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Generalizing the Pappus and Reye configurations

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Abstract

We give a group theoretic construction which yields the Veblen-Young configuration, the Pappus configuration, and the Reye configuration, among others. That description allows to find the full group of automorphisms of the configuration in question, and also to determine all polarities of the Pappus configuration. The Pappus configuration allows a natural completion which forms the affine plane of order three. We determine all embeddings of that affine plane into projective Moufang planes explicitly. Related questions are also studied for the Reye configuration.

Introduction

The Pappus configuration is (together with the Desargues configuration) one of the most important configurations in the foundations of (projective) geometry; it is used to secure that coordinates from a commutative field can be introduced for a given projective space. The Reye configuration (see [29], cf. [12, § 22] or the English translation [13, § 22]) is sometimes vaguely considered as a generalization of the Pappus configuration. We give group-theoretic descriptions of both configurations that make this vague impression more precise and also allow to determine the full groups of automorphisms for these configurations.

1 A general construction

We start with an abstract point of view, the Pappus and Reye configurations will occur as special cases in Section 3 and in Section 4 below, respectively.

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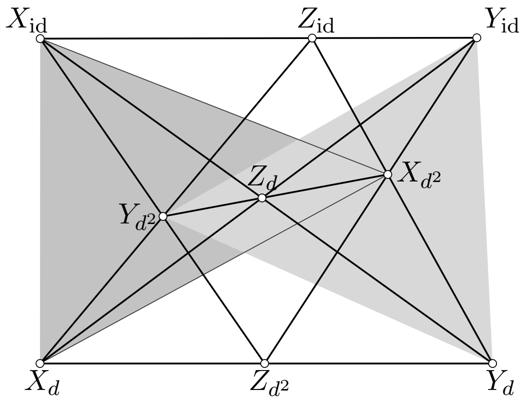

Figure 1: Pappus configuration: two triplets in perspective from three centers.

1.1 Definition. Let Δ be a group, and let Φ denote the group of automorphisms of Δ. We use multiplicative notation in Δ, and denote the neutral element of that group by 1. The application of a homomorphism φ to $a \in \Delta$ will be written as a^φ.

We form three disjoint copies of the set Δ, and write these as families $X := \{X_a \mid a \in \Delta\}$, $Y := \{Y_a \mid a \in \Delta\}$, and $Z := \{Z_a \mid a \in \Delta\}$, respectively. The incidence structure $\mathcal{R}_\Delta := (P, \mathcal{B}, \in)$ has point set $P := X \cup Y \cup Z$ and block set $\mathcal{B} := \{\{X_a, Y_b, Z_{ab}\} \mid a, b \in \Delta\}$.

The application of maps to points or blocks will be considered as action from the right, we write $U_a.\varphi$ and $B.\varphi := \{X_a.\varphi, Y_b.\varphi, Z_c.\varphi\}$ for $U \in \{X, Y, Z\}$, for $a, b, c \in \Delta$, and for $B = \{X_a, Y_b, Z_c\}$.

Note that every block $B \in \mathcal{B}$ contains exactly three points. We may regard the incidence structure as a picture of two sets (namely X and Y) that are in perspective from several centers (namely, every point in Z). This is a well-known interpretation for both the Pappus and the Reye configuration.

1.2 Definitions. **(a)** Two bijections τ_0 and τ_1 of P are defined by $X_a.\tau_0 := X_{a^{-1}}$, $Y_b.\tau_0 = Z_b$, $Z_c.\tau_0 = Y_c$, and $X_a.\tau_1 := Z_a$, $Y_b.\tau_1 = Y_{b^{-1}}$, $Z_c.\tau_1 = X_c$, respectively. For the sake of symmetry, we abbreviate $\tau_2 := \tau_0\tau_1\tau_0$; then $X_a.\tau_2 = Y_{a^{-1}}$, $Y_b.\tau_2 = X_{b^{-1}}$, and $Z_c.\tau_2 = Z_{c^{-1}}$.

(b) For each $(\varphi, k, m, r) \in \Phi \times \Delta \times \Delta \times \Delta$ the bijection ${}^k\varphi_r^m$ of P is defined by $X_a.{}^k\varphi_r^m := X_{k^{-1}a^\varphi m}$, $Y_b.{}^k\varphi_r^m = Y_{m^{-1}b^\varphi r}$, and $Z_c.{}^k\varphi_r^m = Z_{k^{-1}c^\varphi r}$.

1.3 Lemma. **(a)** *Both τ_0 and τ_1 are involutory automorphisms of $\mathcal{R}_\Delta$, and $\Sigma := \langle\tau_0, \tau_1\rangle \cong \mathrm{Sym}_3$.*

(b) *For each $(\varphi, k, m, r) \in \Phi \times \Delta \times \Delta \times \Delta$ the map ${}^k\varphi_r^m$ is an automorphism of $\mathcal{R}_\Delta$.*

(c) $\Xi := \{{}^k\varphi_r^m \mid (\varphi, k, m, r) \in \Phi \times \Delta \times \Delta \times \Delta\}$ *is a normal subgroup of* $\mathrm{Aut}(\mathcal{R}_\Delta)$*; the multiplication is given by the formula* $({}^k\varphi_r^m)({}^{k'}\varphi'^{m'}_{r'}) = {}^{k^{\varphi'}k'}(\varphi\varphi')^{m^{\varphi'}m'}_{r^{\varphi'}r'}$.

(d) *For all* $k, m, r \in \Delta$*, the inner automorphism* $\iota_m \colon a \mapsto a^m := m^{-1}am$ *of* Δ *gives* ${}^m\mathrm{id}_m^m = {}^1(\iota_m)_1^1$ *and* $({}^1\iota_{m1}^1)\,({}^k\mathrm{id}_r^1) = {}^{mk}\mathrm{id}_{mr}^m$.
Thus $\Xi = \left\{ {}^1\varphi_1^1 \,\middle|\, \varphi \in \Phi \right\} \left\{ {}^k\mathrm{id}_r^1 \,\middle|\, k, r \in \Delta \right\} \cong \Phi \ltimes (\Delta \times \Delta)$.

(e) *The action of the generators* τ_0 *and* τ_1 *of* Σ *on* Ξ *is given by* $({}^k\varphi_r^m)^{\tau_0} = {}^m\varphi_r^k$ *and* $({}^k\varphi_r^m)^{\tau_1} = {}^k\varphi_m^r$*, respectively.*

Proof: It is straightforward to verify $\mathcal{B}.\tau_0 = \mathcal{B} = \mathcal{B}.\tau_1$ and the relation $\tau_0\tau_1\tau_0 = \tau_1\tau_0\tau_1$. Clearly τ_0 and τ_1 are involutions. We conclude $\langle \tau_0, \tau_1 \rangle \cong \mathrm{Sym}_3$. The remaining assertions are checked by straightforward calculations. □

1.4 Theorem. *Let* Δ *and* Ψ *be groups.*

(a) *The partition* $P = X \cup Y \cup Z$ *of the point set is invariant under* $\mathrm{Aut}(\mathcal{R}_\Delta)$.

(b) *The subgroup* $\Sigma := \langle \tau_0, \tau_1 \rangle \cong \mathrm{Sym}_3$ *of* $\mathrm{Aut}(\mathcal{R}_\Delta)$ *permutes* $\{X, Y, Z\}$ *in the natural way.*

(c) *The group* Ξ *is the kernel of the action of* $\mathrm{Aut}(\mathcal{R}_\Delta)$ *on* $\{X, Y, Z\}$.

(d) *We have* $\mathrm{Aut}(\mathcal{R}_\Delta) = \Sigma \ltimes \Xi$.

(e) *The configurations* $\mathcal{R}_\Delta$ *and* $\mathcal{R}_\Psi$ *are isomorphic (as incidence structures) if, and only if, the groups* Δ *and* Ψ *are isomorphic.*

(f) *The group* $\mathrm{Aut}(\mathcal{R}_\Delta)$ *acts transitively on the point set* P*, and transitively on the block set* $\mathcal{B}$.

Proof: Let $U, V \in \{X, Y, Z\}$. Then a point of U and a point of V are on a common block if either they are equal or $U \neq V$. Thus the partition is invariant under $\mathrm{Aut}(\mathcal{R}_\Delta)$. The assertion about Σ is obvious.

Clearly Ξ acts trivially on $\{X, Y, Z\}$. Let ξ be an element of the kernel of the action on $\{X, Y, Z\}$. We claim that ξ belongs to Ξ. For $U \in \{X, Y, Z\}$ and $a \in \Delta$ there exists $a_U \in \Delta$ such that $U_a.\xi = U_{a_U}$. From $\{X_a, Y_b, Z_{ab}\}.\xi \in \mathcal{B}$ we infer $a_X b_Y = (ab)_Z$ for all $a, b \in \Delta$. Using ${}^k\mathrm{id}_r^1 \in \Xi$ we may assume $X_1.\xi = X_1$ and $Y_1.\xi = Y_1$; then $1_X = 1 = 1_Y = 1_Z$. We obtain $a_X = a_X 1_Y = (a\,1)_Z = a_Z = (1\,a)_Z = 1_X a_Y = a_Y$, and mapping a to a_X is an automorphism φ of Δ. Now $\xi = {}^1\varphi_1^1 \in \Xi$, and assertion (c) is proved.

As Σ acts faithfully on $\{X, Y, Z\}$ and induces the full permutation group $\mathrm{Sym}_{\{X,Y,Z\}}$, we have $\mathrm{Aut}(\mathcal{R}_\Delta) = \Sigma \ltimes \Xi$.

If $\mathcal{R}_\Delta$ and $\mathcal{R}_\Psi$ are isomorphic we start as in the proof of assertion (c) by the remark that an isomorphism ξ may be chosen such that X_1 and Y_1 in $\mathcal{R}_\Delta$ are mapped to their respective counterparts in $\mathcal{R}_\Psi$. Then ξ induces a bijection from Δ onto Ψ, and we see as above that this map is a group homomorphism.

The set $\left\{ {}^0\mathrm{id}_r^0 \;\middle|\; r \in \Delta \right\} \subseteq \mathrm{Aut}(\mathcal{R}_\Delta)$ forms a transitive subgroup on Y, and the group generated by $\left\{ {}^0\mathrm{id}_r^0 \;\middle|\; r \in \Delta \right\} \cup \Sigma$ is transitive on $X \cup Y \cup Z$. The subgroup generated by $\left\{ {}^1\mathrm{id}_r^m \;\middle|\; m, r \in \Delta \right\}$ acts transitively on the set of blocks. This completes the proof of the last assertion (f). □

1.5 Remarks. Our argument proving invariance of the partition $P = X \cup Y \cup Z$ of the point set as noted in 1.4 (a) actually is the observation that the incidence structure $\mathcal{R}_\Delta$ is a group divisible design (see [2, p. 45]; we may write "group" instead of the term "groop" introduced there because Δ is indeed a group) with parameter $\lambda = 1$, and in fact a transversal design $\mathrm{TD}[3, |\Delta|]$, i.e., the dual of a $(|\Delta|, 3)$-net (cf. [2, p. 51]). In [22] and [19], such nets are called 3-nets realizing the group Δ; this should not be confused with the notion of 3-net used in [21].

If Δ is finite then the transversal designs $\mathcal{R}_\Delta$ that we consider here are obtained from $(|\Delta|, 2)$-difference matrices, in a rather trivial special case of the construction described in [2, VIII, § 3]: labeling the columns of a $2 \times |\Delta|$ matrix D by the elements of Δ, we use entries $d_{1,j} = 1 \in \Delta$ and $d_{2,j} = j$. Then D is a $(|\Delta|, 2, 1)$-difference matrix over Δ in the sense of [2, VIII, 3.4], and leads to the transversal design T_Δ with point set $Y \cup Z$ and blocks $B_{a,b} = \{Y_b, Z_{ab}\}$; so T_Δ forms a $\mathrm{TD}[2, |\Delta|]$. (Actually, this transversal design T_Δ is a complete bipartite graph.) The substructure T_Δ of $\mathcal{R}_\Delta$ is invariant under the subgroup $\Theta := \left\{ {}^1\mathrm{id}_r^1 \;\middle|\; r \in \Delta \right\} \cong \Delta$ of $\mathrm{Aut}(\mathcal{R}_\Delta)$, and that subgroup acts regularly both on Y and on Z. The general extension procedure described in [2, VIII, 3.8] now just amounts to the (re-)construction of $\mathcal{R}_\Delta$: each Θ-invariant parallel class in T_Δ is of the form $C_a := \left\{ B_{a,b} \;\middle|\; b \in \Delta \right\}$ with fixed $a \in \Delta$, and one adds the set X of points "at infinity" in such a way that X_a becomes incident with each member $B_{a,b}$ of the class C_a. (It appears that this idea dates back to the very last remark in [14].)

1.6 Definition. Let $\mathbb{J} = (P, \mathcal{B}, \in)$ and $\mathbb{J}' = (P', \mathcal{B}', \in)$ be incidence geometries. A *lineation* from $\mathbb{J}$ to $\mathbb{J}'$ is a map $\lambda\colon P \to P'$ such that for each $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}'$ with $\left\{ p^\lambda \;\middle|\; p \in B \right\} \subseteq B'$.

In general, the block B' will not be unique (for instance, think of a constant map λ). However, if each block in $\mathbb{J}$ has at least two points and there are no digons (i.e., if two points are joined by a block in $\mathbb{J}'$ then that block is unique) in $\mathbb{J}'$ then injectivity of λ yields the existence of a unique map $\beta\colon \mathcal{B} \to \mathcal{B}'$ such that $\left\{ p^\lambda \;\middle|\; p \in B \right\} \subseteq B^\beta$ holds for each $B \in \mathcal{B}$. For each group Δ, there are no digons in $\mathcal{R}_\Delta$.

A lineation λ from $\mathbb{J}$ to $\mathbb{J}'$ is called an *embedding* if it is injective, has an injective block map β, and $p \in B \iff p^\lambda \in B^\beta$ holds for $(p, B) \in P \times \mathcal{B}$.

As $\mathrm{Aut}(\mathcal{R}_\Delta)$ acts transitively on the set $\mathcal{B}$ of blocks in $\mathcal{R}_\Delta$ (see 1.4 (f)), the following observation suffices to understand injective lineations that are not embeddings:

1.7 Theorem. *Let Δ be a group, let $\mathbb{J}' = (P', \mathcal{B}', \in)$ be an incidence geometry without digons, let λ be an injective lineation from $\mathcal{R}_\Delta = (P, \mathcal{B}, \in)$ to $\mathbb{J}'$, let $\beta\colon \mathcal{B} \to \mathcal{B}'$ denote the corresponding block map, and let $C := \{X_1, Y_1, Z_1\}$.*

Then the set $\left\{u \in \Delta \,\middle|\, X_u^\lambda \in C^\beta\right\}$ *is a subgroup of* Δ*, and* $\left\{u \in \Delta \,\middle|\, X_u^\lambda \in C^\beta\right\} = \left\{u \in \Delta \,\middle|\, Y_u^\lambda \in C^\beta\right\} = \left\{u \in \Delta \,\middle|\, Z_u^\lambda \in C^\beta\right\}$.

Proof: Consider $a \in \left\{u \in \Delta \,\middle|\, X_u^\lambda \in C^\beta\right\}$. Then $\{X_a^\lambda, Y_1^\lambda\} \subseteq C^\beta$ yields $C^\beta = (X_a \vee Y_1)^\beta \ni Z_a^\lambda$ and analogously $Y_a^\lambda \in (X_1 \vee Z_a)^\beta$. This shows $\left\{u \in \Delta \,\middle|\, Z_u^\lambda \in C^\beta\right\} \supseteq \left\{u \in \Delta \,\middle|\, X_u^\lambda \in C^\beta\right\} \subseteq \left\{u \in \Delta \,\middle|\, Y_u^\lambda \in C^\beta\right\}$. The reverse inclusions follow by analogous arguments, so the three sets coincide.

For $a, c \in \left\{u \in \Delta \,\middle|\, X_u^\lambda \in C^\beta\right\}$ we now find $X_{a^{-1}}^\lambda \in (Y_a \vee Z_1)^\beta = C^\beta$ and $Z_{ac}^\lambda \in (X_a \vee Y_c)^\beta = C^\beta$. This shows that $\left\{u \in \Delta \,\middle|\, X_u^\lambda \in C^\beta\right\}$ is a subgroup of Δ. □

1.8 Corollary. *If* Δ *is a group of prime order then every injective lineation from* $\mathcal{R}_\Delta$ *into an incidence geometry without digons has either constant or injective block map. In the latter case, the lineation is an embedding.* □

The assertion of 1.8 does not remain valid if we drop the condition that Δ has prime order. See 4.4 below for an injective lineation from $\mathcal{R}_{\mathrm{V}_4}$ to $\mathbb{P}_2(\mathbb{R})$ which is not an embedding.

1.9 Theorem. *Let* Δ *be any group, let* $\mathbb{P}$ *be a projective space, and let* $\lambda\colon \mathcal{R}_\Delta \to \mathbb{P}$ *be an injective lineation. For elements* $\alpha_1, \ldots, \alpha_n \in \Delta$ *and* $j \le n$ *consider the subgroup* $\Delta_j := \langle \alpha_1, \ldots, \alpha_j \rangle$*. Then the image* $\mathcal{R}_{\Delta_n}^\lambda$ *of* $\mathcal{R}_{\Delta_n}$ *is contained in a projective subspace of dimension at most* $n+1$*. In particular, if* Δ *is cyclic then* $\mathcal{R}_\Delta^\lambda$ *is contained in some plane of* $\mathbb{P}$.

Proof: Let S be any projective subspace of $\mathbb{P}$ with $\{X_1^\lambda, Y_1^\lambda, Z_1^\lambda\} \subseteq S$. For $U \in \{X, Y, Z\}$, consider $\Gamma_{S,U} := \left\{\gamma \in \Delta \,\middle|\, U_\gamma^\lambda \in S\right\}$. The collinear sets $\{X_\gamma^\lambda, Y_1^\lambda, Z_\gamma^\lambda\}$ and $\{X_1^\lambda, Y_\gamma^\lambda, Z_\gamma^\lambda\}$ yield $\Gamma_{S,X} = \Gamma_{S,Z} = \Gamma_{S,Y}$; we abbreviate $\Gamma_S := \Gamma_{S,U}$. Now the blocks $\{X_\gamma^\lambda, Y_{\gamma^{-1}\delta}^\lambda, Z_\delta^\lambda\}$ give $(\Gamma_S)^{-1}\Gamma_S = (\Gamma_{S,X})^{-1}\Gamma_{S,Z} \subseteq \Gamma_{S,Y} = \Gamma_S$, and we obtain that Γ_S is a subgroup of Δ.

Let $\Delta_0 = \{1\}$ be the trivial subgroup, and let S_0 be the subspace generated by $\mathcal{R}_{\Delta_0}^\lambda = \{X_1^\lambda, Y_1^\lambda, Z_1^\lambda\}$; this is a line. For $j \ge 0$, consider the subspace S_{j+1} generated by $\mathcal{R}_{\Delta_j}^\lambda \cup \{Z_{\alpha_{j+1}}^\lambda\}$. As the subgroup Γ_{S_j} contains the set $\Delta_j \cup \{\alpha_{j+1}\}$, it contains the subgroup Δ_{j+1}, and S_{j+1} contains $\mathcal{R}_{\Delta_{j+1}}^\lambda$. Clearly S_j has codimension at most one in S_{j+1}, and inductively we obtain $\dim S_n \le n+1$. □

2 The Veblen-Young configuration $\mathcal{R}_{\mathrm{C}_2}$

We use a cyclic group $\mathrm{C}_2 = \{1, t\}$ of order two for Δ; then Φ is trivial. We obtain the Veblen-Young configuration (which plays its role in the axioms for projective spaces, and—under the name "O'Nan configuration"—in the theory of hermitian unitals) as $\mathcal{R}_{\mathrm{C}_2}$, see Figure 2.

From 1.4 (d) we infer that $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_2}) = \Sigma \ltimes \Xi \cong \mathrm{Sym}_3 \ltimes \mathrm{C}_2^2 \cong \mathrm{Sym}_4$ has order $6 \cdot 2^2 = 2^3 \cdot 3 = 24$. This group acts faithfully on the set $\mathcal{B}$ of (four) blocks.

3 The Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$

We use a cyclic group $\mathrm{C}_3 = \langle d \rangle$ of order three for Δ; then $\Phi = \langle x \mapsto x^{-1} \rangle$ has order two. From 1.4 (d) we infer $\left|\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_3})\right| = 3! \cdot 2 \cdot 3^2 = 108$.

The resulting geometry $\mathcal{R}_{\mathrm{C}_3}$ is known as the *Pappus configuration.* See Figure 1 which also exhibits the two sets in perspective from three centers. We can interpret that figure as a drawing in the euclidean plane such that each one of the two sets forms a triangle, and so does the set of centers. However, it is also possible to draw the figure in such a way that the set $\{Y_1, Y_d, Y_{d^2}\}$ is collinear (see Figure 2).

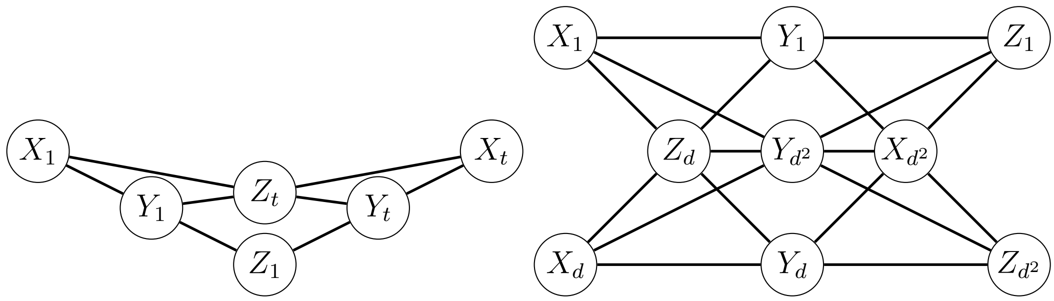

Figure 2: The Veblen-Young configuration $\mathcal{R}_{\mathrm{C}_2}$, and the Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$.

3.1 Example (The Pappus configuration in the affine plane of order three)**.**
Let us "draw" the Pappus configuration in the projective plane $\mathbb{P}_2(\mathbb{F}_3)$; i.e. consider an injective lineation from $\mathcal{R}_{\mathrm{C}_3}$ to $\mathbb{P}_2(\mathbb{F}_3)$. To each point U_a of the configuration we thus assign some point $\hat{U}_a = v\mathbb{F}_3$ with $v \in \mathbb{F}_3^3 \smallsetminus \{(0,0,0)\}$ such that the resulting map from P to the set of one-dimensional subspaces is injective, and each block is mapped into some two-dimensional subspace. From 1.8 we know that the block map β will be injective because there is no injection from nine points into a line of $\mathbb{P}_2(\mathbb{F}_3)$. Then 1.7 secures that $\hat{U}_a$ will lie on B^β only if $U_a \in B$.

The group $\mathrm{PGL}_3(\mathbb{F}_3)$ acts transitively on the set of quadrangles in $\mathbb{P}_2(\mathbb{F}_3)$. Therefore, we may assume that $\hat{Y}_d = \mathbb{F}_3(1,1,1)$, $\hat{Y}_1 = \mathbb{F}_3(1,1,0)$, $\hat{Z}_1 = \mathbb{F}_3(1,2,0)$, and $\hat{Z}_{d^2} = \mathbb{F}_3(1,2,1)$, see Figure 3. Then $\hat{X}_{d^2} = (\hat{Y}_1 \vee \hat{Z}_{d^2}) \wedge (\hat{Y}_d \vee \hat{Z}_1) = \mathbb{F}_3(1,0,2)$. From $\hat{X}_1 \in (\hat{Y}_1 \vee \hat{Z}_1)$ we infer $\hat{X}_1 \in \{\mathbb{F}_3(1,0,0), \mathbb{F}_3(0,1,0)\}$, but $\mathbb{F}_3(0,1,0)$ also lies on $(\hat{Y}_d \vee \hat{Z}_{d^2})$, leading to a contradiction. So $\hat{X}_1 = \mathbb{F}_3(1,0,0)$. Similarly, we find $\hat{X}_d = \mathbb{F}_3(1,0,1)$. Intersecting suitable lines, we now obtain $\hat{Y}_{d^2} = \mathbb{F}_3(1,1,2)$, and $\hat{Z}_d = \mathbb{F}_3(1,2,2)$. We have thus proved that there is only one way to draw the Pappus configuration in $\mathbb{P}_2(\mathbb{F}_3)$, up to the action of $\mathrm{PGL}_3(\mathbb{F}_3)$.

The points of the configuration are just those *not* on $\mathbb{F}_3(0,1,0) + \mathbb{F}_3(0,0,1)$, and the blocks are induced by the lines *not* through $\mathbb{F}_3(0,0,1)$. The stabilizer of the configuration is thus induced by the stabilizer of a flag in $\mathrm{GL}_3(\mathbb{F}_3)$, in fact, by the group $\left\{ \left(\begin{smallmatrix} r & x & z \\ 0 & s & y \\ 0 & 0 & t \end{smallmatrix} \right) \,\middle|\, r,s,t,x,y,z \in \mathbb{F}_3, rst \neq 0 \right\}$. The induced group has order $2^2 \cdot 3^3 = 3! \cdot 2 \cdot 3^2$, and coincides with $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_3})$; see 1.4 (d). Actually, the induced group is a Borel subgroup (i.e., a minimal parabolic subgroup) in $\mathrm{PGL}_3(\mathbb{F}_3)$.

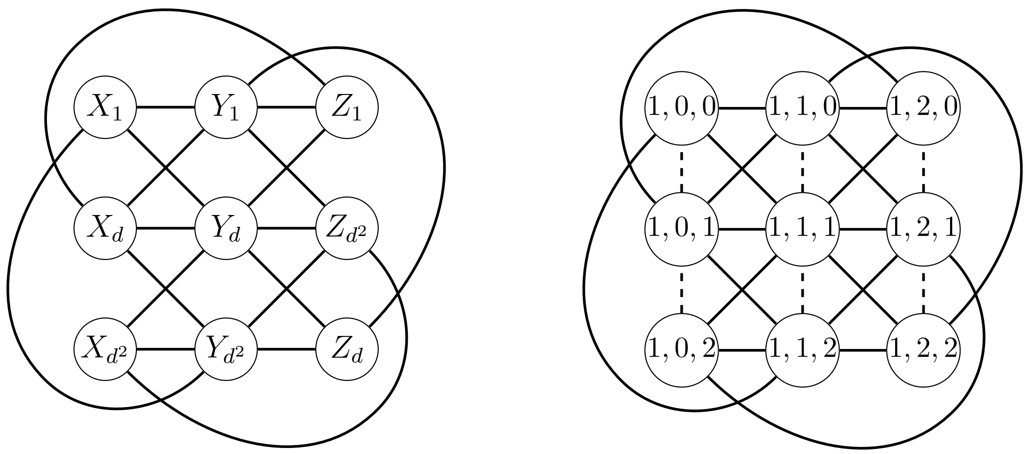

Figure 3: The Pappus configuration $\mathcal{R}_{C_3}$ (left), and a Pappus configuration drawn in $\mathbb{A}_2(\mathbb{F}_3)$ (right).

We have thus proved the following:

3.2 Theorem. $\mathrm{Aut}(\mathcal{R}_{C_3})$ *is isomorphic to the stabilizer of a flag in* $\mathrm{PGL}_3(\mathbb{F}_3)$. □

See 5.12 below for an alternative proof of 3.2, using the (unique) resolution of the transversal design $\mathcal{R}_{C_3}$.

We are going to study embeddings of $\mathcal{R}_{C_3}$ into more general projective planes (see [26] for an approach different from the present one). The class of Moufang planes is suited well for such a purpose.

3.3 Definition (Coordinates in Moufang planes)**.** Recall that a Moufang plane is a projective plane $\mathbb{P}$ where for each flag (p, L) in $\mathbb{P}$ the group of all collineations with center p and axis L acts transitively on $M \setminus \{p\}$ for any line $M \neq L$ through p. (In other words, the plane $\mathbb{P}$ is a translation plane with respect to any one of its lines.) These are the planes that can be coordinatized by alternative fields, see [25, Sect. 7] or [33, 17.2]. Each alternative field is either a (not necessarily commutative) field, or an octonion field ([25, Sect. 6] or [33, 17.3]). The distributive laws hold in an octonion field, but multiplication is not associative. However, any two elements of an alternative field lie in an associative subalgebra which also contains the inverse of each of its non-zero elements. That subalgebra is a (not necessarily commutative) field.

Using a suitable alternative field $\mathbb{K}$, we introduce inhomogeneous coordinates for $\mathbb{P}$: points are pairs $(x, y) \in \mathbb{K}^2$ and symbols (s) for points at infinity (with $s \in \mathbb{K} \cup \{\infty\}$); lines are sets $[s, t] := \big\{(x, sx + t) \,\big|\, x \in \mathbb{K}\big\} \cup \{(s)\}$ for $(s, t) \in \mathbb{K}^2$, sets $[c] := \big\{(c, y) \,\big|\, y \in \mathbb{K}\big\} \cup \{(\infty)\}$ for $c \in \mathbb{K}$, and the line $[\infty] := \big\{(s) \,\big|\, s \in \mathbb{K} \cup \{\infty\}\big\}$ at infinity.

The projective group of $\mathbb{P}$ acts transitively on quadrangles ([25, 7.3.14], see also [18, 2.7]).

3.4 Lemma. *Let $\mathbb{P}$ be a Moufang plane, coordinatized over an alternative field $\mathbb{K}$, and consider an embedding of $\mathcal{R}_{C_3}$ into $\mathbb{P}$. Then the points of the configuration are contained in a pappian subplane which is coordinatized by a commutative subfield of $\mathbb{K}$.*

Proof: For $U \in \{X, Y, Z\}$ and $c \in \langle d \rangle$, let $\hat{U}_c$ denote the point in $\mathbb{P}$ assigned to the point U_c of the configuration $\mathcal{R}_{C_3}$. The points X_1, X_d, Y_1, Y_d are mapped to the vertices of a quadrangle in $\mathbb{P}$. As the projective group of $\mathbb{P}$ acts transitively on quadrangles, we may assume $\hat{X}_1 = (0)$, $\hat{X}_d = (0, 1)$, $\hat{Y}_1 = (\infty)$, and $\hat{Y}_d = (1, 0)$. Then $\hat{Z}_d = (\hat{X}_1 \vee \hat{Y}_d) \wedge (\hat{X}_d \vee \hat{Y}_1) = [0, 0] \wedge [0] = (0, 0)$.

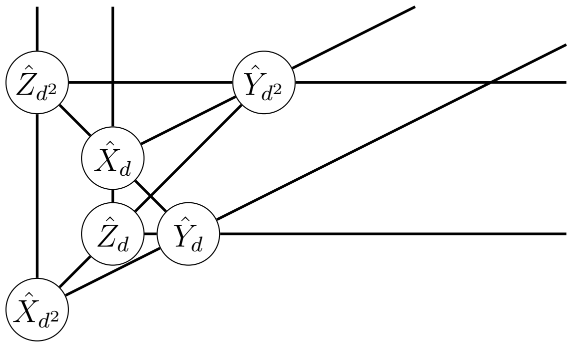

Figure 4: Embedding of $\mathcal{R}_{C_3}$ in $\mathbb{P}$.

The point $\hat{Z}_1 \in \hat{X}_1 \vee \hat{Y}_1 = [\infty]$ is of the form $\hat{Z}_1 = (u)$ for some $u \in \mathbb{K} \setminus \{0\}$. Now $\hat{X}_{d^2} \in \hat{Y}_d \vee \hat{Z}_1 = [u, -u]$ gives $\hat{X}_{d^2} = (x, ux - u)$ for some $x \in \mathbb{K}$. We find $\hat{Z}_{d^2} = (\hat{X}_{d^2} \vee \hat{Y}_1) \wedge (\hat{X}_d \vee \hat{Y}_d) = [x] \wedge [-1, 1] = (x, 1 - x)$ and $\hat{Y}_{d^2} = (\hat{X}_d \vee \hat{Z}_1) \wedge (\hat{X}_1 \vee \hat{Z}_{d^2}) = [u, 1] \wedge [0, 1 - x] = (-u^{-1}x, 1 - x)$.

Finally, the condition $\hat{Y}_{d^2} \in \hat{Z}_d \vee \hat{X}_{d^2} = [u - ux^{-1}, 0]$ yields that $(u - ux^{-1})(-u^{-1}x) = -x + ux^{-1}u^{-1}x$ equals $1 - x$. This means that u and x commute. All coordinates used are in the commutative subfield generated by u and x in $\mathbb{K}$. □

3.5 Lemma. *Assume that mapping T_c to $\hat{T}_c$ gives an embedding of $\mathcal{R}_{C_3}$ into a Moufang plane $\mathbb{P}$. Let $\{U, V, W\} = \{X, Y, Z\}$. If each one of the two sets $\{\hat{U}_1, \hat{U}_d, \hat{U}_{d^2}\}$ and $\{\hat{V}_1, \hat{V}_d, \hat{V}_{d^2}\}$ is collinear then $\{\hat{W}_1, \hat{W}_d, \hat{W}_{d^2}\}$ is collinear, as well, and the embedding of $\mathcal{R}_{C_3}$ extends to an embedding of the affine plane $\mathbb{A}_2(\mathbb{F}_3)$ of order 3 into $\mathbb{P}$.*

Proof: The configuration is contained in a pappian subplane by 3.4. So Pappus' Theorem applies to the hexagon $(\hat{U}_1, \hat{V}_1, \hat{U}_d, \hat{V}_d, \hat{U}_{d^2}, \hat{V}_{d^2})$, and gives the collinearity of $\{\hat{Z}_1, \hat{Z}_d, \hat{Z}_{d^2}\}$.

If $\{\hat{T}_1, \hat{T}_d, \hat{T}_{d^2}\}$ is collinear for each $T \in \{X, Y, Z\}$ then these three extra blocks turn the configuration into an incidence geometry isomorphic to $\mathbb{A}_2(\mathbb{F}_3)$. Thus we obtain an embedding of $\mathbb{A}_2(\mathbb{F}_3)$ into $\mathbb{P}$ from every embedding of the Pappus configuration such that (at least) two sets $\{U_1, U_d, U_{d^2}\}$ and $\{V_1, V_d, V_{d^2}\}$ become collinear in the plane. □

3.6 Remark. In 3.5 (and in 4.6 below) we consider embeddings of dual 3-nets, with a collinearity assumption on the images of at least two sets of pairwise non-connected points. For dual k-nets with $k \geq 4$, it has been shown in [20, Thm 5.2] that collinearity of a single maximal set of pairwise non-connected points already imposes serious restrictions on the embeddings in question.

3.7 Theorem. *Let $\mathbb{P}$ be a Moufang projective plane, coordinatized by an alternative field $\mathbb{K}$. There exists an embedding of $\mathbb{A}_2(\mathbb{F}_3)$ into $\mathbb{P}$ if, and only if, there exists a root u of X^2+X+1 in $\mathbb{K}$. Inhomogeneous coordinates may then be introduced in such a way that the points of $\mathbb{A}_2(\mathbb{F}_3)$ are the following:*

$$(0,0),\ (1,0),\ (0,1),\ (-u,1),\ (1,-u^2),\ (-u,-u^2),\ (0),\ (u),\ (\infty);$$

note that $u^2=-(1+u)=u^{-1}$ is also a root of X^2+X+1. Any two such embeddings are conjugates under the group of automorphisms of $\mathbb{P}$. If $\mathbb{K}$ is associative, then these embeddings are conjugates even under the projective group $\mathrm{PGL}_3(\mathbb{K})$.

Proof: The affine plane $\mathbb{A}_2(\mathbb{F}_3)$ contains a Pappus configuration. We know from the proof of 3.4 that, up to a choice of a quadrangle of reference for the coordinates, the image of that configuration under the embedding consists of the points $\hat{X}_1=(0)$, $\hat{X}_d=(0,1)$, $\hat{X}_{d^2}=(x,ux-u)$, $\hat{Y}_1=(\infty)$, $\hat{Y}_d=(1,0)$, $\hat{Y}_{d^2}=(-u^{-1}x,1-x)$, $\hat{Z}_1=(u)$, $\hat{Z}_d=(0,0)$, and $\hat{Z}_{d^2}=(x,1-x)$, where $u,x\in\mathbb{K}$ commute with each other. For the following arguments, Figure 5 may be helpful.

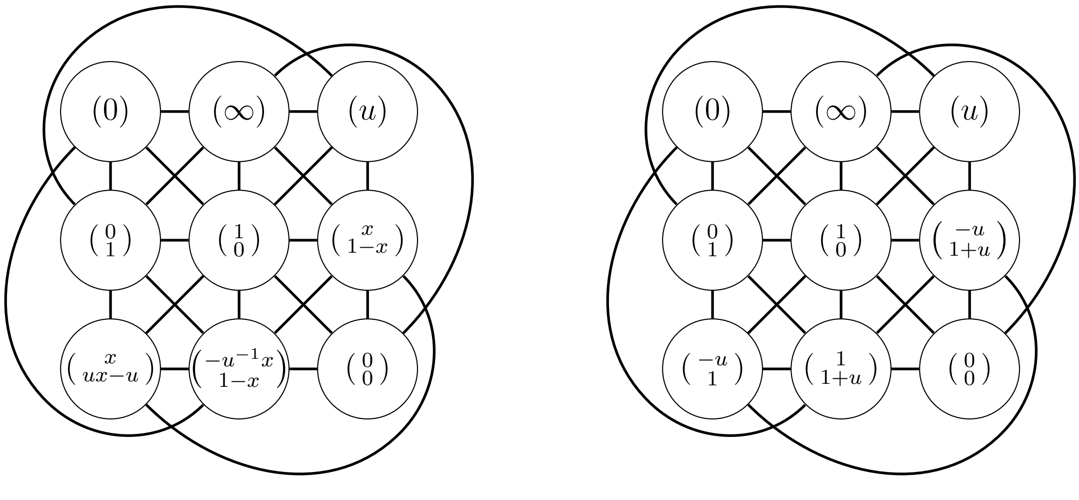

Figure 5: Embedding of $\mathbb{A}_2(\mathbb{F}_3)$ in $\mathbb{P}$ (for the sake of readability, coordinates are given as columns).

The embedding of the affine plane implies that $\{\hat{W}_1,\hat{W}_d,\hat{W}_{d^2}\}$ is collinear for each $W\in\{X,Y,Z\}$. From $(-u^{-1}x,1-x)=\hat{Y}_{d^2}\in\hat{Y}_1\vee\hat{Y}_d=[1]$ we obtain $x=-u$. Both $\hat{X}_{d^2}\in\hat{X}_1\vee\hat{X}_d$ and $\hat{Z}_{d^2}\in\hat{Z}_1\vee\hat{Z}_d$ now yield $u^2+u+1=0$.

If the center of $\mathbb{K}$ contains a root of X^2+X+1 (in particular, if $\operatorname{char}\mathbb{K}=3$ and thus $u=1$) then $\{u,u^2\}$ is the set of *all* roots of X^2+X+1 in $\mathbb{K}$. We also observe $u^2 = u^{-1}$. The affine collineation interchanging (x, y) with (y, x) extends to a projective collineation mapping $(0, 0)$, $(1, 0)$, $(0, 1)$, $(-u, 1)$, $(1, -u^2)$, $(-u, -u^2)$, (0), (u), (∞) to $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, -u)$, $(-u^2, 1)$, $(-u^2, -u)$, (∞), (u^2), (0), respectively. This amounts to interchanging the roles of u and u^2.

Now assume that $\mathbb{K}$ is associative but u is not contained in the center of $\mathbb{K}$. Then X^2+X+1 is the minimal polynomial of both u and u^2 over the center of $\mathbb{K}$, and it is known (see [7, 3.4.5]) that all roots of this polynomial are conjugates in $\mathbb{K}$. For any other root v of $X^2 + X + 1$ in $\mathbb{K}$, there exists therefore a semilinear bijection (with inner companion) of $\mathbb{K}^3$ inducing a projective collineation mapping $(0, 0)$, $(1, 0)$, $(0, 1)$, $(-u, 1)$, $(1, -u^2)$, $(-u, -u^2)$, (0), (u), (∞) to $(0, 0)$, $(1, 0)$, $(0, 1)$, $(-v, 1)$, $(1, -v^2)$, $(-v, -v^2)$, (0), (v), (∞), respectively.

It remains to handle the non-associative case. Then $\mathbb{K}$ is an octonion field. We give the argument for a slightly more general case, namely the case where $\mathbb{K}$ is a non-commutative non-split composition algebra. Then $\mathbb{K}$ is a quaternion or an octonion field. We claim that any two roots of $X^2 - X + 1$ in $\mathbb{K}$ are in the same orbit under the group of all automorphisms of $\mathbb{K}$ considered as an algebra over its center. This is a well-known fact for quaternion fields (see [5, 5.1], or use [7, 3.4.5] again as above). In an octonion field, we use Artin's result (that any two elements lie in an associative subfield, see [32, Prop. 1.5.2]), the observation that this subfield is (contained in) a quaternion subalgebra (cf. [17, 1.4]), and the fact that every inner automorphism of a quaternion subalgebra extends to an algebra automorphism of the octonion field (see [5, 5.3]). It remains to note that the stabilizer of a quadrangle in the projective group of the projective plane over $\mathbb{K}$ induces the group of all linear automorphisms of $\mathbb{K}$. □

3.8 Remarks. The existence of embeddings of $\mathbb{A}_2(\mathbb{F}_3)$ into $\mathbb{P}_2(\mathbb{C})$ is a well known fact; the nine points of $\mathbb{A}_2(\mathbb{F}_3)$ form the set of inflection points of a nonsingular cubic (see [8, Thm. 2, Thm. 3]). Actually, the group structure of the elliptic curve associated with a cubic curve over a commutative field $\mathbb{K}$ yields embeddings (of the dual) of $\mathcal{R}_\Delta$ into the plane over $\mathbb{K}$ whenever Δ is isomorphic to a subgroup of the additive or the multiplicative group of $(\mathbb{K}, +)$, see [34, Prop. 5.6]. Alternative ways to embed $\mathcal{R}_\Delta$ for such subgroups are given in 5.10 and 5.14 below.

More generally, for finite groups Δ the embeddings of the dual of $\mathcal{R}_\Delta$ (i.e., of $(3, |\Delta|)$-nets realizing the group Δ, cf. 1.5) into pappian projective planes have been studied in [22] and [19]: if $4 \le |\Delta| < \infty$ and the characteristic of the field coordinatizing that plane is 0 or greater[1] than $|\Delta|$ then Δ is cyclic, or a direct product of two cyclic groups, or dihedral, or a quaternion group of order 8, or isomorphic to one of the groups Alt_4, Sym_4, Alt_5. According to a computer-aided search reported in [22], the latter three cases do not occur in characteristic 0. Embeddings of $\mathcal{R}_\Delta$ for any dihedral group Δ have been constructed in [24, Sect. 6.2].

The construction of the embedding in 3.7 was inspired by [16, Thm. 3.4], cf. also [6] and [22]. In the latter paper, it is also noted that any set of nine points in a

[1] Some lower bound on the characteristic is needed; see 5.10 below for examples.

projective plane such that any line joining two of these points contains a third one is either contained in a line, or forms a subgeometry isomorphic to $\mathbb{A}_2(\mathbb{F}_3)$; cf. 1.8.

For desarguesian planes, our result 3.7 is known, see [23, Thm. 2] for the finite case and [30] and [4] for embeddings into possibly infinite desarguesian planes. In the latter paper, it is also proved that every embedding of a finite affine plane of order $r \geq 4$ into a (not necessarily finite) desarguesian plane extends to an embedding of the projective closure of the affine plane. In particular, such an embedding entails an embedding of coordinatizing fields. There is also a version [15] for embeddings of the configuration obtained by deleting a point and all lines through it from $\mathbb{A}_2(\mathbb{F}_3)$.

3.9 Remark. In 3.1, we have given an embedding of the Pappus configuration into the (Moufang) plane $\mathbb{P}_2(\mathbb{F}_3)$; the Pappus configuration uses all points off the line $\mathbb{F}_3(0,1,0) + \mathbb{F}_3(0,0,1)$. That embedding extends to an embedding of $\mathbb{A}_2(\mathbb{F}_3)$ into $\mathbb{P}_2(\mathbb{F}_3)$ which is the embedding in 3.7 for $\mathbb{P} = \mathbb{P}_2(\mathbb{F}_3)$. See also 5.10.

4 The Reye configuration $\mathcal{R}_{\mathrm{V}_4}$ and the configuration $\mathcal{R}_{\mathrm{C}_4}$

We use the elementary abelian group $\mathrm{V}_4 = \{1, a, b, ab\}$ of order four for Δ, then Φ is isomorphic to Sym_3. Explicitly, we may identify V_4 with the (normal) subgroup $\{\mathrm{id}, (0,1)(2,3), (0,2)(1,3), (0,3)(1,2)\}$ of Sym_4, then Φ is induced by the subgroup $\langle (1,2), (1,3) \rangle \cong \mathrm{Sym}_3$ of Sym_4. A realization of $\mathcal{R}_{\mathrm{V}_4}$ in the projective completion of euclidean three-space (or of the euclidean plane) is shown in Figure 6.

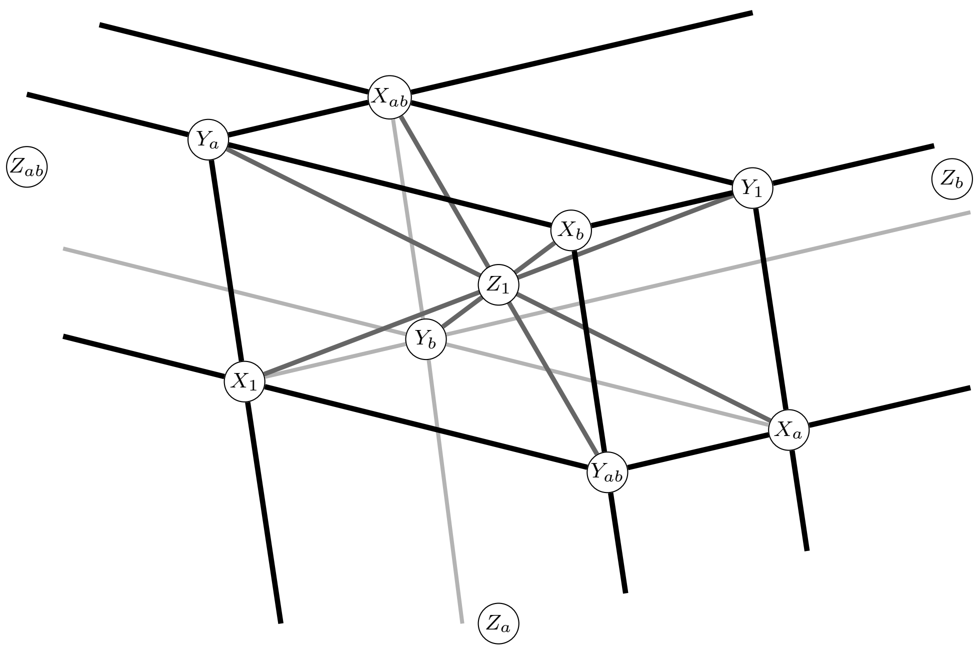

Figure 6: Spatial (or planar) embedding of Reye's configuration; Z_a, Z_b, and Z_{ab} lie at infinity.

The configuration $\mathcal{R}_{\mathrm{V}_4}$ is named to honor Theodor Reye, who studied it in [29], cf. [28]. Reye himself in [29] attributes the configuration to Poncelet's book of 1822. See [27, No. 633, p. 401 f] for a reprint of the second edition from 1865, or http://books.google.de/books?id=82ISAAAAIAAJ for the first edition.

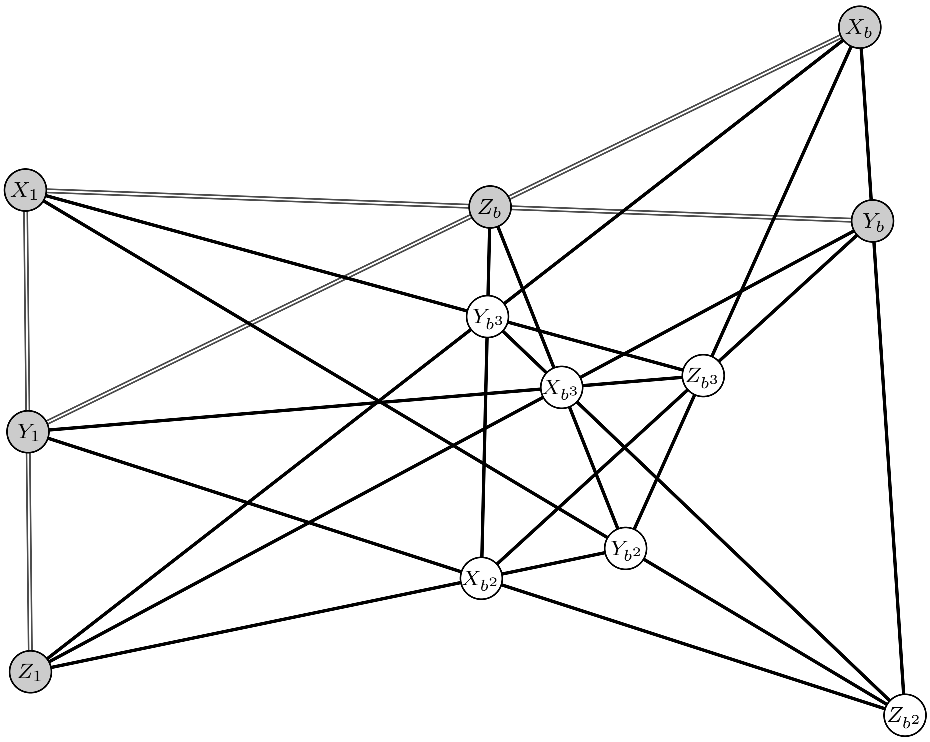

Figure 7: A planar embedding of $\mathcal{R}_{\mathrm{C}_4}$.

4.1 Remarks. We have $|\operatorname{Aut}(\mathcal{R}_{\mathrm{V}_4})| = |\operatorname{Sym}_4 \times \operatorname{Sym}_4|$ but $\operatorname{Aut}(\mathcal{R}_{\mathrm{V}_4})$ is not isomorphic to $\operatorname{Sym}_4 \times \operatorname{Sym}_4$. In order to see this, consider an automorphism φ of order 3 in Φ. Then ${}^1\varphi_1^1$ is an element of order 3 in $\operatorname{Aut}(\mathcal{R}_{\mathrm{V}_4})$; we have $U_i.{}^1\varphi_1^1 = U_{i\varphi}$ for each $U \in \{X, Y, Z\}$ and each $i \in \mathrm{V}_4$. From 1.4 (d) and 1.3 (c), (d) we infer that the centralizer of ${}^1\varphi_1^1$ in $\operatorname{Aut}(\mathcal{R}_{\mathrm{V}_4})$ is generated by $\{{}^1\varphi_1^1, \tau_0, \tau_1\}$, and has order 18.

In $\operatorname{Sym}_4 \times \operatorname{Sym}_4$, every element of order 3 is a conjugate of an element of the set $\{(\delta, 1), (\delta, \delta), (1, \delta)\}$, where $\delta = (1, 2, 3)$. The corresponding centralizers have order 72, 9, and 72, respectively.

Apart from the elementary abelian groups (isomorphic to V_4), there is one more isomorphism type of groups of order 4; the cyclic ones. We have $\operatorname{Aut}(\mathrm{C}_4) \cong \mathrm{C}_2$ and obtain $|\operatorname{Aut}(\mathcal{R}_{\mathrm{C}_4})| = 2^6 \cdot 3 = 192$ from 1.4. Figure 7 shows an embedding of $\mathcal{R}_{\mathrm{C}_4}$ in the euclidean plane, and thus in the real projective plane. From 1.4 (e) we know that the configurations $\mathcal{R}_{\mathrm{V}_4}$ and $\mathcal{R}_{\mathrm{C}_4}$ are not isomorphic.

4.2 Remark. The automorphism group of the real projective plane does not contain any subgroups isomorphic to $\mathrm{C}_4 \times \mathrm{C}_4$. Therefore, it is impossible to produce an embedding of $\mathcal{R}_{\mathrm{C}_4}$ in that plane in such a way that the subgroup $\{{}^k\mathrm{id}_r^m \mid k, r, m \in \Delta\} =$

$\left\{ {}^{s}\mathrm{id}_{t}^{1} \,\middle|\, s,t \in \Delta \right\} \cong \Delta \times \Delta = \mathrm{C}_4 \times \mathrm{C}_4$ of $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_4})$ extends to a group of automorphisms of the plane. As every embedding of $\mathcal{R}_{\mathrm{C}_4}$ in a projective space is planar (see 1.9), this observation yields that there is no embedding of $\mathcal{R}_{\mathrm{C}_4}$ in any real projective space such that $\left\{ {}^{k}\mathrm{id}_{r}^{1} \,\middle|\, k,r \in \Delta \right\} \cong \Delta \times \Delta$ of $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_4})$ extends to a group of automorphisms of that projective space.

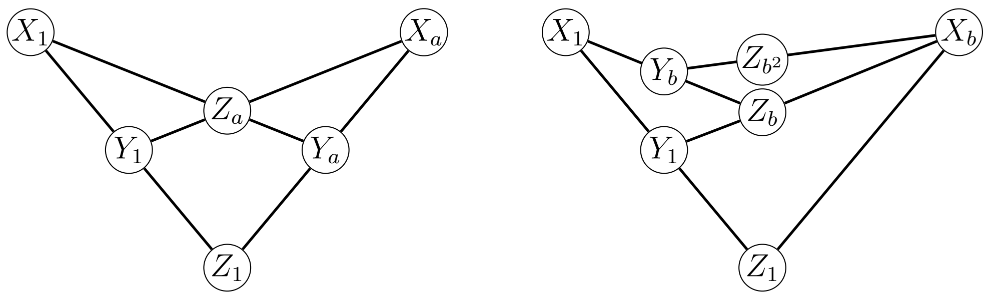

Figure 8: A Veblen-Young configuration in both $\mathcal{R}_{\mathrm{V}_4}$ and $\mathcal{R}_{\mathrm{C}_4}$ (left, with $a = (0,2)(1,3) = (0,1,2,3)^2$), and a "failed" (non-closing) configuration in $\mathcal{R}_{\mathrm{C}_4}$ (right, with $b = (0,1,2,3)$).

4.3 Remark. Note that $\mathrm{Aut}(\mathcal{R}_{\mathrm{V}_4})$ is transitive on the set $\left\{ (p,B) \in P \times \mathcal{B} \,\middle|\, p \in B \right\}$ of flags, and the stabilizer of the flag $(X_1, \{X_1, Y_1, Z_1\})$ still acts transitively on $\left\{ B \in \mathcal{B} \,\middle|\, Y_1 \notin B \ni X_1 \right\}$. Thus we can see from Figure 8 that the configurations $\mathcal{R}_{\mathrm{V}_4}$ and $\mathcal{R}_{\mathrm{C}_4}$ are not isomorphic: for each $a \in \mathrm{V}_4 \smallsetminus \{1\}$ the set $\{X_1, Y_1, Z_1, X_a, Y_a, Z_a\}$ forms a Veblen-Young configuration in $\mathcal{R}_{\mathrm{V}_4}$ but $\{X_1, Y_1, Z_1, X_i, Y_i, Z_i\}$ forms such a configuration in $\mathcal{R}_{\mathrm{C}_4}$ only if i is the involution in C_4. This alternative to an application of 1.4 (e) is still based on our knowledge of $\mathrm{Aut}(\mathcal{R}_{\Delta})$. Since $\mathcal{R}_{\mathrm{V}_4}$ has a non-planar embedding in euclidean three-space (see Figure 6), our result 1.9 gives a geometric reason.

4.4 Examples. We have promised an example of an injective lineation which is not an embedding. Here it is (actually, we get two birds with one stone): Let $\hat{X}_b$, $\hat{Y}_b$, $\hat{X}_{ab}$, $\hat{Y}_{ab}$ be the vertices of a rectangle in the euclidean plane, choose a point $\hat{Z}_b$ not lying on any line joining two of these four points, and let $\hat{Z}_{ab}$ be the image of $\hat{Z}_b$ under the half turn around the midpoint of the rectangle (see Figure 9). The lines

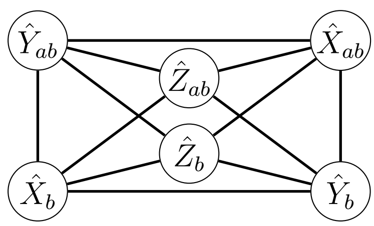

Figure 9: Constructing an injective lineation from $\mathcal{R}_{\mathrm{V}_4}$ (or $\mathcal{R}_{\mathrm{C}_4}$) into $\mathbb{P}_2(\mathbb{R})$.

$\hat{X}_b \vee \hat{Y}_b$ and $\hat{X}_{ab} \vee \hat{Y}_{ab}$ are parallel, so they meet in a point at infinity which we call $\hat{Z}_1$. Analogously, the lines $\hat{X}_{ab} \vee \hat{Y}_b$ and $\hat{X}_b \vee \hat{Y}_{ab}$ meet in a point named $\hat{Z}_{ab^2}$, the lines

$\hat{X}_b \vee \hat{Z}_b$ and $\hat{X}_{ab} \vee \hat{Z}_{ab}$ meet in a point $\hat{Y}_1$, the lines $\hat{X}_b \vee \hat{Z}_{ab}$ and $\hat{X}_{ab} \vee \hat{Z}_b$ meet in a point $\hat{Y}_a$, the lines $\hat{Y}_b \vee \hat{Z}_b$ and $\hat{Y}_{ab} \vee \hat{Z}_{ab}$ meet in a point $\hat{X}_1$, and the lines $\hat{Y}_b \vee \hat{Z}_{ab}$ and $\hat{Y}_{ab} \vee \hat{Z}_b$ meet in a point $\hat{X}_a$, respectively.

We have constructed an injective lineation from $\mathcal{R}_\Delta$ into $\mathbb{P}_2(\mathbb{R})$ *if* the group Δ is generated by commuting non-trivial elements a and $b \neq a$ such that $a^{-1} = a$ and $b^2 \in \{1, a\}$. The groups V_4 (generated by commuting involutions a, b) and C_4 (generated by b, with $a := b^2$) both satisfy these requirements. This lineation is not an embedding because $\{X_1, Y_1, Z_1\}^\beta = \{X_1, Y_a, Z_a\}^\beta = \{X_a, Y_1, Z_a\}^\beta = \{X_a, Y_a, Z_1\}^\beta$ is the line at infinity.

4.5 Theorem. *Consider any group Φ and a projective space $\mathbb{P}$. If $\mathbb{P}$ has dimension two, assume that $\mathbb{P}$ is a Moufang plane. Assume that mapping T_c to $\hat{T}_c$ gives an embedding of $\mathcal{R}_\Phi$ into $\mathbb{P}$, and that there are $U \in \{X, Y, Z\}$ and a non-cyclic subgroup $\{1, a, b, ab\}$ of order four in Φ such that the image of $\mathcal{R}_\Phi$ is not contained in the subspace H generated by $\{\hat{U}_1, \hat{U}_a, \hat{U}_b, \hat{U}_{ab}\}$. Then the projective space $\mathbb{P}$ has characteristic two (i.e., each of its elations has order two).*

Proof: Without loss of generality, we may assume $U = Z$. It suffices to consider the case where $\Phi = \{1, a, b, ab\} \cong \mathrm{V}_4$. Let H denote the subspace spanned by $\{\hat{U}_1, \hat{U}_a, \hat{U}_b, \hat{U}_{ab}\}$. Then H is a hyperplane in the subspace S generated by H and $\hat{X}_1$, and S contains the image of $\mathcal{R}_\Phi$ under the embedding. We take an affine point of view, with H at infinity. Then $\{\hat{X}_1, \hat{X}_a, \hat{X}_b, \hat{X}_{ab}\} \cup \{\hat{Y}_1, \hat{Y}_a, \hat{Y}_b, \hat{Y}_{ab}\}$ consists of the vertices of a parallelogram or a parallelepiped because $\hat{Z}_a$, $\hat{Z}_b$, and $\hat{Z}_{ab}$ lie at infinity, cf. Fig. 6. The fact that $\hat{Z}_1$ also lies at infinity means that the parallelogram with vertices $\hat{X}_1$, $\hat{X}_a$, $\hat{Y}_1$, $\hat{Y}_a$ has parallel diagonals, and $\mathbb{P}$ has characteristic two. □

4.6 Theorem. *Assume that mapping T_c to $\hat{T}_c$ gives an embedding of $\mathcal{R}_{\mathrm{V}_4}$ into a Moufang plane $\mathbb{P}$. Let $\{U, V, W\} = \{X, Y, Z\}$ and $\mathrm{V}_4 = \{1, a, b, ab\}$. If each one of the two sets $\{\hat{U}_1, \hat{U}_a, \hat{U}_b, \hat{U}_{ab}\}$ and $\{\hat{V}_1, \hat{V}_a, \hat{V}_b, \hat{V}_{ab}\}$ is collinear then $\mathbb{P}$ is coordinatized by an alternative field $\mathbb{K}$ of characteristic two, and there are $t \in \mathbb{K} \setminus \{0\}$ and $e \in \mathbb{K} \setminus \{0\}$, $f \in \mathbb{K} \setminus \{0, e\}$ such that with a suitable choice of inhomogeneous coordinates we have*

$$\begin{array}{llllllllllll} \hat{X}_1 & = & (e,0), & \hat{X}_a & = & (f,0), & \hat{X}_b & = & (e+f,0), & \hat{X}_{ab} & = & (0,0), \\ \hat{Y}_1 & = & (e,t), & \hat{Y}_a & = & (f,t), & \hat{Y}_b & = & (e+f,t), & \hat{Y}_{ab} & = & (0,t), \\ \hat{Z}_1 & = & (\infty), & \hat{Z}_a & = & ((e+f)^{-1}), & \hat{Z}_b & = & (f^{-1}), & \hat{Z}_{ab} & = & (e^{-1}). \end{array}$$

In particular, the set $\{\hat{W}_1, \hat{W}_a, \hat{W}_b, \hat{W}_{ab}\}$ is collinear, as well.

If $\mathbb{P}$ is desarguesian, we may assume $t = 1 = e$.

Proof: Without loss of generality, we assume $U = X$ and $V = Y$. Transitivity properties of the little projective group of $\mathbb{P}$ allow to assume $\{X_1, X_a, X_b, X_{ab}\} \subseteq \mathbb{K} \times \{0\}$ and $\{Y_1, Y_a, Y_b, Y_{ab}\} \subseteq \mathbb{K} \times \{t\}$, with suitable $t \in \mathbb{K} \setminus \{0\}$. We then have $x_j, y_j \in \mathbb{K}$ such that

$$\begin{array}{llllllllllll} \hat{X}_1 & = & (x_1,0), & \hat{X}_a & = & (x_a,0), & \hat{X}_b & = & (x_b,0), & \hat{X}_{ab} & = & (x_{ab},0), \\ \hat{Y}_1 & = & (x_1,t), & \hat{Y}_a & = & (x_a,t), & \hat{Y}_b & = & (x_b,t), & \hat{Y}_{ab} & = & (x_{ab},t). \end{array}$$

Note that this implies $\hat{Z}_1 = (\infty)$, and that the point $\hat{Z}_a$ is the intersection of the lines $[t(x_a - x_1)^{-1}, -(t(x_a - x_1)^{-1})x_1]$ and $[t(x_1 - x_a)^{-1}, -(t(x_1 - x_a)^{-1}x_a]$.

From 4.5 we know char $\mathbb{K} = 2$, so $\hat{Z}_a = ((x_1 + x_a)^{-1})$ lies on the line $[\infty]$. Applying an elation with that axis and center $\hat{Z}_1$ we may assume $x_{ab} = 0$, and then compute $\hat{Z}_a = (\hat{X}_b \vee \hat{Y}_{ab}) \wedge (\hat{X}_{ab} \vee \hat{Y}_b) = (t(x_b)^{-1})$. This gives $x_b = x_1 + x_a$. Further computations yield $\hat{Z}_b = (t(x_a)^{-1})$, and $\hat{Z}_{ab} = (t(x_1)^{-1})$. Every choice of $e := x_1 \in \mathbb{K} \setminus \{0\}$ and $f := x_a \in \mathbb{K} \setminus \{0, e\}$ gives an embedding. If $\mathbb{P}$ is desarguesian, we apply the collineation $(x, y) \mapsto (e^{-1}x, t^{-1}y)$ to achieve $t = 1 = e$. □

4.7 Remark. The embedding indicated by Figure 6 shows that the collinearity assumptions in 4.6 cannot be relaxed.

5 Some parallelisms, and some embeddings

Considering $\mathcal{R}_{\mathrm{V}_4}$ and $\mathcal{R}_{\mathrm{C}_4}$ as transversal designs (cf. 1.5) sheds more light on these remarkable configurations. Clearly these two are the only possible class regular transversal designs TD[3, 4]. Extending the difference matrix for $\mathcal{R}_{\mathrm{V}_4}$ by two more rows leads to embeddings of $\mathcal{R}_{\mathrm{V}_4}$ into the dual of the affine plane $\mathbb{A}_2(\mathbb{F}_4)$, which is a TD[5, 4]. We have given such an embedding in 4.6; see also 5.10 below. From 5.2 below, we infer that there is no transversal design TD[4, 4] extending $\mathcal{R}_{\mathrm{C}_4}$.

5.1 Definition. A *parallel class* of an incidence structure $\mathbb{J} = (P, \mathcal{B}, \in)$ is a set $\mathcal{E} \subseteq \mathcal{B}$ of blocks such that each point in P belongs to exactly one member of $\mathcal{E}$ (i.e., a partition $\mathcal{E}$ of P into blocks). If there exists a partition of $\mathcal{B}$ such that each member of that partition is a parallel class then that partition is called a *parallelism* or a *resolution* of $\mathbb{J}$, and $\mathbb{J}$ is called *resolvable* if such a resolution exists.

5.2 Proposition. *The configurations $\mathcal{R}_{\mathrm{C}_2}$ and $\mathcal{R}_{\mathrm{C}_4}$ do not have any parallel classes.*

PROOF: This is obvious for $\mathcal{R}_{\mathrm{C}_2}$, so consider $\mathrm{C}_4 = \{1, b, b^2, b^3\}$. If there exists any parallel class, then there exist one containing the block $\{X_1, Y_1, Z_1\}$ because $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_4})$ acts transitively on the set of blocks (cf. 1.4 (f)). For the following argument, see also Figure 7. If the blocks $\{X_1, Y_1, Z_1\}$, $\{X_b, Y_u, Z_{bu}\}$ and $\{X_{b^2}, Y_v, Z_{b^2v}\}$ are in a parallel class then $u \notin \{1, b^3\}$ and $v \notin \{1, u, b^2, b^3u\}$. This implies that $(u, v) = (b^j, b^3)$ holds for some $j \in \{1, 2\}$. In any case, the remaining set $\{X_{b^3}, Y_{b^{3-j}}, Z_{b^{-j}}\}$ does not form a block, and we do not have a parallel class. □

5.3 Theorem. *If Δ admits an automorphism α such that $x \mapsto x^{-1}x^\alpha$ is a bijection (in particular, if Δ is a finite group admitting a fixed-point-free automorphism) then $\mathcal{R}_\Delta$ is resolvable.*

PROOF: Assume that $\alpha \in \mathrm{Aut}(\Delta)$ has the required property. The blocks $B_m := \{X_1, Y_1, Z_1\}.{}^1\mathrm{id}^m_{m^\alpha} = \{X_m, Y_{m^{-1}m^\alpha}, Z_{m^\alpha}\}$ with $m \in \Delta$ form a parallel class $\mathcal{E}$ because both α and the map $m \mapsto m^{-1}m^\alpha$ are bijections of Δ onto itself. Applying the subgroup $\left\{{}^1\mathrm{id}^1_r \;\middle|\; r \in \Delta\right\}$ of $\mathrm{Aut}(\mathcal{R}_\Delta)$ we obtain that $\left\{\mathcal{E}.{}^1\mathrm{id}^1_r \;\middle|\; r \in \Delta\right\}$ is a resolution of $\mathcal{R}_\Delta$; in fact, an arbitrary block $\{X_a, Y_b, Z_{ab}\}$ is obtained as $B_m.{}^1\mathrm{id}^1_r$ with the unique pair $(m, r) = (a, (a^\alpha)^{-1}ab)$. □

5.4 Remarks. The set $\Gamma_\alpha := \left\{{}^1\mathrm{id}^m_{m^\alpha} \;\middle|\; m \in \Delta\right\}$ forms a subgroup of $\mathrm{Aut}(\mathcal{R}_\Delta)$, and that subgroup Γ_α acts regularly on X, on Y, and on Z, respectively. An application of the construction described in [2, VIII, § 3] now yields an extension of $\mathcal{R}_\Delta$ to a transversal design; that extension coincides with the one obtained from the resolution in 5.3.

We note that Γ_α stabilizes the parallel class $\mathcal{E}$, and acts transitively on $\mathcal{E}$. By construction, the group $\left\{{}^1\mathrm{id}^m_r \;\middle|\; m, r \in \Delta\right\}$ acts transitively on the set $\left\{\mathcal{E}.{}^1\mathrm{id}^1_r \;\middle|\; r \in \Delta\right\}$ of parallel classes.

A finite abelian group admits a fixed-point-free automorphism precisely if its Sylow 2-subgroup admits such an automorphism. A finite abelian 2-group is of the form $\prod_{j=1}^{e} \mathrm{C}_{2^j}^{d_j}$ with non-negative integers d_j, and admits a fixed-point-free automorphism precisely if $d_j \neq 1$ for each j (cf. [10, Thm. 4.2]). See [9] for a classification of certain infinite (namely, direct products of cyclic) abelian groups admitting a fixed-point-free automorphism of prime order.

If Δ is abelian then the next result is a special case of 5.3 because squaring is an automorphism in the abelian case.

5.5 Theorem. *If the squaring map $a \mapsto a^2$ is a bijection of Δ then $\mathcal{R}_\Delta$ is resolvable.*

Proof: The blocks $B_a := \{X_a, Y_a, Z_{a^2}\}$ with $a \in \Delta$ form a parallel class $\mathcal{E}$. As in the proof of 5.3 we apply the elements of $\left\{{}^1\mathrm{id}^1_r \;\middle|\; r \in \Delta\right\}$ and obtain the parallel classes $\mathcal{E}.{}^1\mathrm{id}^1_r = \left\{\{X_a, Y_{ar}, Z_{a^2r}\} \;\middle|\; a \in \Delta\right\}$ for $r \in \Delta$. An arbitrary block $\{X_a, Y_b, Z_{ab}\}$ lies in precisely one of these classes; namely the one with $r = a^{-1}b$. Thus we see that $\left\{\mathcal{E}.{}^1\mathrm{id}^1_r \;\middle|\; r \in \Delta\right\}$ is a resolution of $\mathcal{R}_\Delta$. □

5.6 Corollary. *The configuration $\mathcal{R}_\Delta$ is resolvable whenever Δ is a finite group of odd order.* □

5.7 Remark. In the proofs of 5.3 and of 5.5, the subgroup $\left\{{}^1\mathrm{id}^1_r \;\middle|\; r \in \Delta\right\}$ has been chosen somewhat arbitrarily in $\mathrm{Aut}(\mathcal{R}_\Delta)$. E.g., we could have used the group $\left\{{}^k\mathrm{id}^1_1 \;\middle|\; k \in \Delta\right\}$ instead. As the two subgroups are conjugates under $\tau_2 \in \mathrm{Aut}(\mathcal{R}_\Delta)$, this will also result in a resolution of $\mathcal{R}_\Delta$. That resolution coincides with the one constructed in 5.3 or in 5.5, respectively, precisely if Δ is commutative.

The next result is also a special case of 5.3 (using the fixed-point-free automorphism $x \mapsto xu$ with $u \in \mathbb{K} \smallsetminus \{0, -1\}$). However, we shall use the explicit description of the resolution in 5.10 below.

5.8 Theorem. *If Δ is isomorphic to the additive group of a field with more than two elements (in particular, if Δ is a finite elementary abelian group with $|\Delta| > 2$) then $\mathcal{R}_\Delta$ is resolvable.*

PROOF: We identify Δ with the additive group of the field $\mathbb{K}$ in question. In particular, we use additive notation in this proof.

For each $u \in \mathbb{K} \smallsetminus \{0,-1\}$, and any $s,t \in \mathbb{K}$ with $s \neq t$, the blocks $B^u_s := \{X_s, Y_{us}, Z_{(1+u)s}\}$ and $B^u_t := \{X_t, Y_{ut}, Z_{(1+u)t}\}$ have no point in common. So $\mathcal{E}^u := \{B^u_s \mid s \in \mathbb{K}\}$ forms a parallel class in $\mathcal{R}_{(\mathbb{K},+)}$.

Now fix $u \in \mathbb{K} \smallsetminus \{0,-1\}$, and consider $v \in \mathbb{K}$. The automorphism ${}^0\mathrm{id}^0_v$ fixes each point X_a, maps Y_b to Y_{b+v}, and maps Z_c to Z_{c+v}. So the orbit $\mathcal{E}^u.{}^0\mathrm{id}^0_v = \big\{\{X_s, Y_{us+v}, Z_{(1+u)s+v}\} \mid s \in \mathbb{K}\big\}$ is another parallel class in $\mathcal{R}_{(\mathbb{K},+)}$. An arbitrary block $\{X_a, Y_b, Z_{a+b}\}$ lies in precisely one of these classes; namely the one with $v = b - ua$. So the orbit of $\mathcal{E}^u$ under the subgroup $\{{}^0\mathrm{id}^0_v \mid v \in \mathbb{K}\}$ of $\mathrm{Aut}(\mathcal{R}_{(\mathbb{K},+)})$ forms a resolution of $\mathcal{R}_{(\mathbb{K},+)}$. □

5.9 Corollary. *The Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$ and the Reye configuration $\mathcal{R}_{\mathrm{V}_4}$ are resolvable.* □

5.10 Examples. Let $\mathbb{K}$ be any (not necessarily commutative) field, and consider the group $(\mathbb{K},+)$, written additively. Using homogeneous coordinates, we embed $\mathcal{R}_{(\mathbb{K},+)}$ into the projective plane $\mathbb{P}_2(\mathbb{K})$ over $\mathbb{K}$, as follows.

Points are one-dimensional subspaces of the left vector space $\mathbb{K}^3$ of *rows*, and lines are kernels of linear forms, given as columns $(x,y,z)^\intercal$ obtained by transposition of non-zero elements in $\mathbb{K}^3$. For $a,b,c \in \mathbb{K}$, put $\hat{X}_a := \mathbb{K}(1,a,0)$, $\hat{Y}_b := \mathbb{K}(1,-b,1)$, and $Z_c := \mathbb{K}(0,-c,1)$. In fact, for $B = \{X_a, Y_b, Z_{a+b}\} \in \mathcal{B}$, the line $\hat{B} := \ker(-a,1,a+b)^\intercal$ of the plane contains $\{\hat{X}_a, \hat{Y}_b, \hat{Z}_{a+b}\}$. This embedding shows that the lower bound on the characteristic in the (non-)embeddability results of [19] (see 3.8) cannot be dispensed with completely. Note that this embedding of $\mathcal{R}_{(\mathbb{K},+)}$ entails an embedding of $\mathcal{R}_\Delta$ for each subgroup $\Delta \leq (\mathbb{K},+)$ because $\mathcal{R}_\Delta$ is a substructure of $\mathcal{R}_{(\mathbb{K},+)}$.

Each one of the sets $\{\hat{X}_a \mid a \in \mathbb{K}\}$, $\{\hat{Y}_b \mid b \in \mathbb{K}\}$, and $\{\hat{Z}_c \mid c \in \mathbb{K}\}$ is collinear; they are contained in the lines $\hat{X} := \ker(0,0,1)^\intercal$, $\hat{Y} := \ker(-1,0,1)^\intercal$, and $\hat{Z} := \ker(1,0,0)^\intercal$, respectively[2]. Note also that each member of $\{{}^k\mathrm{id}^m_r \mid k,m,r \in \mathbb{K}\} \leq \mathrm{Aut}(\mathcal{R}_{(\mathbb{K},+)})$ extends to an automorphism of the projective plane over $\mathbb{K}$. The elements of $\{{}^0\mathrm{id}^0_v \mid v \in \mathbb{K}\}$ are restrictions of elations with axis $\ker(0,0,1)^\intercal$, those of $\{{}^v\mathrm{id}^0_0 \mid v \in \mathbb{K}\}$ are restrictions of elations with axis $\ker(-1,0,1)^\intercal$, and those of $\{{}^0\mathrm{id}^v_0 \mid v \in \mathbb{K}\}$ are restrictions of elations with axis $\ker(1,0,0)^\intercal$; the center is $\mathbb{K}(0,1,0)$, in any case.

If $|\mathbb{K}| = 2$ then the image of $\mathcal{R}_{(\mathbb{K},+)}$ has all points of the plane except for $\mathbb{K}(0,1,0)$, and the lines $\hat{B}$ with $B \in \mathcal{B}$ are just those lines of the plane that do not pass through $\mathbb{K}(0,1,0)$; the remaining lines are $\hat{X}$, $\hat{Y}$, and $\hat{Z}$.

Now assume $|\mathbb{K}| > 2$, and choose $u \in \mathbb{K} \smallsetminus \{0,-1\}$. We use the parallel class $\mathcal{E}^u := \{B^u_s \mid s \in \mathbb{K}\}$ introduced in the proof of 5.8. For each $v \in \mathbb{K}$, the line set $\{\hat{B} \mid B \in \mathcal{E}^u.{}^0\mathrm{id}^0_v\}$ is confluent; each member of that set passes through the point

[2] The union $\hat{X} \cup \hat{Y} \cup \hat{Z}$ is a degenerate cubic. As such, it does not carry a natural group structure (as opposed to the elliptic curve obtained in the non-degenerate case).

$\mathbb{K}(1+u,-v,1)$. So the parallel classes obtained in 5.8 are induced by "ideal" points on the line $\ker(-1,0,1+u)^\intercal$; each point on that line is used as an ideal point, except for the point $\mathbb{K}(0,1,0)$. Different choices of $u \in \mathbb{K} \smallsetminus \{0,-1\}$ lead to different sets of points to be added. The point $\mathbb{K}(0,1,0)$ is the only point that does not occur either as $\hat{U}_a$ with $U \in \{X,Y,Z\}$ and $a \in \mathbb{K}$ or as an ideal point for some $u \in \mathbb{K} \smallsetminus \{0,-1\}$; and every line of the projective plane apart from those through $\mathbb{K}(0,1,0)$ is of the form $\hat{B}$ with $B \in \mathcal{B}$. In other words, we reconstruct the dual affine plane over $\mathbb{K}$ from $\mathcal{R}_{(\mathbb{K},+)}$ and the parallelism considered here. In general, however, only a part of $\mathrm{Aut}(\mathcal{R}_{(\mathbb{K},+)})$ can be seen in this embedding (cf. 5.11 below); e.g., we see only the subgroup of $\Phi = \mathrm{Aut}(\mathbb{K},+)$ that is generated by field automorphisms and multiplications by field elements.

If $\mathrm{char}\,\mathbb{K} = 3$, we use $u = 1$ and obtain the embeddings of the Pappus configuration $\mathcal{R}_{(\mathbb{F}_3,+)}$ and the affine plane of order 3 that have been discussed in 3.7, up to projective equivalence.

If $\mathbb{K}$ is a field of characteristic 3 and we embed the Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$ into $\mathbb{P}_2(\mathbb{K})$ as in 5.10, every automorphism of $\mathcal{R}_{(\mathbb{F}_3,+)}$ extends to a collineation of $\mathbb{P}_2(\mathbb{K})$ (cf. 3.2). We put this observation into a general context:

5.11 Theorem. *Let $\mathbb{K}$ be a commutative field of finite degree over its prime field. Assume that $\mathcal{R}_{(\mathbb{K},+)}$ is embedded into $\mathbb{P}_2(\mathbb{K})$, as in 5.10. Then the stabilizer of the point set $\hat{X} \cup \hat{Y} \cup \hat{Z}$ in the group of all collineations of $\mathbb{P}_2(\mathbb{K})$ induces the full group $\mathrm{Aut}(\mathcal{R}_{(\mathbb{K},+)})$ if, and only if, either $\mathbb{K}$ is a prime field or $|\mathbb{K}| = 4$.*

Proof: We use $\mathrm{Aut}(\mathcal{R}_\Delta) \cong \mathrm{Sym}_3 \ltimes (\mathrm{Aut}(\Delta) \ltimes \Delta^2)$, see 1.4 (d) together with 1.3 (d). If Δ is the additive group of a field $\mathbb{K}$ of finite degree $d := \dim_{\mathbb{F}} \mathbb{K}$ over its prime field $\mathbb{F} \in \{\mathbb{Q}\} \cup \{\mathbb{F}_p \mid p \text{ prime}\}$ then $\mathrm{Aut}(\Delta) \cong \mathrm{GL}_d(\mathbb{F})$. That group is solvable exactly if either $d = 1$ (and $\mathbb{K} = \mathbb{F}$) or $|\mathbb{K}| \in \{4,9\}$; in fact, $\mathrm{Aut}(\mathbb{F}_p,+) = \mathbb{F}_p^\times \cong \mathrm{C}_{p-1}$ and $\mathrm{Aut}(\mathbb{Q},+) = \mathbb{Q}^\times$ (cf. [31, 31.10]) are abelian, while $\mathrm{Aut}(\mathbb{F}_4,+) \cong \mathrm{GL}_2(\mathbb{F}_2)$ and $\mathrm{Aut}(\mathbb{F}_9,+) \cong \mathrm{GL}_2(\mathbb{F}_3)$ are solvable, but $\mathrm{GL}_d(\mathbb{F})$ has the simple subquotient $\mathrm{PSL}_d(\mathbb{F})$ in all other cases.

The stabilizer Ψ of $\hat{X} \cup \hat{Y} \cup \hat{Z}$ in the group of all collineations of $\mathbb{P}_2(\mathbb{K})$ induces the full symmetric group on this set of three lines through the point $\mathbb{K}(0,1,0)$, and the kernel of the action on this set induces a group isomorphic to the automorphism group $\mathrm{Aut}(\mathbb{K})$ of the field $\mathbb{K}$ on the pencil of lines through $\mathbb{K}(0,1,0)$. The kernel of the action on that pencil is a semidirect product $\mathbb{K}^\times \ltimes \mathbb{K}^2$.

Each simple subquotient of Ψ is, therefore, isomorphic to a subquotient of $\mathrm{Aut}(\mathbb{K})$. As we assume the degree d to be finite, we find that every simple subquotient of Ψ is a subquotient of Sym_d, and thus finite. If $\mathbb{F} = \mathbb{Q}$ and $d > 1$ then $\mathrm{Aut}(\mathbb{K},+)$ has an infinite simple subquotient (namely, $\mathrm{PSL}_d(\mathbb{Q})$). If $\mathbb{F}$ is finite then $\mathrm{Aut}(\mathbb{F})$ is cyclic, and Ψ is a solvable subgroup of $\mathrm{Aut}(\mathcal{R}_{(\mathbb{K},+)})$. It remains to study the cases where $|\mathbb{K}| = p^2 \in \{4,9\}$. We then have $|\Psi| = 12(p^2-1)p^4$, while $|\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_p^2})| = 6 \cdot |\mathrm{GL}_2(\mathbb{F}_p)| \cdot p^4 = 6(p^2-1)(p-1)p^5$. These orders coincide if $p = 2$ but they are different if $p = 3$. □

5.12 Remark. The Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$ has exactly one resolution (namely, the one discussed in 5.10, for $u = 1$). Adding a point for each parallel class, we obtain a *unique* extension to the dual of $\mathbb{A}_2(\mathbb{F}_3)$. This gives an alternative proof of 3.2.

5.13 Remark. As in the proof of 5.8, we see that the orbit of the set $\mathcal{F}^u := \big\{\{X_s, Y_{su}, Z_{s(1+u)}\} \bigm| s \in \mathbb{K}\big\}$ under the subgroup $\big\{{}^0\mathrm{id}_v^0 \bigm| v \in \mathbb{K}\big\}$ of $\mathrm{Aut}(\mathcal{R}_{(\mathbb{K},+)})$ forms a resolution of $\mathcal{R}_{(\mathbb{K},+)}$. If $\mathbb{K}$ is not commutative, this resolution may differ from the one constructed in the proof of 5.8; in fact, the embedding in 5.10 will not map the parallel class $\mathcal{F}^u$ to a set of confluent lines unless u lies in the center of $\mathbb{K}$.

5.14 Examples. For any (not necessarily commutative) field $\mathbb{K}$, we consider a subgroup Δ of the multiplicative group $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$, and embed the structure $\mathcal{R}_\Delta$ into the projective plane over $\mathbb{K}$, as follows. For $a \in \Delta$, let $\hat{X}_a := \mathbb{K}(a, 1, 0)$, $\hat{Y}_b := \mathbb{K}(1, 0, b)$, and $\hat{Z}_c := \mathbb{K}(0, 1, -c)$. The image of the block $B_{a,b} = \{X_a, Y_b, Z_{ab}\}$ is then contained in the line $\hat{B}_{a,b} = \ker(-b, ab, 1)^\intercal$. Each set $U \in \{X, Y, Z\}$ is mapped into a line $\hat{U}$; we have $\hat{X} = \ker(0, 0, 1)^\intercal$, $\hat{Y} = \ker(0, 1, 0)^\intercal$, and $\hat{Z} = \ker(1, 0, 0)^\intercal$.

Note that the action of the subgroup $\big\{{}^k\mathrm{id}_r^m \bigm| k, m, r \in \Delta\big\}$ extends to an action by collineations of the projective plane; the extension of ${}^k\mathrm{id}_r^m$ is induced by the linear map $(x, y, z) \mapsto (xm, yk, zr)$. Extensions of the automorphisms τ_0, τ_1, and τ_2 (see 1.3) are induced by $(x, y, z) \mapsto (y, x, -z)$, by $(x, y, z) \mapsto (z, -y, x)$, and by $(x, y, z) \mapsto (-x, -z, -y)$, respectively.

This construction can also be found (for the special case of a commutative field $\mathbb{K}$ of characteristic 0, and then a finite cyclic group Δ) in [19, Prop. 6]. The involutory homologies in [19, Prop. 7] are conjugates of the extension of our automorphism τ_0.

5.15 Remark. Every finite subgroup of the multiplicative group of a commutative field $\mathbb{K}$ is cyclic, and this extends to the case of non-commutative fields of positive characteristic, see [11, Thm. 6]. If Δ is a finite subgroup of the multiplicative group of a non-commutative field $\mathbb{K}$ of characteristic 0 then the Sylow p-subgroups of Δ are cyclic for each odd prime p, and the Sylow 2-subgroup is either cyclic or a generalized quaternion group; see [1, Thm. 2]. Finite groups with only cyclic Sylow subgroups have been determined by Zassenhaus [35, Satz 5], see [1, Lemma 1]; here we have metacyclic groups satisfying an additional arithmetic constraint, and of course cyclic groups. If the Sylow 2-subgroups of Δ are generalized quaternion groups but all other Sylow subgroups are cyclic then [1] gives a complete and explicit classification. In particular, two-fold coverings of dihedral groups, of Alt_4, of Sym_4 and of Alt_5 (i.e., the binary dihedral, tetrahedral, octahedral, and icosahedral groups which occur as subgroups of the multiplicative group of the quaternion field over the real numbers) play an essential role; the group Δ is a direct product of one of these binary groups and some suitable group with all Sylow subgroups cyclic.

5.16 Remark. Note that 5.14 yields embeddings of the Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$ into the projective plane $\mathbb{P}_2(\mathbb{F}_4)$, and of $\mathcal{R}_{\mathrm{C}_4}$ into $\mathbb{P}_2(\mathbb{F}_5)$. Any embedding of $\mathcal{R}_{\mathrm{C}_3}$ obtained in 5.14 extends to an embedding of the affine plane $\mathbb{A}_2(\mathbb{F}_3)$ of order 3 because each one of the sets X, Y, and Z is mapped into a line. See 3.7, and note that $\mathbb{K}^\times$ contains a group of order 3 precisely if $\mathbb{K}$ contains a root $u \neq 1$ of $X^2 + X + 1$. The parallel classes $\mathcal{E}^1.{}^0\mathrm{id}_v^0$ from the proof of 5.8 will not become confluent under the embedding in 5.14. In fact, if they were confluent, the present embedding of $\mathbb{A}_2(\mathbb{F}_3)$ would extend to an embedding of $\mathbb{P}_2(\mathbb{F}_3)$ into $\mathbb{P}_2(\mathbb{K})$, and lead to an embedding of the field $\mathbb{F}_3$ into $\mathbb{K}$. Such an embedding is impossible because the multiplicative group of a field of characteristic 3 will never contain a group of order 3.

6 Remarks on $12_4 16_3$ configurations and their automorphisms

An incidence structure is called a $p_s\,b_t$ configuration if it has p points and b blocks such that s blocks pass through each point and t points are on each block, and there are no digons (i.e., if two points are joined by a block then that block is unique). A p_s configuration is a $p_s\,b_t$ configuration with $p_s = b_t$.

The Pappus configuration is a 9_3 configuration, and there are three isomorphism types of such configurations (cf. [12, § 17] or [13, § 17]). Only two of them admit an automorphism group transitive on the set of points (see [12, p. 96] or [13, p. 109]).

The Reye configuration is a $12_4\,16_3$ configuration. In [3] the results of a computer search are used to show that only three isomorphism types of $12_4\,16_3$ configurations admit an automorphism group transitive on the set of points. Those configurations have automorphism groups of order 12, 192, and 576, respectively. The latter two types are the configurations $\mathcal{R}_{\mathrm{C}_4}$ and $\mathcal{R}_{\mathrm{V}_4}$, respectively.

7 Incidence graphs and dualities

For any incidence structure (with points and blocks), the *incidence graph* has as vertex set the disjoint union of the set of points with the set of blocks, a subset $\{x, y\}$ of that vertex set forms an edge precisely if either (x, y) or (y, x) is a flag. By its very construction, every incidence graph is bi-partite; we will draw the points as white vertices, and the blocks as black ones.

The incidence graph of the Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$ is shown in Figure 10; incidence graphs of the Reye configuration $\mathcal{R}_{\mathrm{V}_4}$ and the configuration $\mathcal{R}_{\mathrm{C}_4}$, respectively, are shown in Figure 11. Note that only three edges are changed when passing from the incidence graph of $\mathcal{R}_{\mathrm{V}_4}$ to that of $\mathcal{R}_{\mathrm{C}_4}$; the changed edges are marked in Figure 11.

An abstract automorphism of the incidence graph of an incidence geometry induces an automorphism of that structure if, and only if, it preserves the sets of points and the set of blocks, respectively (i.e., if it preserves the colors white and black in the vertex set). If an automorphism swaps the colors then it induces a *duality* of the incidence structure (swapping points and lines and reversing incidence).

7.1 Theorem. *The incidence structure $\mathcal{R}_\Delta$ admits a duality precisely if $\Delta \cong \mathrm{C}_3$. In that case, one even has polarities (i.e., dualities that are involutions). In particular, there exist $2^2 \cdot 3^3 = 108$ many dualities of the Pappus configuration $\mathcal{R}_{\mathrm{C}_3}$. Among* *these dualities, we have* 18 *polarities. The polarities of* $\mathcal{R}_{\mathrm{C}_3}$ *form a single conjugacy class under* $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_3})$.

Proof: Every block has three points, but the cardinality of the line pencil $\mathcal{B}_{X_1} = \big\{\{X_1, Y_b, Z_b\} \bigm| b \in \Delta\big\}$ equals the cardinality of Δ. So the existence of a duality implies $\Delta \cong \mathrm{C}_3$. In order to verify that $\mathcal{R}_{\mathrm{C}_3}$ does admit polarities, it remains to look at Figure 10 which clearly exhibits (involutory) automorphisms of the incidence graph of the Pappus configuration that swap the colors.

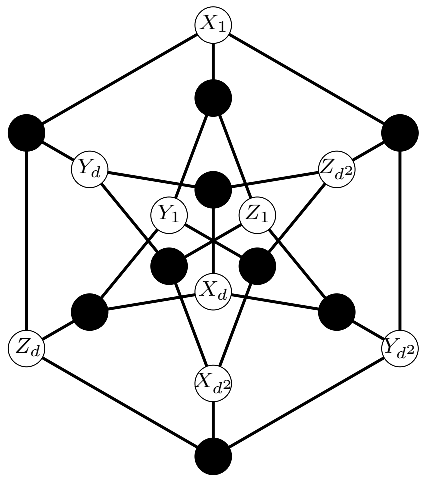

Figure 10: Incidence graph of the Pappus configuration.

We also give a polarity π explicitly in terms of the embedding of $\mathcal{R}_{\mathrm{C}_3}$ into $\mathbb{P}_2(\mathbb{F}_3)$, as described in 3.1. Interchanging subspaces of $\mathbb{F}_3^3$ with their orthogonal spaces with respect to the quadratic form $q(x, y, z) := y^2 - xz$ gives a polarity of the projective plane, and π maps the point $p_\infty := \mathbb{F}_3(0, 0, 1)$ to the line $L_\infty := \mathbb{F}_3(0, 1, 0)+\mathbb{F}_3(0, 0, 1)$. Therefore, the embedding given in 3.1 is invariant under π, and we may regard π as a polarity of $\mathcal{R}_{\mathrm{C}_3}$. Using $\left(\begin{smallmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{smallmatrix}\right)$ as a Gram matrix for q and the fact (see 3.1) that $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_3})$ is induced by the group $\left\{\left(\begin{smallmatrix} r&x&z\\ 0&s&y\\ 0&0&t \end{smallmatrix}\right) \middle| \ r, s, t, x, y, z \in \mathbb{F}_3, rst \neq 0\right\}$, it is easy to see that the centralizer of π is induced by $\left\{\left(\begin{smallmatrix} r&x&rx\\ 0&s&-rsx\\ 0&0&r \end{smallmatrix}\right) \middle| \ r, s, x \in \mathbb{F}_3, rs \neq 0\right\}$. Therefore, the conjugacy class of π under $\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_3})$ contains 18 elements.

The number of dualities of $\mathcal{R}_{\mathrm{C}_3}$ equals the number $\left|\mathrm{Aut}(\mathcal{R}_{\mathrm{C}_3})\right| = 2^2 \cdot 3^3 = 108$ of automorphisms, cf. 1.4 (d). In fact, if δ is any duality of $\mathcal{R}_{\mathrm{C}_3}$ then $\pi^{-1}\delta$ is an automorphism, and extends to an automorphism α of $\mathbb{P}_2(\mathbb{F}_3)$ by 3.1. Now $\delta = \pi\alpha$ is (induced by) a duality of $\mathbb{P}_2(\mathbb{F}_3)$. We obtain that every polarity of $\mathcal{R}_{\mathrm{C}_3}$ is induced by a polarity of $\mathbb{P}_2(\mathbb{F}_3)$ having (π_∞, L_∞) as one of its absolute flags. Every such polarity has three more absolute flags, say (p_j, L_j) with $j \in \{0, 1, 2\}$. The four absolute points form a non-degenerate conic (that is, a quadrangle) in $\mathbb{P}_2(\mathbb{F}_3)$. We infer that the points p_0, p_1, p_2 lie in $\mathcal{R}_{\mathrm{C}_3}$, and they are not collinear but pairwise collinear. There are just 18 such sets of points in $\mathcal{R}_{\mathrm{C}_3}$, and (together with the absolute point p_∞ at infinity) any such set determines the respective polarity. We obtain that the polarities of $\mathcal{R}_{\mathrm{C}_3}$ are just those in the conjugacy class of π. □

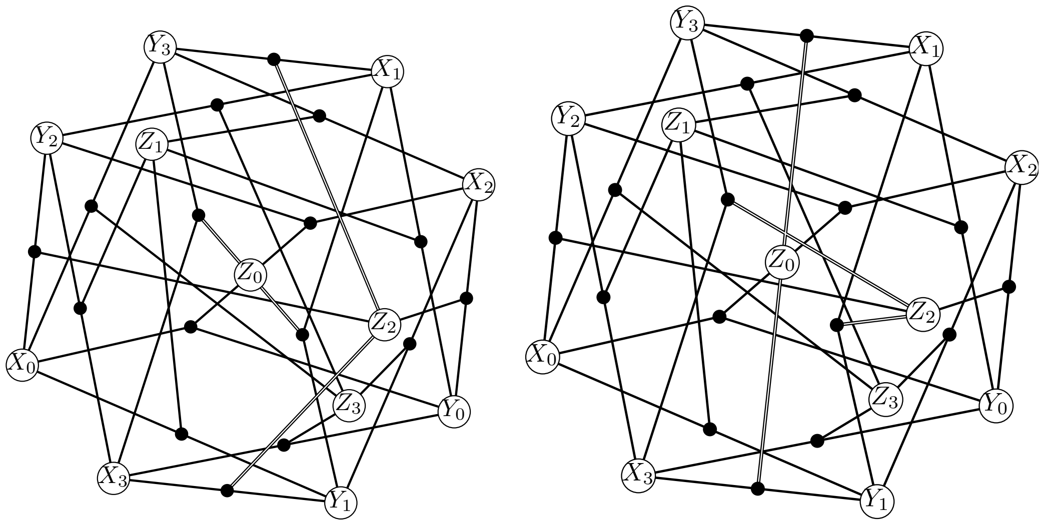

Figure 11: Incidence graphs of the Reye configuration $\mathcal{R}_{\mathrm{V}_4}$ (left) and the configuration $\mathcal{R}_{\mathrm{C}_4}$ (right).

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