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A short proof of the characterization of binary matroids with no 4-wheel minor

S. R. Kingan

Department of Mathematics, Brooklyn College
City University of New York, Brooklyn, NY 11210
U.S.A.
skingan@brooklyn.cuny.edu

Abstract

The Strong Splitter Theorem is used to give a short proof that the class of binary matroids with no 4-wheel minor consists of a few small matroids and the infinite family of binary spikes.

1 Introduction

The class of binary matroids with no minor isomorphic to $M(W_4)$ was characterized as follows by Oxley [2], Theorem 2.1:

Theorem 1.1. *Let M be a 3-connected binary matroid. Then M has no minor isomorphic to $M(W_4)$ if and only if M is isomorphic to $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,3}$, $M(W_3)$, F_7, F_7^*, or Z_r, Z_r^*, $Z_r \backslash a_r$ or $Z_r \backslash c_r$, for $r \geq 4$.*

Besides the small matroids that are trivially in the class, there is one infinite family Z_r (subsequently named the binary spike). Matrix representations for Z_r and Z_r^* are shown below, where we use the name of the matroid to also stand for the matrix representing it:

$$Z_r = \begin{array}{c} \begin{array}{ccccccccc} b_1 & \cdots & b_r & a_1 & a_2 & \cdots & a_{r-1} & a_r & c_r \end{array} \\ \left(\begin{array}{ccc|cccccc} & & & 0 & 1 & \cdots & 1 & 1 & 1 \\ & & & 1 & 0 & \cdots & 1 & 1 & 1 \\ & I_r & & 1 & 1 & \cdots & 1 & 1 & 1 \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & 1 & 1 & \cdots & 0 & 1 & 1 \\ & & & 1 & 1 & \cdots & 1 & 0 & 1 \end{array}\right) \end{array} \qquad Z_r^* = \begin{array}{c} \begin{array}{cccccccc} b_1 & \cdots & b_{r+1} & a_1 & a_2 & \cdots & a_{r-1} & a_r \end{array} \\ \left(\begin{array}{ccc|ccccc} & & & 0 & 1 & \cdots & 1 & 1 \\ & & & 1 & 0 & \cdots & 1 & 1 \\ & I_{r+1} & & 1 & 1 & \cdots & \cdots & \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & 1 & 1 & \cdots & 0 & 1 \\ & & & 1 & 1 & \cdots & 1 & 0 \\ & & & 1 & 1 & \cdots & 1 & 1 \end{array}\right) \end{array}$$

Observe that Z_r has two non-isomorphic 3-connected single-element deletions $Z_r \backslash a_r$ and $Z_r \backslash c_r$, both of which are self-dual. Moreover, $Z_r \backslash \{a_r, c_r\} = Z_{r-1}^*$, $Z_r^*/b_{r+1} = Z_r \backslash c_r$, $Z_r^*/b_r \cong Z_r \backslash a_r$, and $Z_r^*/\{b_r, b_{r+1}\} \cong Z_{r-1}$. Since $Z_r \backslash c_r / b_r \cong Z_{r-1}$ and Z_4 has no minor isomorphic to the self-dual matroid $M(W_4)$, neither does Z_r nor Z_r^*.

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The main technique used in [2] was the Splitter Theorem [4]. The main technique used here is the Strong Splitter Theorem [1].

Theorem 1.2. *Suppose N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or a whirl, then M has no larger minor isomorphic to a wheel or whirl, respectively. Let $j = r(M) - r(N)$. Then there is a sequence of 3-connected matroids $M_0, M_1, \ldots, M_t$ such that $M_0 \cong N$, $M_t = M$, M_{i-1} is a minor of M_i for $1 \leq i \leq n$, and for some $j \leq t$:*

(i) For $1 \leq i \leq j$, $r(M_i) - r(M_{i-1}) = 1$ and $|E(M_i) - E(M_{i-1})| \leq 3$; and

(ii) For $j < i \leq t$, $r(M_i) = r(M)$ and $|E(M_i) - E(M_{i-1})| = 1$.

Moreover, when $|E(M_i) - E(M_{i-1})| = 3$, for some $1 \leq i \leq j$, $E(M_i) - E(M_{i-1})$ is a triad of M_i.

Let $\mathcal{M}$ be a class of matroids closed under minors. We may focus on the 3-connected members of $\mathcal{M}$ since matroids that are not 3-connected can be pieced together from 3-connected matroids using the operations of 1-sum and 2-sum [3], 8.3.1. Let us denote a *simple* single-element extension of M by an element e as $M + e$ and a *cosimple* single-element coextension of M by an element f as $M \circ f$. Note that a simple extension of a 3-connected matroid is also 3-connected. Likewise for cosimple coextensions.

Suppose N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or a whirl, then M has no larger minor isomorphic to a wheel or whirl, respectively. The Splitter Theorem states that there is a sequence of 3-connected matroids $M_0, M_1, \ldots, M_t$ such that $M_0 \cong N$, $M_t = M$, and for $1 \leq i \leq t$ either $M_i = M_{i-1} + e$ or $M_i = M_{i-1} \circ f$ [3], Cor. 12.2.1. Thus to reach a matroid isomorphic to M, one may start with N and perform simple single-element extensions and cosimple single-element coextensions. The Splitter Theorem imposes no conditions to restrict how N can grow to (a matroid isomorphic to) M. Theorem 1.2 extends the Splitter Theorem by proving that after two simple single-element extensions a cosimple single-element coextension must be performed, and it puts additional restrictions on how the coextensions are obtained.

A 3-connected rank k matroid in $\mathcal{M}$ that has no further 3-connected extensions in $\mathcal{M}$ is called a *monarch* for $\mathcal{M}$. Note that $\mathcal{M}$ may have several monarchs of varying sizes. (The class under consideration has just one monarch and that makes things very easy.) Theorem 1.2 implies that every 3-connected rank r monarch in $\mathcal{M}$ is a simple extension of a 3-connected rank r matroid M_r, where M_r is obtained from a 3-connected rank $r-1$ matroid M_{r-1} in the following ways: $M_r = M_{r-1} \circ f$ or $M_r = M_{r-1} \circ f + e$ or $M_r = M_{r-1} \circ f + \{e_1, e_2\}$ or $M_r = M_{r-1} + e \circ f$, where f is added in series to an element in M_{r-1} or $M_r = M_{r-1} + \{e_1, e_2\} \circ f$, where $\{e_1, e_2, f\}$ is a triad. There is no reason to asssume *a priori* that M_r is unique for a specific excluded minor class. However, if M_r happens to be unique, we get a recursive way of defining it, and consequently a recursive way of defining the corresponding rank r monarch.

2 The proof

The proof of Theorem 1.1 essentially comes down to the following result [2], Theorem 2.2. The class of binary matroids with no minor isomorphic to P_9 or P_9^* is denoted as $EX[P_9, P_9^*]$. The matroids P_9 and P_9^* are shown below:

$$P_9 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad P_9^* = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 \end{array}\right]$$

Theorem 2.1. *Let M be a binary non-regular 3-connected matroid. Then M is in $EX[P_9, P_9^*]$ if and only if M is isomorphic to F_7, F_7^*, or Z_r, Z_r^*, $Z_r\backslash a_r$ or $Z_r\backslash c_r$, for $r \geq 4$.*

Proof. The proof is by induction on the rank. It is easy to check that the binary non-regular 3-connected rank 4 matroids in $EX[P_9, P_9^*]$ are $F_7^* = Z_4\backslash\{a_4, c_4\}$, $Z_4\backslash a_4$, and $Z_4\backslash c_4$, and Z_4. Assume a binary non-regular 3-connected matroid with rank at most r is in $EX[P_9, P_9^*]$ if and only if it, or its dual, is isomorphic to a member of the known classes of matroids. Thus Z_{r-3}^* has no coextensions and its simple single-element extensions $Z_{r-2}\backslash a_{r-2}$ and $Z_{r-2}\backslash c_{r-2}$ both coextend only to Z_{r-2}^* and Z_{r-2}^* extends only to Z_{r-1} in $EX[P_9, P_9^*]$ (see Figure 1).

The next two claims complete the proof.

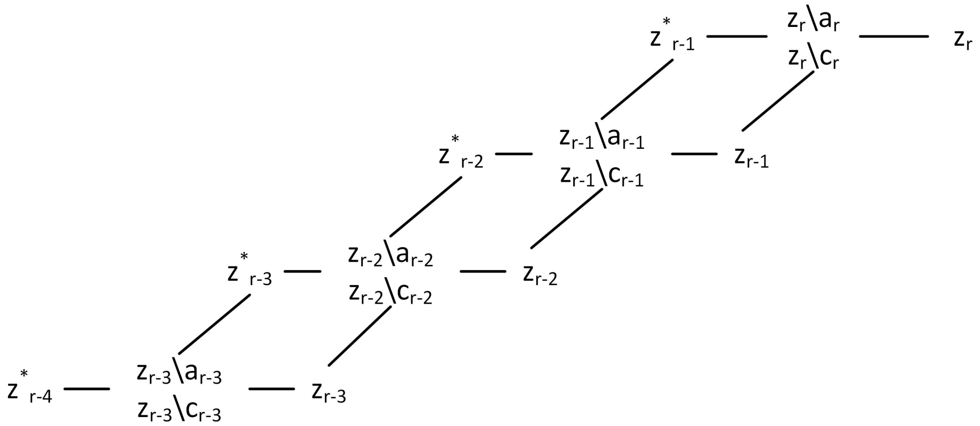

Figure 1: Growth of $EX[P_9, P_9^*]$

Claim A. *Z_{r-2}^* has no coextensions and its simple single-element extensions $Z_{r-1}\backslash a_{r-1}$ and $Z_{r-1}\backslash c_{r-1}$ both coextend only to Z_{r-1}^* in $EX[P_9, P_9^*]$.*

Proof. Suppose M is a cosimple coextension of Z_{r-2}^* in $EX[P_9, P_9^*]$. Theorem 1.2 implies that M must be a cosimple single-element coextension of Z_{r-2}^*, $Z_{r-1}\backslash c_{r-1}$, $Z_{r-1}\backslash a_{r-1}$, or Z_{r-1}. Moreover, if M is a cosimple single-element coextension of Z_{r-1}, then $\{b_r, a_r, c_r\}$ forms a triad in M. By the induction hypothesis the only rows that can be added to Z_{r-3} are $[11\ldots10]$ and $[11\ldots11]$ (see Figure 1). Adding $[11\ldots10]$ gives $Z_{r-2}\backslash c_{r-2}$ and adding $[11\ldots11]$ gives $Z_{r-2}\backslash a_{r-2}$. Adding both gives Z^*_{r-2}. Therefore Z^*_{r-2} has no further cosimple coextensions in $EX[P_9, P_9^*]$.

The only simple single-element extensions of Z^*_{r-2} in $EX[P_9, P_9^*]$ are obtained by adding columns $a_{r-1} = [11\ldots10]^T$ and $c_{r-1} = [11\ldots11]^T$ giving respectively, $Z_{r-1}\backslash c_{r-1}$ and $Z_{r-1}\backslash a_{r-1}$. However, $Z_{r-1}\backslash c_{r-1}$ and $Z_{r-1}\backslash a_{r-1}$ are also single-element coextensions of Z_{r-2} by rows $[11\ldots10]$ and $[11\ldots11]$, respectively. Adding both these rows to Z_{r-2} gives Z^*_{r-1}.

Adding to Z^*_{r-2} both columns c_{r-1} and a_{r-1} gives Z_{r-1}. The only cosimple single-element coextension of Z_{r-1} we must check is the matroid Z'_{r-1} formed by adding row $[00\ldots011]$. The matroid $Z'_{r-1}/\{b_5, b_6, \ldots b_{r-1}\}\backslash\{a_5, a_6, \ldots a_{r-1}\}$ shown below has a P_9^*-minor.

$$Z'_{r-1}/\{b_5, b_6, \ldots b_{r-1}\}\backslash\{a_5, a_6, \ldots a_{r-1}\} = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Claim B. Z^*_{r-1} extends only to Z_r in $EX[P_9, P_9^*]$.

Proof. We will prove that the only columns that can be added to Z^*_{r-1} are $c_r = [11\ldots11]^T$ and $a_r = [11\ldots10]^T$. First observe that $Z^*_{r-1}/b_r = Z_{r-1}\backslash c_{r-1}$ and $Z^*_{r-1}/b_{r-1} \cong Z_{r-1}\backslash a_{r-1}$. By the induction hypothesis applied to Z^*_{r-1}/b_r, the only columns that can be added are c_{r-1} with a zero or one in the last position, $b_1, b_2, \ldots b_{r-2}, b_{r-1}$ with a one in the last position, and $a_1, a_2, \ldots a_{r-2}, a_{r-1}$ with the entry in the last position switched. They are:

1. $c^0_{r-1} = [11\ldots10]^T$ and $c^1_{r-1} = [11\ldots11]^T$;
2. $b^1_1 = [100\ldots01]^T$, $b^1_2 = [010\ldots01]^T$ up to $b^1_{r-2} = [000\ldots0101]^T$, $b^1_{r-1} = [000\ldots011]^T$; and
3. $a^0_1 = [0111\ldots1110]^T$, $a^0_2 = [1011\ldots1110]^T$ up to $a^0_{r-2} = [111\ldots1010]^T$, $a^0_{r-1} = [111\ldots1100]^T$.

Similary, the only columns that can be added to Z^*_{r-1}/b_{r-1} are a_{r-1} with a zero or one, $b_1, b_2, \ldots b_{r-2}, b_r$ with a one in the second-last position, and $a_1, a_2, \ldots a_{r-2}, c_{r-1}$ with the entry in the second-last position switched. They are:

(4) $a^0_{r-1} = [11\ldots00]^T$ and $a^1_{r-1} = [11\ldots10]^T$;

(5) $b^1_1 = [100\ldots10]^T$, $b^1_2 = [010\ldots10]^T$ up to $b^1_{r-2} = [000\ldots0110]^T$, $b^1_r = [000\ldots011]^T$; and

(6) $a_1^0 = [0111\dots1101]^T$, $a_2^0 = [1011\dots1101]^T$ up to $a_{r-2}^0 = [111\dots1001]^T$, and $a_{r-1}^1 = [111\dots1111]^T$.

Observe that the only overlapping columns among the first set of columns in (1), (2), and (3) and in the second set of columns in (4), (5), and (6) are $[11\dots10]^T$, $[11\dots11]^T$, and $[00\dots011]$. The first is a_r and the second is c_r. They give the single-element extensions $Z_r\backslash c_r$ and $Z_r\backslash a_r$, and together the double-element extension Z_r. Lastly, let $Z_{r-1}^* + b_r^1$ be the matroid obtained by adding $b_r^1 = [00\dots11]$ to Z_{r-1}^*. Observe that

$$(Z_{r-1}^* + b_{r-1}^1)/\{b_4,\dots,b_{r-2}\}\backslash\{a_4,\dots,a_{r-2}\} = Z_5^* + b_4^1.$$

The matroid $Z_5^* + b_4^1$ shown below has a P_9^*-minor.

$$Z_5^* + b_4^1 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

This completes the proof of Theorem 1.1. □

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