)^{2r}\nu(n,r) \\ &\ll n^{5/2}r^{-1/2}\nu(n,r). \end{aligned}$$

It remains to prove the formula (1.15). Let $u_1 = u_1(\varepsilon)$ be such a sufficiently large positive constant depending on the parameter $\varepsilon > 0$ that if $u \geq u_1$ applying (3.4) to (4.20) it follows that

$$\begin{aligned} R &\ll \nu(n,r)\exp\left\{-\frac{u^{1-4/r}}{4\pi^2(\log u)^2}\left(1 + O\left((\log u)^3 u^{2/r-1/2}\right)\right)\right\} \\ &\ll_\varepsilon \nu(n,r)\exp\left\{-\frac{u^{1-4/r}(1-\varepsilon)}{4\pi^2(\log(u+1))^2}\right\}. \end{aligned}$$

Actually, this estimate is also correct in the case of $u \leq u_1$. Recalling (4.9) and (1.1) we finish the proof of (1.15). □

Acknowledgments

I am grateful to Eugenijus Manstavičius who inspired me to write this paper and provided some nice remarks and suggestions that encouraged me to make some improvements.

References

[1] R. Arratia, A. D. Barbour and S. Tavaré, *Logarithmic Combinatorial Structures: A Probabilistic Approach*, EMS Monogr. Math., EMS Publishing House, Zürich, 2003.

[2] R. Arratia and S. Tavaré, The cycle structure of random permutations, *Ann. Probab.* **20** (3) (1992), 1567–1591.

[3] E. A. Bender, A. Mashatan, D. Panario and L. B. Richmond, Asymptotics of combinatorial structures with large smallest component, *J. Combin. Theory Ser. A* **107** (2004), 117–125.

[4] M. Bóna, *Combinatorics of Permutations*, CRC Press, Boca Raton, London, New York, Washington D.C., 2004.

[5] A. Granville, Cycle lengths in a permutation are typically Poisson, *Electron. J. Combin.* **13** (2006), #R107.

[6] A. Hildebrand and G. Tenenbaum, Integers without large prime factors, *J. Théorie des Nombres de Bordeaux* **5** (1993), 411–484.

[7] E. Manstavičius, On permutations missing short cycles, *Lietuvos matem. rink.* (spec. issue) **42** (2002), 1–6.

[8] E. Manstavičius and R. Petuchovas, Local probabilities and total variation distance for random permutations, *Ramanujan J.* **43** (3) (2017), 679–696.

[9] E. Manstavičius and R. Petuchovas, Local probabilities for random permutations without long cycles, *Electron. J. Combin.* **23** (1) (2016), #P1.58, 25 pp.

[10] P. R. de Montmort, Essay d'analyse sur les jeux de hazard, Seconde Edition, *Revue & augmente de plusieurs Lettres. Paris: Jacque Quillau,* 1713.

[11] M. R. Spiegel, S. Lipschutz and J. Liu, *Mathematical Handbook of Formulas and Tables, Third Ed.*, Schaum's Outline Series, 2009.

[12] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, 1995.

[13] G. Tenenbaum, Crible d'Ératosthène et modèle de Kubilius, In: *Number Theory in Progress, Proc. Conf. in Honor of Andrzej Schinzel*, Zakopane, Poland, 1997. (K. Gyory, H. Iwaniec, J. Urbanowicz, Eds.), Walter de Gruyter, Berlin, New York, 1999, 1099–1129.

[14] A. Weingartner, On the degrees of polynomial divisors over finite fields, *Math. Proc. Cambridge Philos. Soc.* **161** (3) (2016), 469–487.

(Received 5 Feb 2017; revised 17 July 2017, 21 May 2018)