AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **72(2)** (2018), Pages 164–184

A Generalization of a result of Catlin: 2-factors in line graphs

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Abstract

A 2-factor of a graph G consists of a spanning collection of vertex disjoint cycles. In particular, a hamiltonian cycle is an example of a 2-factor consisting of precisely one cycle. Harary and Nash-Williams described graphs with hamiltonian line graphs. Gould and Hynds generalized this result, describing those graphs whose line graphs contain a 2-factor with exactly k ($k \geq 1$) cycles. With this tool, we show that certain properties of a graph G, that were formerly shown to imply the hamiltonicity of the line graph, $L(G)$, are actually strong enough to imply that $L(G)$ has a 2-factor with k cycles for $1 \leq k \leq f(n)$, where n is the order of the graph G.

1 Introduction

We present an extension and then a broader generalization of the following result of Catlin [3], which gives specific conditions on a graph G that imply that the line graph $L(G)$ is hamiltonian.

Theorem 1.1 [3] *If G is a 2-edge-connected simple graph of order n such that $\delta(G) \geq n/5$, then $L(G)$ is hamiltonian.*

The subgraph H of G is said to be a 2-*factor of* G if for every $v \in V(G)$, $\deg_H v = 2$. A trivial consequence of the definition is that every 2-factor of a

ISSN: 2202-3518

graph G consists of a spanning collection of vertex disjoint cycles. In particular, a hamiltonian cycle is an example of a 2-factor consisting of precisely one cycle.

We begin by proving the following extension of Theorem 1.1.

Theorem 1.2 *If G is a 2-edge-connected simple graph of order n such that $\delta(G) \geq n/5$, then $L(G)$ has a 2-factor with k cycles for each $k \in \{1, 2, \ldots, \lfloor n/10 \rfloor\}$.*

The remainder of the paper is dedicated to proving the following broader generalization for graphs with large independence number.

Theorem 1.3 *If G is a 2-edge-connected simple graph of order $n > 65$ with $\delta(G) \geq n/5$ and $\alpha(G) = cn$, for some c, $1/4 \leq c < 1$, then $L(G)$ contains a 2-factor with k cycles, for each $k = 1, 2, \ldots, cn\lfloor \frac{\delta}{3} \rfloor - [(1-c)n - \delta]$.*

All graphs considered in this paper are simple graphs. For terms or notation not defined here, see [4]. For a graph G, let $N(v)$ denote the neighborhood of vertex v. A set $S \subseteq V(G)$ is said to be *independent* if for all $u, v \in S, uv \notin E(G)$. The *independence number* of a graph G, denoted $\alpha(G)$, is the size of a largest independent set of vertices of G. For a set $S \subseteq V(G)$ we use $\langle S \rangle$ to denote the subgraph induced by S.

A *circuit* of G is an alternating sequence $C : v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1$ of vertices and edges of G, such that $e_i = v_i v_{i+1}$, $i = 1, 2, \ldots, m-1$, $e_m = v_m v_1$, and $e_i \neq e_j$ if $i \neq j$. A circuit whose m vertices v_i are distinct is called a *cycle*.

We define a *dominating circuit* of a graph G to be a circuit of G with the property that every edge of G either belongs to the circuit or is adjacent to an edge of the circuit.

A *star* is a complete bipartite graph, $K_{1,n}$. The vertex of degree n is termed the *center* of the star and the vertices of degree 1 are the *leaves*. If a star has center w we often denote it as S_w. Further, if we wish to specify a star centered at w with some specific leaves, say a, b, c, we will denote it by $S_w(a, b, c)$. Note that there may be other leaves in S_w not specified.

Early studies of 2-factors centered on the question of existence, often of simply a hamiltonian cycle. More recently, the focus in the area of 2-factors has shifted from the problem of showing the existence of a 2-factor to that of showing the existence of 2-factors with specific structural features. In 1978, Sauer and Spencer made the following conjecture along those lines.

Conjecture 1.1 [8] *Let H be any graph on n vertices with maximum degree $\Delta \leq 2$. If G is a graph on n vertices with minimum degree $\delta(G) > 2n/3$ then G contains an isomorphic copy of H.*

In 1993, Aigner and Brandt settled Conjecture 1.1 with a slight improvement.

Theorem 1.4 [1] *Let G be a graph of order n with $\delta(G) \geq (2n-1)/3$. Then G contains any graph H of order at most n with $\Delta(H) \leq 2$.*

In the above result, the minimum degree must be very high to guarantee that a graph contains all possible 2-factors or 2-factors with a particular structure. Thus, a more relaxed question would be: is there a lesser degree condition that will imply the existence of 2-factors with k cycles for a range of k? The following was shown.

Theorem 1.5 [2] *Let k be a positive integer and let G be a graph of order n. If $\deg(x) + \deg(y) \geq n$ for all $x, y \in V(G)$ such that $xy \notin E(G)$, then G contains a 2-factor with k cycles for all k, $1 \leq k \leq \lfloor n/4 \rfloor$.*

Note that Theorem 1.5 is a generalization of the classic hamiltonian result of Ore [7] for the case when $n \geq 4k$. The complete bipartite graph $K_{n/2,n/2}$ shows that this result is best possible.

This type of result naturally leads to the question of whether or not other hamiltonian results can be extended in a similar manner.

The following is the well-known result of Harary and Nash-Williams [6] describing graphs with hamiltonian line graphs.

Theorem 1.6 [6] *Let G be a graph without isolated vertices. Then $L(G)$ is hamiltonian if, and only if, $G \simeq K_{1,n}$, for some $n \geq 3$, or G contains a dominating circuit.*

Given a graph G, we say that G contains a *dominating k-system* if G contains a collection of k edge disjoint circuits and stars $(K_{1,n_i}, n_i \geq 3)$ such that each edge of G is either contained in one of the circuits or stars, or is adjacent to one of the circuits.

We will use a generalization of Theorem 1.6 that allows us to describe those graphs whose line graphs contain a 2-factor with exactly k $(k \geq 1)$ cycles.

Theorem 1.7 [5] *Let G be a graph with no isolated vertices. The graph $L(G)$ contains a 2-factor with k $(k \geq 1)$ cycles if, and only if, G contains a dominating k-system.*

Recall that Catlin's result, stated in Theorem 1.1, gives specific conditions on a graph G that imply that the line graph $L(G)$ is hamiltonian. We will now show that these same conditions actually imply much more, by proving the extension stated in Theorem 1.2.

2 Extension: Proof of Theorem 1.2

Recall that the hypothesis of Theorem 1.2 gives us a 2-edge-connected simple graph of order n such that $\delta(G) \geq n/5$. From Theorem 1.1, we know that $L(G)$ is hamiltonian. Now since $\delta(G) \geq n/5$, we know that $|E(G)| \geq n^2/10$. Therefore, by proving the following theorem, we achieve the desired result.

Theorem 2.1 *Let G be a graph of order n and k be an integer $1 \leq k \leq \lfloor cn \rfloor$, for some constant $c > 0$. If $|E(G)| \geq cn^2$ and $L(G)$ is hamiltonian, then $L(G)$ contains a 2-factor with k cycles.*

Proof: Let G be a graph as in the theorem. We know by assumption that $L(G)$ is hamiltonian and so we will proceed by induction on the number of cycles in a 2-factor. Suppose that $L(G)$ has a 2-factor with $k-1$ cycles for $k \leq cn$, but $L(G)$ does not have a 2-factor with k cycles. By Theorem 1.7 we know that the graph G does have a dominating $(k-1)$-system and G does not have a dominating k-system.

Consider a dominating $(k-1)$-system in G. Every edge of G is either in a star in this system, a circuit in this system, or is dominated by a circuit in the system (called a *dominated edge*). We will let i be the number of stars in the system and thus $k-1-i$ is the number of circuits. If there is a star in the system with six or more edges, we can separate the star into two smaller stars with at least 3 edges each. This gives us a dominating k-system in G, which means that all stars in the system have at most 5 edges. If there is a circuit in the system that is not a cycle, then we can separate the circuit into 2 edge disjoint circuits. This again gives us a dominating k-system which means all of the circuits in this system must be cycles. Finally, there can be at most $n-1$ dominated edges. Consider the subgraph of G induced by these dominated edges. No vertex in this subgraph can have degree 3 or higher. Such a vertex would allow us to form another star and thus a dominating k-system. Consequently this subgraph has maximum degree 2. Now suppose all of the vertices in the subgraph have degree 2. Then we can find a cycle in the subgraph which, when added to the $(k-1)$-system, gives us a dominating k -system in G. Hence, we have a subgraph which must be a collection of disjoint paths, and thus can contribute at most $n-1$ edges to our graph G

Combining these three results, we see that $|E(G)| \leq 5i + n(k-1-i) + (n-1)$. For $n > 5$ this is maximized at $i = 0$. So, $|E(G)| \leq 5i + n(k-1-i) + (n-1) \leq nk - 1 \quad (n > 5)$. By our original assumption, $|E(G)| \geq cn^2$ which implies that $cn^2 \leq nk - 1$. On the other hand, we know that $k \leq cn$. But, $k \leq cn$ implies that $k < cn + \frac{1}{n}$ which in turn implies that $cn^2 > nk - 1$, a contradiction. Thus, it must be the case that $L(G)$ has a 2-factor with k cycles. ■

In an effort to improve the result of Theorem 1.2, we continued our study of line graphs obtained from graphs G of order n with $\delta(G) \geq n/5$. We concentrated on such graphs with large independence number. This is where we moved from an extension to a generalization. The proof of this generalization, stated in Theorem 1.3, will be the focus of the remainder of the paper.

3 Generalization: Proof of Theorem 1.3

The proof of Theorem 1.3 will follow from Theorem 1.2 and the two theorems we will prove in this section. Before proving these two theorems, we state two technical lemmas that will be useful. The proofs of the lemmas are not included as they are lengthy and the techniques are very similar to those found in the included proofs. However, proofs are available upon request.

In Lemma 1 we will consider four types of dominating $(k-2)$-systems. For example, a type 1 dominating $(k-2)$-system is one that can be formed from a dominating k-system that contains three stars which can be combined to form a cycle as pictured in Figure 1. Similarly, dominating $(k-2)$-systems of types 2, 3, and 4 are those that can be formed from dominating k-systems by combining three particular elements as pictured in Figures 2, 3, and 4. In types 1, 2, and 3, the star with center u must have exactly three or four leaves, hence the dotted line for uv.

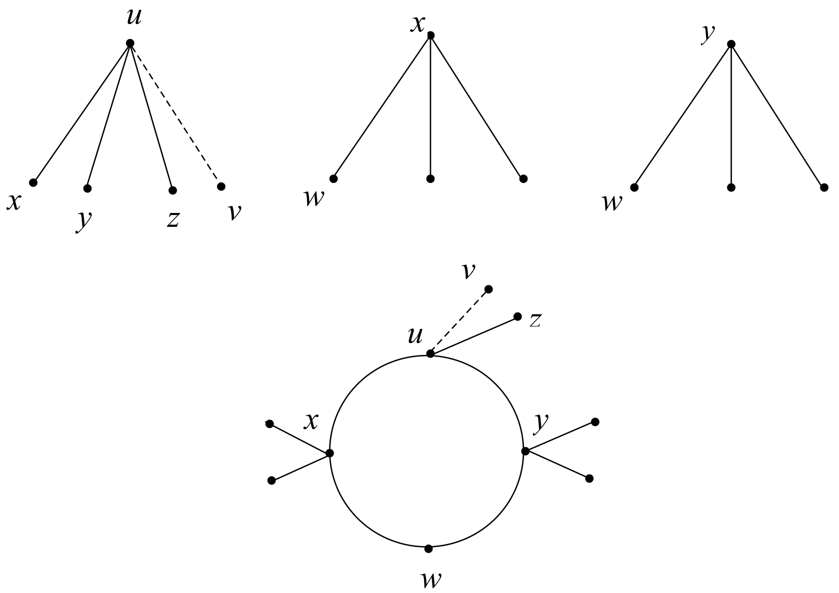

Figure 1: Lemma 1 — Type 1

Lemma 1 *Let G be a graph of order $n > 65$, such that $\delta(G) \geq n/5$. If G contains a dominating k-system and a dominating $(k-2)$-system of type $1, 2, 3$ or 4, then G contains a dominating $(k-1)$-system.*

Lemma 2 *Let G be a graph of order $n > 65$ with $\delta(G) \geq n/5$. If G contains at least one dominating k-system, but does not contain a dominating k-system consisting entirely of circuits, and G does not contain a dominating $(k-1)$-system, then any two degree one vertices x and y in the same star in a dominating k-system of G cannot have a common neighbor in G other than the center of the star.*

Recall that given a graph G, we say that G contains a *dominating k-system* if G contains a collection of k edge disjoint circuits and stars $(K_{1,n_i}, n_i \geq 3)$ such that each edge of G is either contained in one of the circuits or stars, or is adjacent to one of the circuits. An edge that is simply adjacent to one of the circuits is called a *dominated edge.*

The term *dominating system* will denote a collection of edge disjoint circuits and stars with possible dominated edges, but unknown size. The term $k-system$ will denote a collection of k edge disjoint circuits and stars with possible dominated edges, but that doesn't necessarily dominate. The term *system* will denote a collection of edge disjoint circuits and stars with possible dominated edges, that doesn't necessarily dominate and whose size is unknown.

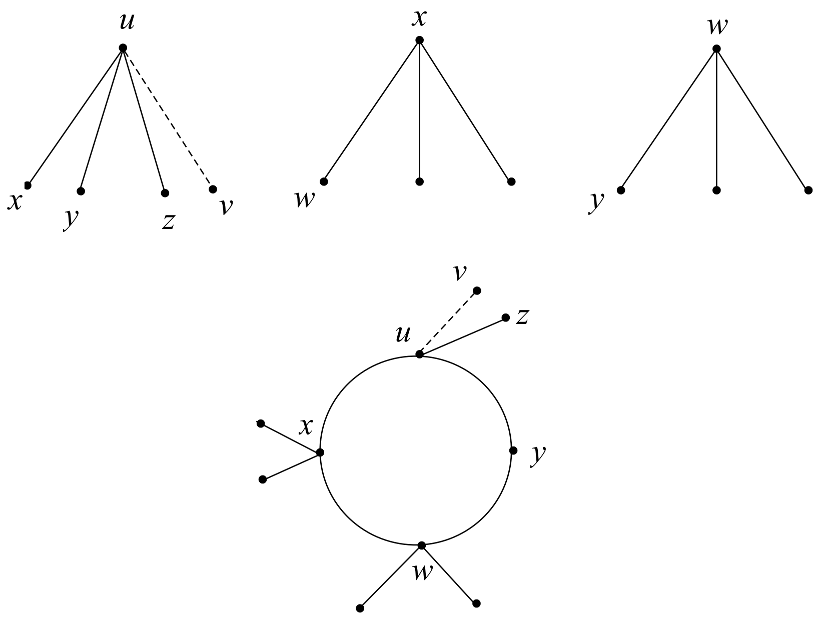

Figure 2: Lemma 1 — Type 2

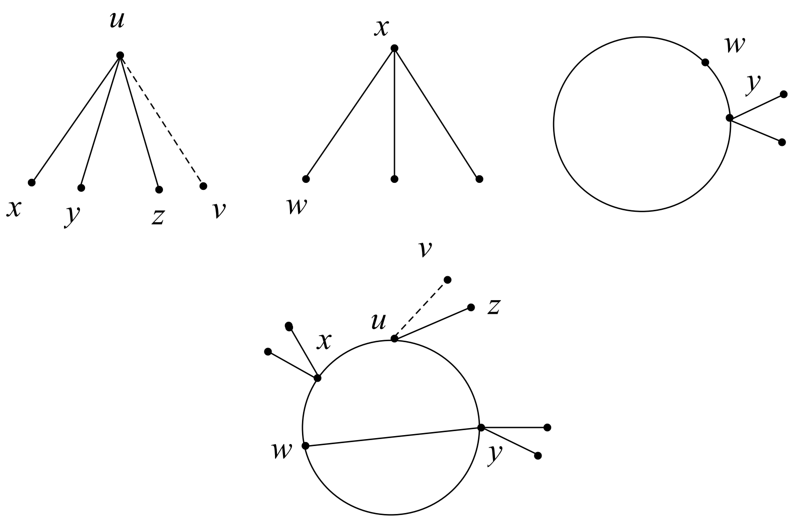

Figure 3: Lemma 1 — Type 3

Theorem 3.1 *Let G be a 2-edge-connected simple graph of order $n > 65$ such that $\delta(G) \geq n/5$ and $\alpha(G) = cn$ for some c, $1/4 \leq c < 1$. Then $L(G)$ has a 2-factor with at least $k = cn\lfloor \frac{\delta}{3} \rfloor - [(1-c)n - \delta]$ cycles.*

Proof: Since G has an independent set of vertices, say I, of size cn ($c \geq 1/4$), let $R = V(G) \setminus I$ and $x \in I$. As $\deg(x) \geq \delta(G)$, we can form at least $\lfloor \frac{\delta(G)}{3} \rfloor$ stars

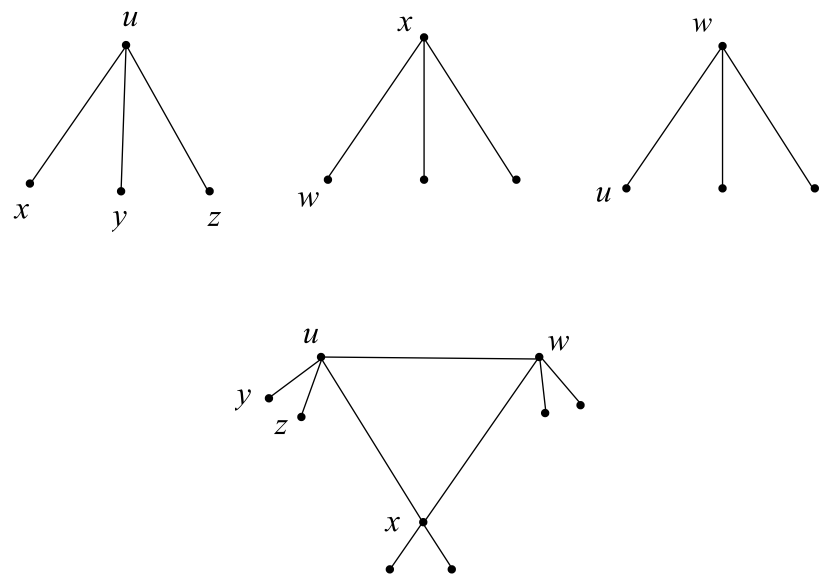

Figure 4: Lemma 1 — Type 4

centered at the vertex x, each with at least 3 leaves. Doing this for each vertex in I, we build a collection of at least $cn\lfloor\frac{\delta}{3}\rfloor$ stars that collectively use all edges between the sets R and I. Place these stars in $\mathcal{S}$, which represents a system that we will make into a dominating system. Let $|\mathcal{S}|$ represent the number of stars and circuits in $\mathcal{S}$. If there are any stars or circuits contained in $\langle R\rangle_G$, then we add them at random to $\mathcal{S}$.

At this point, we will say an edge is *part of a system* if it is in a star or circuit in the system or is a dominated edge. Removing from G the edges that are already part of the system, $\mathcal{S}$, we are left with at most some disjoint paths in $\langle R\rangle_G$, $P_1, P_2, \ldots, P_s$. We want to add all of the edges of these paths to $\mathcal{S}$ so that $\mathcal{S}$ dominates. This will be done through a finite number of steps, using an iterative process, during which we modify $\mathcal{S}$ in such a way that it remains a system but ultimately dominates. During the process we might reduce the size of $\mathcal{S}$, but by at most one in each step.

Suppose that in the original graph, one of the paths P_i in $\langle R\rangle_G$ contains two or more vertices that share a common neighbor z in I. Choose two such vertices $x_i \neq y_i$, such that any other neighbors of z on P_i lie between x_i and y_i on P_i. Form the cycle $C : x_i, z, y_i, \ldots, x_i$ where $y_i, \ldots, x_i$ denotes the portion of the path P_i between y_i and x_i. The edges x_iz and y_iz are the only edges of C that were already in $\mathcal{S}$. Because of the way we are building $\mathcal{S}$, x_iz and y_iz must either be dominated edges or edges of stars with center z in $\mathcal{S}$. If there exists any star with center z in $\mathcal{S}$ then modify $\mathcal{S}$, if necessary, to form the star $S_z(x_i, y_i)$ without decreasing $|\mathcal{S}|$. In $\mathcal{S}$, replace $S_z(x_i, y_i)$ with C which dominates the remaining edges of $S_z(x_i, y_i)$. If there is no star with center z in $\mathcal{S}$, then x_iz and y_iz are both dominated and z must be on a circuit C^* in $\mathcal{S}$. We can then attach C to C^* making a larger circuit. In either case, we have added edges from $\langle R\rangle_G$ without decreasing $|\mathcal{S}|$. Repeat this process until all that remains in $\langle R\rangle_G$ are disjoint paths which contain no two vertices that have a common neighbor in I.

Now, we must incorporate the remaining edges into $\mathcal{S}$ without affecting $|\mathcal{S}|$ too much. We will begin by showing that any time we incorporate a new edge, $|\mathcal{S}|$ decreases by at most one. By way of contradiction, assume that if we can add one of the remaining edges in $\langle R\rangle_G$ to $\mathcal{S}$, we must lose at least 2 from $|\mathcal{S}|$. Choose an edge pq in $\langle R\rangle_G$ such that pq is not part of the system yet and it is the only edge adjacent to p unused in our system. Consequently, neither p nor q can be the center of any star in $\mathcal{S}$ and neither can be found on any circuit, for otherwise we could easily add pq to $\mathcal{S}$ leaving $|\mathcal{S}|$ unchanged.

As I is an independent set of maximum size, there exists an $x_0 \in I$ such that $px_0 \in E(G)$ and a $y_0 \in I$ such that $qy_0 \in E(G)$. We know $x_0 \neq y_0$ as none of the remaining paths have 2 vertices with a common neighbor in I. Let $N(p) = \{q, x_0, x_1, \ldots, x_r\}$ and $N(q) = \{p, y_0, y_1, \ldots, y_s\}$ where $r, s \geq n/5 - 2$.

Note that we can assume that $\mathcal{S}$ is made up of stars and cycles since any circuit can be divided into edge disjoint cycles, which only increases $|\mathcal{S}|$.

Claim 3.1.1 *The set* $N(p) \cap N(q) = \emptyset$.

Suppose there exists a $z \in N(p) \cap N(q)$. As p and q have no common neighbors in I, $z \in R$. We know that pz is already in our system, which means, because of the properties of p discussed earlier, z is the center of a star or is on a circuit. Consequently, the edge qz must be in our system as well. Remember that the edges pz and qz can only be in a star with center z or dominated by a cycle containing z. If either is in a star with center z then we can clearly rearrange our system, leaving the size the same, to form $S_z(p, q)$. We can then add the edge pq to $S_z(p, q)$ which forms a cycle p, z, q, p that dominates the other edges of $S_z(p, q)$ leaving the system size unchanged, a contradiction. So it must be the case that pz and qz are both dominated by circuits. However, then we can form the cycle p, z, q, p, adding the edge pq to $\mathcal{S}$ and increasing $|\mathcal{S}|$ by 1, another contradiction. Hence, $N(p) \cap N(q) = \emptyset$.

Claim 3.1.2 *The sets* $N(p)$ *and* $N(q)$ *are independent.*

Recall that $N(p) = \{q, x_0, x_1, \ldots, x_r\}$ and that pq is the edge we are trying to add to our system. We know from Claim 3.1.1 that $qx_i \notin E(G)$ for all $i \in \{0, 1, \ldots, r\}$. Suppose, by way of contradiction, that $x_i x_j \in E(G)$ for some $i \neq j$. As in the proof of Claim 3.1.1, px_i (px_j) must appear in the system as an edge of a star with center x_i (or x_j) or as an edge dominated by a cycle containing x_i (or x_j). There are four possible cases for where the edge $x_i x_j$ could be located in relation to our system, $\mathcal{S}$. It could be an edge in a star or cycle in $\mathcal{S}$, a dominated edge in $\mathcal{S}$, or not in $\mathcal{S}$ at all. In each case, similar arguments can be used to show that the edge pq can be added with a loss of at most one from $|\mathcal{S}|$, contradicting our original assumption. Here is an example of one such argument.

Suppose that $x_i x_j$ is an edge in a star in $\mathcal{S}$. Without loss of generality let $S_{x_i}(x_j)$ be this star. Now the cycle x_i, x_j, p, x_i dominates the remaining edges of $S_{x_i}(x_j)$. If the edges px_i and px_j are both either from stars of size 4 or larger or are dominated edges, then we can add that cycle to our system without losing any edges or changing the size of the system. Then we can add the edge pq as a dominated edge without decreasing the system size, a contradiction. Thus, at least one of px_i or px_j is from a star of size 3. If only one of them is in a star of size 3, then we replace that star with our cycle, which dominates the remaining edges of that star, which decreases the system size by one without losing any edges. But, we can still add the edge pq to our system as an edge dominated by the cycle containing p, which is again a contradiction. Therefore the edges px_i and px_j must both be found in stars of size 3. If x_ix_j is in a star of size 4 or larger then we only lose two stars when forming our new cycle. So again, we are able to add the edge pq while only reducing the system size by 1. Consequently, the edge x_ix_j must also be in a star of size 3. If the edge x_ip is not in $S_{x_i}(x_j)$ then interchange edges to form $S_{x_i}(x_j, p)$ without decreasing the size of the system $\mathcal{S}$. Again we lose at most two elements of $\mathcal{S}$ when forming the new cycle containing p, which means $|\mathcal{S}|$ has decreased by at most one. However, since p is now on a cycle in $\mathcal{S}$, we can again add the edge pq to $\mathcal{S}$ giving us another contradiction

As mentioned before, a similar argument shows that if the edge x_ix_j is in a cycle of $\mathcal{S}$, is a dominated edge of $\mathcal{S}$, or is not yet in $\mathcal{S}$, the edge pq can be added to $\mathcal{S}$, without losing any edges from $\mathcal{S}$ and losing at most one from $|\mathcal{S}|$, a contradiction. Therefore, the edge x_ix_j cannot be in $E(G)$. The independence of $N(q)$ can be shown similarly.

Claim 3.1.3 *The set $N(a) \cap N(q) \setminus \{p\}$ is empty for all $a \in N(p) \cap I$ and the set $N(b) \cap N(p) \setminus \{q\}$ is empty for all $b \in N(q) \cap I$.*

Suppose there exists $z \in N(a) \cap N(q) \setminus \{p\}$. Consider the edges pa, qz, and az, which may already be part of $\mathcal{S}$. Note that $|\mathcal{S}|$ can only decrease if we take an edge from a dominating circuit or star in $\mathcal{S}$ and what remains is no longer a circuit or star.

If we can use pa, az, and zq without affecting the system size, as described above, then we form the cycle p, a, z, q, p and add it to the system, a contradiction. Therefore, at least one of these three edges was a vital part of a circuit or star in $\mathcal{S}$, and thus the removal of at least one of these three edges must decrease $|\mathcal{S}|$. First consider the case when each of the three edges has the property that its removal decreases $|\mathcal{S}|$. Note, similar arguments can be used to reach a contradiction no matter which subset of these three edges affects the system when removed.

As before, it must be the case that $S_a(p)$ and $S_z(q)$ are $K_{1,3}$'s in $\mathcal{S}$. The edge az must either be an edge of a $K_{1,3}$ or a cycle edge. Suppose, without loss of generality, that $S_a(z)$ is a $K_{1,3}$ in $\mathcal{S}$. We can then assume that $S_a(p, z) \in \mathcal{S}$. Form the cycle p, a, z, q, p which dominates the edges from $S_a(p, z)$ and $S_z(q)$. Thus we have incorporated pq into $\mathcal{S}$ decreasing $|\mathcal{S}|$ at most 1. Consequently, the edge az must be an edge of a cycle in $\mathcal{S}$. From this cycle we form the new cycle $C : a, p, q, z, \ldots, a$ where $z, \ldots, a$ represents the vertex sequence of the old cycle. Let C dominate the edges from $S_z(q)$ and use the edge az to complete the star from which ap was removed. Again we have incorporated pq decreasing $|\mathcal{S}|$ at most 1, a contradiction. The arguement that $N(b) \cap N(p) \setminus \{q\}$ is empty for all $b \in N(q) \cap I$ is similar.

Claim 3.1.4 *The set $N(a) \cap N(b)$ is empty for all $\{a, b\}$ such that $a \in N(p) \cap I$ and $b \in N(q) \cap I$.*

Suppose there does exist $z \in N(a) \cap N(b)$. We know that p and q do not share any neighbors in I so it follows that $z \neq p$ and $z \neq q$. Consider the edges ap, az, bz, and bq. We know all of these edges are in $\mathcal{S}$. Now $S_a(p)$ and $S_b(q)$ are $K_{1,3}$'s in $\mathcal{S}$. The edges az and bz are either in a $K_{1,3}$ with center other than z or are a cycle edge in $\mathcal{S}$.

Suppose both az and bz are in a $K_{1,3}$ in $\mathcal{S}$. Then, based on the way we formed $\mathcal{S}$ thus far, the centers must be a and b respectively. We can easily trade edges between stars, if necessary, so that $S_a(p, z)$ and $S_b(q, z)$ are in $\mathcal{S}$. Now form the cycle a, z, b, q, p, a. This cycle dominates the edges of $S_a(p, z)$ and $S_b(q, z)$. So we have formed a new cycle that contains the edge pq while only decreasing $|\mathcal{S}|$ by 1, a contradiction.

Now suppose that only az is in a $K_{1,3}$ and that bz is a cycle edge in our system. Again it must be the case that $S_a(z) \in \mathcal{S}$ so we can swap edges, if necessary, to ensure that $S_a(z, p) \in \mathcal{S}$. Remove bz from the cycle that contains it and add it to $S_b(q)$. Now add to the cycle the path b, q, p, a, z to replace the edge bz, thus forming a larger cycle. We can use the edge bq since we added the edge bz to the star which originally contained it. The edges pa and az are from S_a and the new cycle dominates the remaining edge of that star. So we have again decreased $|\mathcal{S}|$ by only 1, a contradiction. The case that bz is in a $K_{1,3}$ and az is a cycle edge in $\mathcal{S}$ is handled similarly.

Thus, az and bz must both be cycle edges in $\mathcal{S}$. Begin by moving az to $S_a(p)$ and bz to $S_b(q)$. Now use the edges ap and bq knowing that what remains after their removal is still a star. If az and bz were on the same cycle in $\mathcal{S}$ we replace the path b, z, a with the path b, q, p, a, which gives a new cycle containing the edge pq. We have left $|\mathcal{S}|$ unchanged. So az and bz must have been on different cycles in $\mathcal{S}$. Both of these cycles contain z so we take what is left of them after the removal of az and bz and join them at the point z. We will then add the path b, q, p, a, giving us a new cycle containing the edge pq. In this case we decreased $|\mathcal{S}|$ by 1, a contradiction.

Thus, in all possible cases we reach a contradiction, proving the claim.

Claim 3.1.5 *If there exists $w \in N(p) \setminus \{x_0\}$ such that $w \in I$, then $N(x_0) \cap N(w) = p$.*

Suppose there exists $w \in N(p) \setminus \{x_0\}$ such that $w \in I$. Also suppose there exists $z \neq p$ such that $z \in N(x_0) \cap N(w)$. We consider the edges x_0p, x_0z, wp, and wz. We know $S_{x_0}(p), S_w(p) \in \mathcal{S}$ and the edges x_0z and wz are either in a star with center other than z or are a cycle edge in $\mathcal{S}$.

Suppose both x_0z and wz are in stars. Then by the way we formed $\mathcal{S}$, the centers must be x_0 and w respectively. We can easily trade edges between stars, if necessary, to form $S_{x_0}(p,z)$ and $S_w(p,z)$. Now form the cycle x_0, z, w, p, x_0. This cycle dominates the remaining edges of the stars with centers x_0 and w which we used to form the cycle. So we have formed a new cycle that contains the vertex p while decreasing $|\mathcal{S}|$ by 1, a contradiction.

Now suppose that only x_0z is in a star and wz is a cycle edge in our system. Again it must be the case that x_0 is the center of the star containing x_0z so we can swap edges, if necessary, to form $S_{x_0}(z,p)$. Remove the edge wz from the cycle that contains it and add it to $S_w(p)$. Add to the cycle the path w, p, x_0, z to replace the edge wz, thus forming a larger cycle. We can use the edge wp since we added the edge wz to the star in which it originally occured. The new cycle dominates the remaining edge of $S_{x_0}(p,z)$. Thus, we have decreased $|\mathcal{S}|$ by 1, a contradiction. Similarly, we arrive at a contradiction if we assume instead that only wz is in a star and x_0z is a cycle edge in $\mathcal{S}$.

Thus, we have that x_0z and wz are both cycle edges in $\mathcal{S}$. Begin by moving x_0z to $S_{x_0}(p)$ and wz to $S_w(p)$. We now use the edges x_0p and wp, since what remains after their removal is still a star. If x_0z and wz were on the same cycle in our system we replace the path w, z, x_0 that we removed with the path w, p, x_0 which gives a new cycle containing the vertex p. Thus, we can incorporate pq and leave $|\mathcal{S}|$ unchanged. Thus, x_0z and wz must have been on different cycles in $\mathcal{S}$. Both of these cycles contain z so take what is left of them after the removal of x_0z and wz and join them at the point z. We then add the path w, p, x_0, giving a new cycle containing the vertex p. In this case we decreased $|\mathcal{S}|$ by 1, a contradiction.

Hence, in all of the possible cases we reach a contradiction, thus proving the claim. The next claim is proved in the same manner.

Claim 3.1.6 *If there exists $u \in N(q) \setminus \{y_0\}$ such that $u \in I$, then $N(y_0) \cap N(u) = q$.*

We now examine the four cases that arise when we consider whether or not the hypotheses of Claims 3.1.5 and 3.1.6 hold.

Case 1 *Suppose the hypotheses of both Claim* 3.1.5 *and Claim* 3.1.6 *hold.*

Then R contains $N(x_0)$, $N(w)$, $N(y_0)$, and $N(u)$. By Claim 1 we know that $u \neq w$. Claim 7 implies that $N(x_0) \cap N(w) = p$. Claim 6 implies that $N(x_0) \cap N(y_0) = \emptyset$, $N(x_0) \cap N(u) = \emptyset$, $N(w) \cap N(u) = \emptyset$, and $N(w) \cap N(y_0) = \emptyset$. Finally, Claim 8 implies that $N(y_0) \cap N(u) = q$. From these it follows that

$$|R| \geq |N(x_0) \cup N(w) \cup N(y_0) \cup N(u)| \geq \frac{4n}{5} - 2$$

which implies that $|I| \leq n/5 + 2$. However, we originally assumed that $|I| \geq n/4$ which means that $n \leq 40$, a contradiction.

Case 2 *Suppose the hypothesis of Claim* 3.1.5 *holds but the hypothesis of Claim* 3.1.6 *does not.*

Then R contains $N(x_0)$, $N(w)$, $N(y_0)$, and $N(q) \setminus \{y_0\}$. Claim 7 implies that $N(x_0) \cap N(w) = p$. Claim 6 implies that $N(x_0) \cap N(y_0) = \emptyset$, and $N(w) \cap N(y_0) = \emptyset$. Claim 4 implies that $N(x_0) \cap N(q) \setminus \{y_0\} = p$, and $N(w) \cap N(q) \setminus \{y_0\} = p$. Finally, Claim 3 implies that $N(y_0) \cap N(q) \setminus \{y_0\} = \emptyset$. From these it follows that

$$|R| \geq |N(x_0) \cup N(w) \cup N(y_0) \cup N(q) \setminus \{y_0\}| \geq \frac{4n}{5} - 3$$

which implies that $|I| \leq n/5 + 3$. However, we originally assumed that $|I| \geq n/4$ which means that $n \leq 60$, a contradiction.

Case 3 *Suppose the hypothesis of Claim* 3.1.6 *holds but the hypothesis of Claim* 3.1.5 *does not.*

Then R contains $N(x_0)$, $N(p) \setminus \{x_0\}$, $N(y_0)$, and $N(u)$. Claim 8 implies that $N(y_0) \cap N(u) = q$. Claim 6 implies that $N(x_0) \cap N(y_0) = \emptyset$, and $N(x_0) \cap N(u) = \emptyset$. Claim 5 implies that $N(y_0) \cap N(p) \setminus \{x_0\} = p$, and $N(u) \cap N(p) \setminus \{x_0\} = q$. Finally, Claim 2 implies that $N(x_0) \cap N(p) \setminus \{x_0\} = \emptyset$. From these it follows that

$$|R| \geq |N(x_0) \cup N(p) \setminus \{x_0\} \cup N(y_0) \cup N(u)| \geq \frac{4n}{5} - 3$$

which implies that $|I| \leq n/5 + 3$. However, we originally assumed that $|I| \geq n/4$ which means that $n \leq 60$, a contradiction.

Case 4 *Suppose neither the hypothesis of Claim* 3.1.5 *nor the hypothesis of Claim* 3.1.6 *holds.*

Then R contains $N(x_0)$, $N(p) \setminus \{x_0\}$, $N(y_0)$, and $N(q) \setminus \{y_0\}$. Claim 6 implies that $N(x_0) \cap N(p) \setminus \{x_0\} = \emptyset$. Claim 5 implies that $N(y_0) \cap N(p) \setminus \{x_0\} = q$. Claim 4 implies that $N(x_0) \cap N(q) \setminus \{y_0\} = p$. Claim 3 implies that $N(y_0) \cap N(q) \setminus \{y_0\} = \emptyset$. Claim 2 implies that $N(x_0) \cap N(p) \setminus \{x_0\} = \emptyset$. Finally Claim 1 implies that $N(p) \setminus \{x_0\} \cap N(q) \setminus \{y_0\} = \emptyset$. From these it follows that

$$|R| \geq |N(x_0) \cup N(p) \setminus \{x_0\} \cup N(y_0) \cup N(q) \setminus \{y_0\}| \geq \frac{4n}{5} - 2$$

which implies that $|I| \leq n/5 + 2$. However, we originally assumed that $|I| \geq n/4$ which means that $n \leq 40$, a contradiction.

In all cases we get a contradiction which means that if we can add one of the remaining edges to $\mathcal{S}$ then we can add it losing at most one from the size of our system. Of course it is important to us that we have a dominating system. In other words, we want to ensure that we can incorporate all remaining edges into our system, $\mathcal{S}$, creating a dominating system. Suppose, again by way of contradiction, that there is an edge pq that we cannot add to the system. Then each of the above claims would hold leading us to the same contradictions, showing that it is impossible to have an edge we cannot incorporate. So we know we can add each of the remaining edges to

$\mathcal{S}$, decreasing $|\mathcal{S}|$ by at most 1 each time. We now know that we have a dominating system in G, but it still remains to show that is the size we claimed it would be.

Recall that we started our process with at least $cn\lfloor\frac{\delta}{3}\rfloor$ stars. It follows then that the size of the system is at least $cn\lfloor\frac{\delta}{3}\rfloor - M$ where M is the maximum number of edges that could have been remaining in R.

Choose $x \in I$ and let $d = \deg(x) \geq \delta \geq n/5$. Let t be the number of vertices in $N(x)$ that are not on one of the remaining paths. The $(d-t)$ vertices that are on paths must be on different paths because no two vertices on our remaining paths have a common neighbor in I. Consider an edge from each of these different paths. Then we have at least $2(d-t)$ vertices and $d-t$ edges so far. We have now accounted for $t+2(d-t) = 2d-t$ vertices of R. Because $|R| \leq (1-c)n$ we know that there are at most $(1-c)n-(2d-t)$ vertices left which can each contribute at most 1 of the remaining edges. When we combine this with the $d-t$ edges from before we see that we have at most $[(1-c)n-(2d-t)]+[d-t]$ edges comprising the disjoint paths. This gives us that the number of edges remaining in R is at most $(1-c)n-\delta$. So we have a dominating system of size at least $cn\lfloor\frac{\delta}{3}\rfloor - [(1-c)n-\delta]$ in G which means we have a 2-factor of size at least $cn\lfloor\frac{\delta}{3}\rfloor - [(1-c)n-\delta]$ in $L(G)$, as claimed. ■

Theorem 3.2 *Let G be a 2-edge-connected simple graph of order $n > 65$ such that $\delta(G) \geq n/5$ and $\alpha(G) = cn$ for some c, $1/4 \leq c < 1$. Then for any $k \geq \lfloor n/10\rfloor + 2$, if $L(G)$ has a 2-factor with k cycles then $L(G)$ has a 2-factor with $k-1$ cycles.*

Proof: Suppose by way of contradiction that, for some $k \geq \lfloor n/10\rfloor + 2$, $L(G)$ has a 2-factor with k cycles but does not have a 2-factor with $k-1$ cycles. Then we know our graph G has a dominating k-system but no dominating $(k-1)$-system.

The following observations are easily seen:

- No vertex can appear on more than one circuit of the dominating k-system or we could connect the two circuits to form a dominating $(k-1)$-system.
- No vertex can be the center of more than one star of the dominating k-system.
- No vertex can be the center of a star and on a circuit of the dominating k-system.
- If there is an edge e between two leaves of a star then it must be a dominated edge in our system.
- No two stars in our system can share a pair of leaves.
- Every star must contain at least one leaf that is not the center of any star and is not on any circuit. Otherwise we could distribute all edges of the star to other elements in the system and form a dominating $(k-1)$-system.

We now consider cases based on the structure of the dominating k-system.

Case 1 *The graph G contains a dominating k-system that consists entirely of circuits.*

Among all such k-systems in G, choose one, say $\mathcal{S}_k$, that has the maximum number of distinct vertices on circuits. In $\mathcal{S}_k$ there must be a circuit with 9 or fewer distinct vertices. Otherwise, we see that $10k \leq n$ or $k \leq \frac{n}{10}$, a contradiction.

Consider a circuit C in $\mathcal{S}_k$ with the fewest number of distinct vertices. Now choose a pair of adjacent vertices x and y on C. Recall that neither x nor y can appear on another circuit and note that they can each have at most 8 neighbors in C. Thus x and y each dominate at least $\frac{n}{5} - 8$ vertices off C in $\mathcal{S}_k$.

Let $N_x = N(x) \setminus C$ and $N_y = N(y) \setminus C$. Then $|N_x| \geq \frac{n}{5} - 8$ and $|N_y| \geq \frac{n}{5} - 8$.

Claim 3.2.1 *The set $N_x \cap N_y$ is empty.*

Suppose there exist $z \in N_x \cap N_y$. Replace the edge xy of C with the path x, z, y to form a new circuit C' that now dominates the edge xy. If z appears on another circuit then we attach this circuit to C' and form a dominating $(k-1)$-system, a contradiction. If z is not on a circuit then we contradict the way we chose the original system.

Claim 3.2.2 *The sets N_x and N_y are independent.*

Suppose this is not the case for N_x. Let $u, v \in N_x, u \neq v$, such that $uv \in E(G)$. The edge uv must appear somewhere in $\mathcal{S}_k$. Note that xu and xv must be dominated edges.

(i) Suppose the edge uv is dominated. Without loss of generality assume v is on a circuit. Then form the cycle u, x, v, u and use it to connect the circuit containing x and the circuit containing v. This produces a dominating $(k-1)$-system, a contradiction.

(ii) Suppose uv is in a circuit. Then replace the edge uv with the path u, x, v to form a new circuit that dominates uv. We now have 2 circuits containing x, which means we can again form a dominating $(k-1)$-system, a contradiction.

The independence of N_y is proved similarly.

Claim 3.2.3 *The set $N_x \cup N_y$ is independent.*

Because N_x and N_y are independent, we need only to show that there is not an edge from N_x to N_y. Again, by way of contradiction, suppose there exists $u \in N_x$ and $v \in N_y$ such that $uv \in E(G)$. The edge uv must appear somewhere in our system.

(i) If the edge uv is dominated, without loss of generality assume that the vertex v is on a circuit. Then in C replace xy with the path x, u, v, y to form a new circuit that now dominates xy. Now 2 circuits contain the vertex v which allows us to form a dominating $(k-1)$-system, a contradiction.

(ii) If uv is on a circuit, then we will use xu and yv to connect the circuits containing uv and xy to form one circuit that now dominates xy and uv, giving us a dominating $(k-1)$-system, a contradiction.

From our claims we see that we have an independent set $I = N_x \cup N_y$ such that $|I| \geq 2n/5 - 16$. Let $a, b \in N_x, (a \neq b)$. Now suppose that there exists a vertex $w \in N(a) \cap N(b)(w \neq x)$.

Subcase 1 *The vertex w is on the circuit C containing x.*

Because a and b are not on C and because w is not on any other circuit except C, it must be the case that the edges aw and bw are dominated edges in our system. We use the edges aw, bw, xa, and xb to form the cycle x, a, w, b, x, which we attach to C. If either a or b are on another circuit in our system then we have 2 circuits that share a vertex, hence we can form a dominating $(k-1)$-system. Thus, neither a nor b was on any circuit in our original system which means by adding them to C we contradict our original choice of system. Consequently, the vertex w cannot be on the circuit that contains x.

Subcase 2 *The vertex w is not on the circuit containing x.*

(i) If both of the edges aw and bw are dominated edges in our system, then again we form the cycle x, a, w, b, x which we attach to the circuit containing x. All of the vertices a, b, w must be on some circuit in our system, or by adding them to the circuit containing x, we contradict our original choice of system. But then we have two circuits with a common vertex which allows us to form a dominating $(k-1)$-system, also a contradiction.

(ii) If exactly one of aw and bw is a circuit edge in our system, then assume without loss of generality that aw is dominated and that bw is in a circuit of our system. Now we replace bw with the path b, x, a, w to form a new circuit that dominates bw. Now two different circuits contain the vertex x which allows us to form a dominating $(k-1)$-system, a contradiction.

(iii) If both of aw and bw are circuit edges in our system, then they must be on the same circuit, as w cannot appear on two different circuits. If the circuit containing aw and bw can be divided in such a way that aw and bw are on different cycles, then we can remove aw and bw from the circuit and attach the path a, x, b to form a new circuit that now dominates aw and bw and contains the vertex x. We now have 2 different circuits that contain the vertex x which allows us to form a dominating $(k-1)$-system, a contradiction.

The only remaining possibility for aw and bw is that they are both edges of the same circuit in our system and that this circuit cannot be divided as above. This is the situation we get if any 2 elements of N_x have a common neighbor other than x.

Similar arguments show that if any two elements of N_y, say a and b, have a common neighbor other than y, say w, then aw and bw are both edges of the same circuit in our system, which is not the circuit containing y, and this circuit cannot be divided in such a way that the edges aw and bw are on different cycles of the circuit.

Now suppose that there exists $a \in N_x$ and $b \in N_y$ such that there exists a vertex $w \in N(a) \cap N(b)$ where $w \neq x, y$. Then, similar arguments show that the edges aw and bw are both edges of the same circuit in our system, which is not the circuit containing x and y, and this circuit cannot be divided in such a way that the edges aw and bw are on different cycles of the circuit. The only difference in the arguments in this case is that where we were attaching a new circuit either to x or y in the previous cases, we are now combining the circuit containing aw and bw with the circuit containing xy to form one large circuit that in addition to dominating aw and bw as before, now dominates xy as well. As before we always get a dominating $(k-1)$-system unless we have the situation described above in (iii).

Our conclusion is that if any two vertices of the set I have a common neighbor other than x or y we get the situation described above in (iii).

Now suppose that three vertices a, b, c of I have a common neighbor w such that $w \neq x, y$. By our previous arguments, aw and bw are both edges of the same circuit in our system, which is not the circuit containing x and y, and that this circuit cannot be factored in such a way that the edges aw and bw are on different cycles of the circuit. But cw must also be on the same cycle of the circuit as aw and bw, which is impossible. So no three vertices of I have a common neighbor other than x or y.

Recall that our set $I = N_x \cup N_y$ is such that $|I| \geq 2n/5 - 16$. Let R be the remaining vertices of $V(G)$ that are not in the set I. Consider the set $R \setminus \{x, y\}$ that has size at most $3n/5 - 14$. By our previous arguments no three vertices of I can have a common neighbor in $R \setminus \{x, y\}$. Let e be the number of edges between the sets I and $R \setminus \{x, y\}$. Since I is independent and each vertex of I is adjacent to only one of x and y, each vertex of I sends at least $n/5 - 1$ edges to the set $R \setminus \{x, y\}$. Thus, there are at least $(n/5 - 1)(2n/5 - 16)$ edges from I to $R \setminus \{x, y\}$, that is,

$$e \geq \frac{2n^2}{25} - \frac{18n}{5} + 16.$$

Now there are at most $3n/5 + 14$ vertices in $R \setminus \{x, y\}$ that must receive these edges but each vertex in $R \setminus \{x, y\}$ receives at most two edges from I. Hence, there are at most $2(\frac{3n}{5} + 14)$ edges from $R \setminus \{x, y\}$ to I. In other words, $e \leq \frac{6n}{5} + 28$.

It follows that

$$\frac{2n^2}{25} - \frac{18n}{5} + 16 \leq \frac{6n}{5} + 28,$$

hence, $n^2 - 60n - 150 \leq 0$, which implies $n < 63$, a contradiction.

Case 2 *Every dominating k-system of G contains at least one star.*

Choose a dominating k-system of G, say $\mathcal{S}_k$, in such a way that the system contains a maximum number of circuits. By our assumption, $\mathcal{S}_k$ must contain at least one star. Suppose, for a contradiction, there exists a star $S_z(x_1, x_2, x_3, x_4, x_5)$ with at least five leaves. Note that for all $i \neq j$, the edge x_ix_j cannot be in our graph. If so, it would be a dominated edge and we could use it to form a k-system with one more circuit, which contradicts the way we chose our original system. Lemma 2 implies that for every $i \neq j$, $N(x_i) \cap N(x_j) = z$. This implies that

$$n \geq |\{x_1, x_2, x_3, x_4, x_5, z\}| + |N(x_1)| + |N(x_2)| + |N(x_3)| + |N(x_4)| + |N(x_5)|$$

and thus $n \geq n + 1$, a contradiction. So it must be the case that each star in our system has 3 or 4 edges. We will show a contradiction is reached in either case. While the setup for these cases is the same, the arguments are different so we will handle them separatetly.

Suppose we do have a star in our system with four edges, say $S_z(x_1, x_2, x_3, x_4)$. Let $N = \{N(x_1) \cup N(x_2) \cup N(x_3) \cup N(x_4)\} \setminus \{z\}$. By Lemma 2 we know that for every $i \neq j$, $N(x_i) \cap N(x_j) = z$ which means that $|N| \geq 4n/5 - 4$. Let $X = \{x_1, x_2, x_3, x_4\}$. We know that $z \notin X$ and $z \notin N$. Again, by the same argument as above, $x_ix_j \notin E(G)$ which means $N \cap X = \emptyset$. This implies that we have accounted for a total of $4n/5 + 1$ vertices in G.

Now z must have at least $n/5 - 4$ neighbors in $G - X$. None of the other edges adjacent to z can be dominated edges in our system or we could add them to S_z giving us a star with 5 edges or more, which we cannot have. The other edges also cannot be in a circuit because as a center of a star, z cannot appear on a circuit. There is not another star with center z, so the remaining edges adjacent to z must be in stars with center other than z. Let $C = \{c_1, c_2, \ldots, c_l\}$ be the neighbors of z not in the set X. Note that $l \geq n/5 - 4$. We know that each of the vertices in C is the center of a star with leaf z. Also we know that $X \cap C = \emptyset$ or we would have two stars in our system that share a pair of vertices. We will now argue that the set $N \cap C$ is empty.

Suppose, by way of contradiction, that $c_ix_j \in E(G)$, for some $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, 3, 4\}$. This edge cannot appear in the star with center c_i or we have two stars that share x_j and z. It cannot be in a circuit or dominated by a circuit containing c_i as c_i cannot be on a circuit. If it is dominated by a circuit containing x_j then we can use the edge c_ix_j to combine the stars containing zx_j and c_iz into a cycle that dominates any edges in the stars not a part of the new cycle, thus forming a dominating $(k-1)$-system. So it must be the case that the edge c_ix_j is found in a star with center x_j. Because we know the star containing zx_j has 4 edges we can use the edge zx_j to combine the stars containing the edges c_ix_j and c_iz into a cycle that dominates the edges of the stars and thus forms a dominating $(k-1)$-system, again a contradiction. It follows that $c_ix_j \notin E(G)$, thus $N \cap C$ is empty.

What we have shown is that none of the vertices in the set C were included in the $4n/5 + 1$ vertices that we had accounted for before. So now, with the sets X, N, C, and the vertex z we have accounted for $n - 3$ different vertices.

Consider one last set of vertices. Recall that for all $i \in \{1, 2, \ldots, l\}$, c_i is the center of a star with z as a leaf. As z is the center of a star, in each of these stars there is a leaf other than z that is not the center of a star and is not on a circuit.

For each $i \in \{1, 2, \ldots, l\}$ let y_i be the leaf in the star with center c_i that meets these qualifications and let $Y = \{y_1, y_2, \ldots, y_l\}$, $l \geq n/5 - 4$. Hence, $|Y| \geq n/5 - 4$. By our choice of each y_i we know that $z \notin Y$. Then $Y \cap C$ is empty or we would have two different stars that contain the pair c_i and z. Also, $Y \cap X$ is empty or we would have two different stars that contain the pair x_i and z. Now we will show that the set $N \cap Y$ is also empty.

Suppose, by way of contradiction, $y_i x_j \in E(G)$, for some $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, 3, 4\}$. This edge cannot appear in a star with center y_i, in a circuit, or as an edge dominated by a circuit containing y_i by the way we chose y_i. If $y_i x_j$ is an edge dominated by a circuit containing x_j then we can use it to combine the stars containing $c_i z$ and $z x_j$ into a cycle that dominates the edges of the stars and thus gives us a dominating $(k-1)$-system. It remains then that the edge $y_i x_j$ must be found in a star with center x_j. Because we know the star containing $z x_j$ has 4 edges we can use the edge $z x_j$ to combine the stars containing the edges $y_i x_j$ and $c_i z$ into a cycle that dominates the edges of the stars and thus forms a dominating $(k-1)$-system, again a contradiction. Thus, $y_i x_j \notin E(G)$ and thus $N \cap Y$ is empty.

So, none of the vertices in the set Y were included in the $n - 3$ vertices that we had previously accounted for in the sets X, N, C, and the vertex z. We now have at least $(n - 3) + (n/5 - 4)$ different vertices that we have accounted for which implies that $n \geq 6n/5 - 7$ or $n \leq 35$ which is a contradiction. Hence, it follows that we cannot have any stars in our system that contain edges edges; thus, every star in our system contains exactly three edges.

We are now in the situation that we know our system must have at least one star and that every star must have exactly three edges. Choose any star $S_z(x_1, x_2, x_3)$ in $\mathcal{S}$. The set-up is similar to the case when we assumed our system contained a star with four edges. Let $N = \{N(x_1) \cup N(x_2) \cup N(x_3)\} \setminus \{z\}$. By Lemma 2 we know that for every $i \neq j$, $N(x_i) \cap N(x_j) = z$ which means that $|N| \geq 3n/5 - 3$. Let $X = \{x_1, x_2, x_3\}$. We know that $z \notin X$ and $z \notin N$. As in the previous case, $N \cap X = \emptyset$. This implies that we already have accounted for a total of $3n/5 + 1$ of the vertices in our graph G.

Now the vertex z must have at least $n/5 - 3$ other neighbors in G besides those in the set X. None of the edges adjacent to z can be dominated edges in our system or we could add them to the star centered at z giving us a star with 4 edges or more which we cannot have. The edges also cannot be in a circuit because as a center of a star the vertex z cannot appear on a circuit. There is not another star with center z so the remaining edges that are adjacent to z must be found in stars with center other than z. Let $C = \{c_1, c_2, \ldots, c_l\}$ be the neighbors of z not in the set X. Note that $l \geq n/5 - 3$. We know that each of the vertices in C is the center of a star with leaf z. Clearly the vertex z is not an element of C. Also we know that $X \cap C = \emptyset$ or we would have two stars in our system that shared a pair of vertices. As before, we will show that the set $N \cap C$ is empty.

Suppose, by way of contradiction, that $c_i x_j \in E(G)$, for some $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, 3\}$. This edge cannot appear in the star with center c_i or we have two stars that share x_j and z. It cannot be in a circuit or dominated by a circuit containing c_i as c_i cannot be on a circuit. If it is dominated by a circuit containing x_j then we can add the edge c_ix_j to S_{c_i} to form a star with four edges which we have shown we cannot have. So it must be the case that the edge c_ix_j is found in a star with center x_j. We can now use the three stars which contain the edges x_jc_i, c_iz, and zx_j to form a type 4 dominating $(k-2)$-system, which by Lemma 1 implies that G contains a dominating $(k-1)$-system. It follows that the edge c_ix_j cannot be in our graph G and thus the set $N \cap C$ is empty.

What we have shown is that none of the vertices in the set C were included in the $3n/5+1$ vertices that we had accounted for before. So now, with the sets X, N, C, and the vertex z we have accounted for $4n/5-3$ different vertices.

We will now consider two more sets of vertices. Recall that for all $i \in \{1, 2, \ldots, l\}$, the vertex c_i is the center of a star with vertex z as a leaf. We know that each star in our system has 3 degree one vertices which means that each of these stars centered at c_i contains 2 other vertices besides c_i and z. So for each $i \in \{1, 2, \ldots, l\}$ let v_i and y_i be the degree one vertices in the star centered at c_i other than z. And let $V = \{v_1, v_2, \ldots, v_l\}$ and $Y = \{y_1, y_2, \ldots, y_l\}$. Recall that $l \geq n/5-3$ which means that $|V| \geq n/5-3$ and $|Y| \geq n/5-3$. By our choice of each v_i and y_i we know that $z \notin V$ and $z \notin Y$. It must be the case that $V \cap X$ is empty and $Y \cap X$ is empty or we would have two different stars in our system that contain the pair of vertices x_i and z. It also must be the case that $V \cap C$ is empty and $Y \cap C$ is empty or we would have two different stars in our system that contain the pair of vertices c_i and z. What remains for us to show is that the sets $N \cap Y$, $N \cap V$, and $V \cap Y$ are empty.

Starting with $N \cap Y$, suppose, by way of contradiction, that there does exist a vertex $y_i \in N(x_j)$ for some $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, 3\}$. In other words, y_ix_j is an edge of our graph G. This edge must appear somewhere in our system and we will consider each possibility. In each case we will use the stars $S_z(x_j)$ and $S_{c_i}(z, y_i)$.

If the edge y_ix_j is dominated in our system then we will use it to combine S_z and S_{c_i} to form the cycle z, c_i, y_i, x_j, z which dominates the edges of the stars giving us a dominating $(k-1)$-system, a contradiction. If the edge y_ix_j is in a star centered at x_j then we can combine this star with S_z and S_{c_i} to form a type 2 $(k-2)$-system, which by Lemma 1 means G contains a dominating $(k-1)$-system. If the edge y_ix_j is in a star centered at y_i then we can combine this star with S_z and S_{c_i} to form a type 1 $(k-2)$-system, which by Lemma 1 means G contains a dominating $(k-1)$-system. If the edge y_ix_j is in a circuit then we can combine this circuit with S_z and S_{c_i} to form a type 3 $(k-2)$-system, which by Lemma 1 means G contains a dominating $(k-1)$-system. In each case we get a contradiction, which means that the set $N \cap Y$ must be empty. By the same argument we also get that the set $N \cap V$ is empty. If the set $V \cap Y$ is not empty that means that $y_i = v_j$ for some $i \neq j$. But that means that the vertices z and v_j appear together in two different stars which cannot happen, so the set $V \cap Y$ must also be empty.

Let us review what we have shown. We have defined five sets, X, N, C, Y and V, and we have shown that the intersection of any two of them is empty. We have also shown that the vertex z is not an element of any of the five sets. The result is that

$n \geq 1 + |X| + |N| + |C| + |Y| + |V|$ or that

$$n \geq 1 + 3 + \frac{3n}{5} - 3 + \frac{n}{5} - 3 + \frac{n}{5} - 3 + \frac{n}{5} - 3 = \frac{6n}{5} - 8.$$

But this implies that $n \leq 40$ which is a contradiction that arises from our original assumption that each star in our system has exactly three edges. This leads us to the conclusion that G cannot contain a dominating k-system that contains any stars.

We began by supposing that for some $k \geq \lfloor n/10 \rfloor + 2$, that $L(G)$ has a 2-factor with k cycles but does not have a 2-factor with $k-1$ cycles. That led us immediately to the assumption that our graph G has a dominating k-system but no dominating $(k-1)$-system. Based on this assumption we then showed that G could not contain a dominating k-system that consists entirely of circuits and that it also could not contain a dominating k-system that contains any stars. The resulting conclusion then is that G cannot contain a dominating k-system. This is obviously false which implies that our supposition that G contains a dominating k-system but no dominating $(k-1)$-system cannot hold. So for every $k \geq \lfloor n/10 \rfloor + 2$ if $L(G)$ has a 2-factor with k cycles then $L(G)$ has a 2-factor with $k-1$ cycles, thus proving the theorem. ■

4 Conclusion

Our main goal was to prove a generalization of Catlin's orginal theorem which gives conditions on a graph G that imply its line graph is Hamiltonian.

We began with a small extension.

Theorem 1.2 *If G is a 2-edge-connected simple graph of order n such that $\delta(G) \geq n/5$ then $L(G)$ has a 2-factor with k cycles for $k = 1, \ldots, \lfloor n/10 \rfloor$.*

We then set out to prove the larger generalization.

Theorem 1.3 *If G is a 2-edge-connected simple graph of order $n > 65$ with $\delta(G) \geq n/5$ and $\alpha(G) = cn$, for some c, $1/4 \leq c < 1$, then $L(G)$ contains a 2-factor with k cycles, for each $k = 1, 2, \ldots, cn\lfloor \frac{\delta}{3} \rfloor - [(1-c)n - \delta]$.*

We finish with a graph that shows our bound in Theorem 1.3 is the best possible. Consider the complete bipartite graph $G = K_{n/2,n/2}$. We know that $\alpha(G) = n/2$ and $\delta(G) = n/2$. From Theorem 1.3 we see that $L(G)$ has a 2-factor with k cycles for each $k = 1, 2, \ldots, n^2/12$. Note that this is best possible since $|E(G)| = n^2/4$ which implies that $|V(L(G))| = n^2/4$ and so it would be impossible to have any more than $n^2/12$ disjoint cycles in $L(G)$.

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(Received 10 Oct 2015; revised 26 June 2017, 5 July 2018)